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Attractors for the Two-Dimensional Navier–Stokes-α Model: An α-Dependence Study

A. A. Ilyin[1] **and E. S. Titi**[2]

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The two-dimensional Navier–Stokes-α model is considered on the torus and on the sphere. Upper and lower bounds for the dimension of the global attractors are given. The dependence of the dimension of the global attractors on α is studied. Special attention is given for the limiting cases when $\alpha \to 0$, that is, when the Navier–Stokes-α model tends to the Navier–Stokes equations, and to the case when $\alpha \to \infty$.

KEY WORDS: Navier–Stokes-α models; attractors; lower bounds.

INTRODUCTION

The Euler equations of motion for ideal (inviscid) incompressible fluids can be derived from Hamilton variational principle subject to the incompressibility constraint div $u = 0$, where u is the velocity field. In this framework the pressure term in the Euler equations is the Lagrange multiplier corresponding to this constraint (see, for instance, [1] and the references therein). The Hamiltonian in this case involves the kinetic energy represented by the square of the L_2-norm of the velocity field $\int |u(x,t)|^2\,dx$. In their studies of one-dimensional models of water waves Camassa and Holm [4] derived a new model based on Hamilton variational principle in which the Hamiltonian involves the one-dimensional H^1 Sobolev norm $\int (|u(x,t)|^2 + \alpha^2\,|u_x(x,t)|^2)\,dx$, where α is length scale. Recently, Holm, Marsden, and

[1] Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya Sq. 4, 125047 Moscow, Russia. E-mail: ilyin@spp.keldysh.ru

[2] Department of Mathematics and Department of Mechanical and Aerospace Engineering, University of California, Irvine, California 92697. E-mail: etiti@math.uci.edu. Also: Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, P.O. Box 26, Rehovot, 76100, Israel.

Ratiu [24] combined the above mentioned approaches and derived the following set of equations in $\mathbb{R}^n$, $n \geqslant 2$, based on Hamilton variational principle and subject to the incompressibility constraint div $u = 0$:

$$\begin{aligned} \partial_t v - u \times \operatorname{rot} v + \nabla p &= 0, \\ \operatorname{div} u &= 0, \\ v &= u - \alpha^2 \Delta u. \end{aligned} \tag{1}$$

Here the Hamiltonian involves $\int (|u(x,t)|^2 + \alpha^2 |\nabla u(x,t)|^2)\, dx$, the square of $H^1(\mathbb{R}^n)$ Sobolev norm, $n \geqslant 2$. By adding the viscous/damping term $-\nu\, \Delta u$ to the left hand side of the first equation in (1) one obtains the equations of motion for second-grade visco-elastic non-Newtonian fluids (see, for instance, [10], [11], [18], and [36]). In [19] (see also [6–8]) it is proposed to add the viscous term $-\nu\, \Delta u$ and a forcing term f, in an *ad hoc* fashion, to the left and right hand sides of the first equation in (1), respectively. As a result one obtains the so-called viscous Camassa–Holm equations (they are also known as the Navier–Stokes-α model (NS-α) or the Lagrangian averaged Navier–Stokes-α model (LANS-α)):

$$\begin{aligned} \partial_t v - \nu\, \Delta u - u \times \operatorname{rot} v + \nabla p &= f, \\ \operatorname{div} u &= 0, \\ v &= u - \alpha^2 \Delta u. \end{aligned} \tag{2}$$

Under certain physical hypothesis the inviscid model (1) can be derived as an averaged equation, based on Lagrangian averaging, using rigorous mathematical tools and physical arguments (see, for instance, [7], [23], [25], and [32–34]). In [12], however, another approach connecting Lagrangian and Eulerian formulations for the Navier–Stokes equations (NSE) was introduced. This exact connection between Lagrangian and Eulerian formulations gives another perspective for looking at the relation between the Navier–Stokes equations (NSE) and the Navier–Stokes-α model (NS-α). Due to the lack of rigorous physical derivation of the system (2) the corresponding boundary conditions required for flows confined in domains with boundaries are still not available (see, however [32] and [33] for an attempt in this direction). Therefore the system (2) is considered in [19] subject to periodic boundary conditions and the global regularity of the three-dimensional system is established. Furthermore, upper estimates for the dimension of its global attractor were provided. Based on these interesting upper bounds the authors of [19] proposed a connection between this model (NS-α) and Reynolds averaged Navier–Stokes equations. Indeed, the system (2) was tested successfully against empirical data in [6–8] as a closure model for the Reynolds equations in infinite channels and pipes and for a wide range of Reynolds numbers.

