Name: ____________________________

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MIDTERM

Math 115 - Spring 2002

1. Let $u(x)$ be a function which represents the speed of a particle that is located at the point $x \in I\!R$. The units of x are $[x] = L$ and those of $u(x)$ are $[u(x)] = LT^{-1}$.

   (a) What are the unites of:

   (i) $[u^2(x)]$ (ii) $\left[\frac{d}{dx}u(x)\right]$ (iii) $\left[\int_a^b u(x)dx\right]$ (iv) $[\ |u(x)|\ ]$.

   (b) Do the following equations make sense physically, and why (do not solve the equations):

   (i) $\frac{d^2}{dx^2}u(x) = -u(x)$ (ii) $\frac{d^2}{dx^2}u(x) = -\frac{u(x)}{x^2}$, for $x > 0$.

2. Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s^2, while air resistance provides 0.1 ft/s^2 deceleration for each foot per second of the car's velocity. (a) Find the car's maximum possible (limiting) velocity. (b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

3. (a) Find a homogeneous difference equation with constant coefficients whose characteristic polynomial is:

$$(r-2)^3(r-5).$$

   (b) Find the most general solution to the equation you found in part (a).

4. (a) Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with real coefficients of degree n. Show that if $r = \lambda + i\mu$ is a complex root for $p(x)$ then $\overline{r} = \lambda - i\mu$ is also a root.

   Hint: Use the Fundamental Theorem of Algebra.

   (b) Let $p(x)$ be a polynomial with real coefficients of degree n, where n is an odd positive integer. Show that p(x) has at least one root which is a real number.

5. **Salmon Dynamics**

Suppose we wish to know the size of salmon population at the end of each spawning cycle. Let x_n be the number of hundreds of millions of salmon at the end of cycle n, and therefore also at the beginning of cycle $n+1$. These salmon are all adults. Let $y_{n+1}(t)$ denote the larval population (again in hundreds of millions) during the subsequent cycle (i.e., cycle $n+1$). How is the initial larval population, $y_{n+1}(t_a)$, related to x_n? The more salmon there are, the more females there are, the more eggs they lay, and hence the more larvae there are. The simplest assumption incorporating this observation is the that number of larvae is proportional to the number of adults at the end of the previous cycle, that is

$$y_{n+1}(t_a) = \alpha x_n,$$

where α is a constant.

What happens to these larvae? The adults cannibalize them. The more adults there are, the more larvae that are eaten. Thus the simplest assumption incorporating this observation is that the larval population decays at a rate proportional to the adult population – which, *in short term*, is fixed. Thus

$$\frac{1}{y_{n+1}}\frac{dy_{n+1}}{dt} = -\beta x_n,$$

where β is a constant. Let $t_b > t_a$ be the time need for the larvae that survive to become mature young salmon.

(i) Show that the number (in hundreds of millions) of larvae that survive to mature young salmon is given by:

$$y_{n+1}(t_b) = y_{n+1}(t_a)\exp(-\beta x_n(t_b - t_a)) = \alpha x_n \exp(-\beta x_n(t_b - t_a)).$$

What happens to these young salmon? The ocean is fraught with risk for them and not all will survive to breed. Let us assume that a fraction γ will survive. Then at the end of the cycle, the salmon population is $\gamma y_{n+1}(t_b)$ *plus* the surviving adults. How many adults survive? In the case of Pacific salmon, we can safely assume the answer is zero, because the adults die soon after spawning. To keep things simple let us assume we are interested in Pacific salmon. Then

$$x_{n+1} = \gamma y_{n+1}(t_b).$$

(ii) Show that by the end of the cycle $n+1$ we have

$$x_{n+1} = K x_n e^{-\lambda x_n} \ ,$$

for some positive constants K and λ.

Based on the above the sequence x_{n+1} is given as a special case of the recurrence relation, $x_{n+1} = F(x_n)$, where $F(x) = Kxe^{-\lambda x}$.

A point x^* is called a fixed point, or an equilibrium, if $F(x^*) = x^*$.

(iii) Find all the fixed points of the Salmon Dynamics.

A fixed point x^* is called exponentially asymptotically stable if there exists $\epsilon > 0$ such that for every x_1 with the property $|x_1 - x^*| < \epsilon$ we have $|x_n - x^*| \leq \theta^n M$, for every $n = 1, 2, \cdots$, where $M > 0$ and $0 \leq \theta < 1$ are fixed.

(iv) Let $F(x)$ be continuous with continuous first derivative. Use Taylor expansion to show that if $|F'(x^*)| < 1$ then the fixed point x^* is exponentially asymptotically stable.

(v) Find sufficient conditions on K and λ so that the fixed points for the salmon dynamics are exponentially asymptotically stable.

6. Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary, which is filled with viscous incompressible homogeneous fluid. We will denote by $|\Omega|$ the volume of Ω. Let $u(x,t) = (u_1, u_2, u_3)$ be the velocity field, and $p(x,t)$ be the pressure form a smooth solution to the Navier–Stokes equations in the domain Ω, subject to the no-slip homogeneous boundary condition, i.e. $u = 0$ on the boundary of Ω.

(a) Let $\tau = \nu^\theta |\Omega|^\gamma$ be a typical unit of time. Find what should be the numerical values of θ and γ.

(b) Let τ be as in part (a) we denote by

$$\epsilon = |\Omega|^{-1}\frac{1}{\tau}\int_0^\tau \int\int\int_\Omega \nu \sum_{k,j=1}^{3} |\frac{\partial u_k}{\partial x_j}|^2 dx dt,$$

the mean rate of dissipation of kinetic energy in turbulent flow. Find the units of ϵ.

(c) It is commonly accepted, by the turbulence community, that in homogeneous isotropic turbulent flows the energy cascades down from larger eddies (larger spatial scales) to smaller eddies at a rate which is related to ϵ. This cascade goes on until we reach to smallest possible eddies, the so-called viscous eddies, where the viscosity acts and transfers the kinetic energy into heat. The typical diameter of the viscous eddy is denoted by l_d and is called the Kolmogorov dissipation length scale in turbulent flows. Suppose that $l_d = \nu^\alpha \epsilon^\beta$. Find what are the exact numerical values of α and β.

(d) Use the divergence Theorem, i.e. Stokes Theorem, to show that

$$\int\int\int_\Omega \nabla p(x,t) \cdot u(x,t)\ dx = 0.$$

GOOD LUCK!!