Name: ______________________________

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TAKE HOME – FINAL EXAM

Math 115 - Spring 2002

1. Use asymptotic methods to find an approximate solution $\bar{x}(t)$ to the initial value problem

$$\ddot{x} + x = \epsilon x^2, \qquad x(0) = \sqrt{1+\epsilon}\ , \quad \text{and} \qquad \dot{x}(0) = 0,$$

such that $|x(t) - \bar{x}(t)| = O(\epsilon^3)$, where t lies in an $O(1)$ interval of time.

2. Find the largest positive integer k such that

$$\sin(\sqrt{1+\epsilon} - (1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2)) = O(\epsilon^k), \qquad \text{as} \quad \epsilon \to 0.$$

If there is no such k explain why and justify your answer.

3. Let $f(\epsilon)$ be a continuous function at $\epsilon = 0$.

(i) If $f(\epsilon) = o(\epsilon)$, show that $\frac{df}{d\epsilon}(0)$ exists, and find $\frac{df}{d\epsilon}(0)$.

(i) If $f(\epsilon) = O(\epsilon)$ instead, can you say anything about the questions in part (i). Explain your steps, and provide counter examples if necessary.

4. **Transport and Diffusion**

Let Ω be a bounded domain in $I\!R^3$ with smooth boundary, which is filled with viscous incompressible homogeneous fluid. Let $u(x,t) = (u_1, u_2, u_3)$ be a given smooth velocity field of this fluid, subject to the no-slip homogeneous boundary condition, i.e. $u = 0$ on the boundary of Ω.

(a) Let $\phi(x,t)$ be a smooth scalar function. Show that

$$\nabla \cdot (u\ \phi^2) = 2((u(x,t) \cdot \nabla)\phi(x,t))\ \phi(x,t)$$

(b) Let $\phi(x,t)$ be smooth scalar function show that

$$\int\int\int_\Omega ((u(x,t) \cdot \nabla)\phi(x,t))\ \phi(x,t) dx = 0$$

Hint: Use part (a) and the Divergence Theorem.

(c) Suppose that the Temperature $T(x,t)$ of this incompressible fluid is governed by the following transport heat (diffusion) equation:

$$\frac{\partial T}{\partial t} - \kappa \Delta T + (u(x,t) \cdot \nabla)T = 0,$$

with initial and boudary conditions:

$$T(x,0) = T_0(x) \quad \text{in} \quad \Omega, \text{ and} \qquad T(x,t) = \Psi(x,t) \quad \text{on the boundary of} \quad \Omega.$$

Assume that all the functions and solutions involved are very smooth, and that one can switch integration with differentiation. Show that if the above heat equation has a solution then it is unique.

Hint: Assume that the problem has to solution T_1 and T_2, then find an equation for the difference $\Theta = T_1 - T_2$. After then multiply the equation you got for the variable Θ with Θ and use the above parts to show that the difference $\Theta = T_1 - T_2$ satisfies

$$\frac{d}{dt}\int\int\int_\Omega |\Theta(x,t)|^2 dx + 2\kappa \int\int\int_\Omega |\nabla\Theta(x,t)|^2 dx = 0.$$

As a result of the above show that $\Theta = 0$.

5. **Dynamical Systems**

Consider the map $x_{n+1} = f(x_n)$. Let u, v be two points of period 2, i.e. $v = f(u)$ and $u = f(v)$. We say u, v is stable if u is a stable fixed point to $g(x) = f^2(x)$.

(a) Show that $|f'(u)f'(v)| < 1$ implies u, v is a stable periodic orbit.

(b) Consider the logistic map: $x_{n+1} = f_\mu(x_n) = \mu x_n(1 - x_n)$. Show that for $3 < \mu < 1 + \sqrt{5}$, the period-2 points are $\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu}$ and $\frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu}$.

(c) Apply part (a) to part (b) and show that the period-2 soltions in the interval $3 < \mu < 1 + \sqrt{5}$ are stable.

(d) What is the situation for $2 < \mu < 3$? and what do we call the point $\mu = 3$?