In this paper we consider the two-dimensional version of (2) and study the dependence of the dimension of its global attractors on the parameter α. There are two limiting cases which are of special interest. The first is when $\alpha \to 0^+$. In this case one can easily show that the solutions of (2) converge uniformly, on finite intervals of time, to the corresponding solutions of the Navier–Stokes equations (see [19] for such results concerning weak solutions in the three-dimensional case). Because of this fact one can easily show that the global attractors $\mathscr{A}_{NS-\alpha}$ of the NS-α converge to a subset of the global attractor $\mathscr{A}_{NS}$ of the NSE as $\alpha \to 0^+$ (see, for instance, [3], [21] for general results on upper semi-continuity of attractors depending on a parameter). We also find here that the upper bounds for the dimension of the global attractors, $\mathscr{A}_{NS-\alpha}$ of the NS-α also converge to the corresponding upper bounds for the global attractors, $\mathscr{A}_{NS}$, of NSE. Using a family of Kolmogorov flows as base flows we can deduce also lower bounds on the dimension of the global attractors. Here again our results indicate that the lower bounds for the dimension of $\mathscr{A}_{NS-\alpha}$ converge to the associated lower bounds of the dimension of $\mathscr{A}_{NS}$, as $\alpha \to 0^+$. In particular, in case one accepts the point of view that the dimension of a global attractor for the NSE is associated with the number of degrees of freedom in turbulent flows, then in our case these sharp bounds on the dimension of the global attractor give a rigorous justification for the NS-α model analogue of the Kraichnan approach to the 2-D turbulence (see, for example, [16] and [37]).

The other interesting limit is when $\alpha \to \infty$. In this case it can easily be shown, using energy estimates, that the dynamics is trivial and that all solutions tend exponentially to a unique steady state (which tends to zero as $\alpha \to \infty$). Therefore in order to get a non-trivial dynamics it is necessary to re-scale the forcing term with $(\alpha/L)^2$, that is, to replace f by $(\alpha/L)^2 f$, and let $\alpha \to \infty$ to arrive to the following system:

$$
\begin{aligned}
\partial_t v - \nu\, \Delta u - u \times \operatorname{rot} v + \nabla p &= f, \\
\operatorname{div} u &= 0, \\
v &= -L^2\, \Delta u,
\end{aligned}
\tag{3}
$$

where $L > 0$ is the size of the periodic domain. Here again one can easily establish the global regularity of the NS-∞ system (3) and the existence of the global attractor $\mathscr{A}_{NS-\infty}$. Denoting by $G = \|f\|_{L^2}\, L^2 \nu^{-2}$ the dimensionless Grashof number we prove that in the case of the system (3) there exists a universal constant $C > 0$ such that $\dim_H \mathscr{A}_{NS-\infty} \leqslant \dim_F \mathscr{A}_{NS-\infty} \leqslant C G^{2/3}$.

We observe that the corresponding upper bound for the dimension of the global attractor, $\mathcal{A}_{NS}$, of the classical NSE is $\dim_H \mathcal{A}_{NS} \leqslant \dim_F \mathcal{A}_{NS} \leqslant CG^{2/3}(\log(1+G))^{1/3}$ (see, for instance, [14], [15], and [37]). Hence, there is an improvement of a logarithmic term. Following the usual method of linearizing (3) around the corresponding Kolmogorov flow [3], [31], [35], [39], [40] we find a lower bound for the dimension of the global attractor $\mathcal{A}_{NS-\infty-\mathrm{Kol}}$. Indeed, we show that for this special flow there exists a universal constant $c>0$ such that

$$cG^{2/5} \leqslant \dim \mathcal{A}_{NS-\infty-\mathrm{Kol}},$$

while the corresponding lower bound for the global attractor of the NSE $\mathcal{A}_{NS-\mathrm{Kol}}$ based on the Kolmogorov flow is

$$cG^{2/3} \leqslant \dim \mathcal{A}_{NS-\mathrm{Kol}},$$

Therefore there is a discrepancy between our upper and lower bounds for the global attractor $\mathcal{A}_{NS-\infty}$ of the system (3), while these estimates are sharp (up to a logarithmic term) in the NSE case. We believe that our lower bound estimate using the Kolmogorov flow approach is sharp and there is no room for improvement by using the Kolmogorov flows as base solutions. Therefore this poses the question about how sharp are our estimates, both from above and below, for the system (3), a subject of future research.

1. SPACE PERIODIC NAVIER–STOKES-α MODEL

We shall be dealing with the following two-dimensional Navier–Stokes-α model (also known as viscous Camassa–Holm equations) [19]:

$$\begin{aligned} \partial_t(u-\alpha^2 \Delta u)-\nu\Delta(u-\alpha^2 \Delta u)-u\times\operatorname{rot}(u-\alpha^2 \Delta u) &= -\nabla p+f, \\ \operatorname{div} u &= 0, \\ u(0) &= u_0, \end{aligned} \tag{4}$$

where u is the velocity vector field, p is the modified pressure, and $\nu>0$ is the viscosity coefficient. The spatial variable x belongs to the two-dimensional torus $T=[0, 2\pi L]^2$ and $\alpha^2=\gamma^2 L^2$ is a parameter (L^2 is singled out here so that γ is dimensionless).

Next, since $\operatorname{div} u=0$, we have

$$\Delta u = -\operatorname{rot}\operatorname{rot} u + \nabla \operatorname{div} u = -\operatorname{rot}\operatorname{rot} u.$$

The operator rot has the conventional meaning and in two dimensions maps vectors to scalars and scalars to vectors:

$$\operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1; \qquad \operatorname{rot} \psi = \partial_2 \psi \cdot e_1 - \partial_1 \psi \cdot e_2. \tag{5}$$

It will be convenient in what follows to identify scalars and normal vectors (that is, vectors parallel to e_3). We also set $n = e_3$.

We assume that $\int_T u_0(x)\, dx = \int_T f(x)\, dx = 0$. Then it can easily be shown that $\int_T u(t, x)\, dx = 0$ for all $t > 0$. This shows that there exists a stream function ψ:

$$u = -\operatorname{rot} \psi = n \times \nabla \psi. \tag{6}$$

The unique choice of ψ is fixed by the condition $\int_T \psi\, dx = 0$.

The passage for incompressible equations of hydrodynamics in two dimensions from a vector system to a single scalar equation is classical and well known. This is especially convenient for the analysis of the attractor dimension (see [17], [27], where the Lieb–Thirring inequality has been avoided). We substitute (6) into the first equation in (4) and apply the operator rot.

Using the formula

$$\operatorname{rot} \operatorname{rot} \psi = -\Delta \psi \cdot n = -\Delta \psi \tag{7}$$

we have for $u = -\operatorname{rot} \psi = n \times \nabla \psi$ and $v = n \times \nabla \varphi$

$$\operatorname{rot} u = \Delta \psi, \qquad \operatorname{rot} \Delta u = \Delta^2 \psi, \qquad \operatorname{rot}(u \times \operatorname{rot} v) = -J(\psi, \Delta \varphi) \tag{8}$$

and we obtain

$$\partial_t (\Delta \psi - \alpha^2 \Delta^2 \psi) - \nu \Delta (\Delta \psi - \alpha^2 \Delta^2 \psi) + J(\psi, \Delta \psi - \alpha^2 \Delta^2 \psi) = \operatorname{rot} f \equiv F. \tag{9}$$

Setting $\varphi = \Delta \psi - \alpha^2 \Delta^2 \psi$ and taking into account the condition $\int_T \psi\, dx = 0$ we have $\psi = (\Delta - \alpha^2 \Delta^2)^{-1} \varphi$ and finally obtain

$$\begin{aligned} &\partial_t \varphi - \nu \Delta \varphi + J((\Delta - \alpha^2 \Delta^2)^{-1} \varphi, \varphi) = F, \\ &\qquad\qquad \varphi(0) = \operatorname{rot}(u_0 - \alpha^2 \Delta u_0). \end{aligned} \tag{10}$$

The bilinear operator J (the Jacobian) here is defined as follows

$$J(a, b) = n \times \nabla a \cdot \nabla b = \partial_1 a\, \partial_2 b - \partial_2 a\, \partial_1 b$$

and has the following well-known properties:

$$\int_T J(a,b)\,dx=0,\quad \int_T J(a,b)\,c\,dx=\int_T J(b,c)\,a\,dx,\quad \int_T J(a,b)\,b\,dx=0. \tag{11}$$

We define the phase space $H=L_2\cap\{\varphi\colon \int\varphi\,dx=0\}$ with L_2-norm $\|\cdot\|_H=\|\cdot\|$. It is standard to show (see, for instance, [2], [3], [14], [37]) that for $\varphi_0\in H$ and $F\in H^{-1}$ the Eq. (10) has a unique solution $\varphi\in C([0,T];H)\cap L_2(0,T;H^1)$ (accordingly, in terms of Eq. (4), $u_0\in H^3$, $f\in L_2$). Hence, the semi-group $S_t\colon H\to H$ is defined: $S_t\varphi_0=\varphi(t)$, where $\varphi(t)$ is the solution of (10).

Since $F\in H^{-1}$ the Eq. (10) holds in H^{-1}. Therefore all our a-priori estimates below are formal and can be justified rigorously either by using the usual Galerkin procedure or by generalizations of the inner products and the identities in (11) (see, for example, [3], [14], [37]).

Multiplying (10) by φ and using (11) we obtain

$$\begin{aligned}\partial_t\|\varphi\|^2+2\nu\|\nabla\varphi\|^2&=2(\operatorname{rot} f,\varphi)=2(f,\operatorname{rot}\varphi)\\ &\leqslant 2\|f\|\,\|\operatorname{rot}\varphi\|=2\|f\|\,\|\nabla\varphi\|\leqslant\nu\|\nabla\varphi\|^2+\nu^{-1}\|f\|^2.\end{aligned}$$

Using the Poincaré and Gronwall inequalities and integrating with respect to t we find

$$\begin{aligned}\limsup_{t\to\infty}\|\varphi(t)\|^2&\leqslant\frac{\|f\|^2}{\lambda_1\nu^2},\\ \sup_{\varphi_0\in B}\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|\nabla\varphi(\tau)\|^2\,d\tau&\leqslant\frac{\|f\|^2}{\nu^2}\end{aligned} \tag{12}$$

for bounded subsets B in H. Hence the semi-group S_t has an absorbing ball (of radius $2\|f\|^2/(\lambda_1\nu^2)$) in H and a global attractor $\mathscr{A}=\mathscr{A}_{NS-\alpha}\subset H$ which is a compact ($\mathscr{A}\Subset H$), strictly invariant ($S_t\mathscr{A}=\mathscr{A}$), and globally attracting set in the phase space H. (For the theory of global attractors see [3], [14], [21], [29], [37].)

Along with (10) we consider the variational equation corresponding to (10):

$$\begin{aligned}&\partial_t\Phi=\nu\,\Delta\Phi-J((\Delta-\alpha^2\Delta^2)^{-1}\varphi(t),\Phi)-J((\Delta-\alpha^2\Delta^2)^{-1}\Phi,\varphi(t))=:\mathscr{L}(t,\varphi_0)\,\Phi,\\ &\Phi(0)=\xi.\end{aligned} \tag{13}$$

It is standard to show that this equation has a unique solution denoted by

$$L(t, \varphi_0)\, \xi := \Phi(t).$$

Using the general theorems in [2] , [3] , [37] we can show that the semigroup S_t is differentiable on the attractor $\mathscr{A}_{NS-\alpha}$ and the differential is the linear operator

$$DS_t(\varphi_0) = L(t, \varphi_0)\colon \xi \in H \to \Phi(t) \in H,$$

$$\|S_t(\varphi_0 + \xi) - S_t \varphi_0 - L(t, \varphi_0)\, \xi\| \leqslant h(\|\xi\|, T)\, \|\xi\|,$$

where $t \in [0, T]$ and $h(r, T) \to 0$ as $r \to 0$. Moreover, the differentials $DS_t \varphi_0$ are continuous (even Hölder continuous) with respect to φ_0 in the norm $\|\cdot\|_{\mathscr{L}(H \to H)}$.

We now estimate the fractal dimension of the attractor.

Theorem 1. *The fractal dimension of the attractor $\mathscr{A}_{NS-\alpha}$ of the Eq. (10) is finite and satisfies for an absolute constant c_5 the estimate*

$$\dim_F \mathscr{A} \leqslant c_5 G^{2/3} \min(\log(1 + 1/\gamma^2), \log(1 + G))^{1/3} \tag{14}$$

where $G = \|f\|/(\nu^2 \lambda_1)$ is the Grashof number and $\lambda_1 = 1/L^2$ is the first eigenvalue of the Laplacian and $\gamma = \alpha/L$.

Proof. In the proof below we follow the ideas and estimates in [15] (see also [17] and [27]). We recall the definition of the global Lyapunov exponents. The numbers $q(m)$ (the sums of the first m global Lyapunov exponents) are defined for $m = 1, 2, \ldots$ as follows:

$$q(m) = \lim_{t \to \infty} \frac{1}{t} \log \bar{\omega}_k(t),$$

where

$$\omega_0(t, \varphi) = 1,$$

$$\omega_m(t, \varphi) = \alpha_1(t, \varphi)\, \alpha_2(t, \varphi) \cdots \alpha_m(t, \varphi),$$

$$\bar{\omega}_m(t) = \sup_{\varphi \in \mathscr{A}} \omega_m(t, \varphi),$$

the $\alpha_j(t, \varphi)$ being the eigenvalues of the self-adjoint positive operator $(LL^*)^{1/2}$ ordered according to their magnitude in a monotone non-increasing order. The quantities $q(m)$ play a vital role in the estimates of the dimension of the global attractor and this will be used later (see [13], [14], [37]).

Using the trace formula [13], [37] we have for the numbers $q(m)$ the estimate

$$q(m) \leqslant \limsup_{t\to\infty} \sup_{\varphi_0 \in \mathscr{A}} \sup_{\substack{\xi_i \in H\\ i=1,\ldots,m}} \frac{1}{t}\int_0^t \mathrm{Tr}[\mathscr{L}(\tau,\varphi_0)\circ Q_m(\tau)]\,d\tau, \tag{15}$$

where the operator $\mathscr{L}$ is defined in (10) and $Q_m(\tau)$ is the orthogonal projection in H onto $\mathrm{Span}\{\Phi_1(\tau),\ldots,\Phi_m(\tau)\}$ and Φ_i is the solution of (13) with $\Phi(0)=\xi_i$. Since $\Phi_i(\tau)\in H^1$ for $\tau>0$, we denote by $\theta_i=\theta_i(\tau)\in H\cap H^1$, $i=1,\ldots,m$ an orthonormal (in H) basis of $\mathrm{Span}\{\Phi_1(\tau),\ldots,\Phi_m(\tau)\}$ and set $v_i=v_i(\tau)=n\times\nabla(\varDelta-\alpha^2\varDelta^2)^{-1}\theta_i$. Then using the definition of $\mathscr{L}$ in (13), the properties of the Jacobian (11), and the orthonormality of the θ_j we obtain

$$\begin{aligned}
\mathrm{Tr}[\mathscr{L}(\tau,\varphi_0)\circ Q_m(\tau)] &= \sum_{j=1}^m (\mathscr{L}(t,\varphi_0)\,\theta_j,\theta_j)\\
&= -\nu\sum_{j=1}^m \|\nabla\theta_j\|^2 - \sum_{j=1}^m \int_T J((\varDelta-\alpha^2\varDelta^2)^{-1}\theta_j,\varphi(t))\,\theta_j\,dx\\
&= -\nu\sum_{j=1}^m \|\nabla\theta_j\|^2 - \int_T \sum_{j=1}^m \theta_j v_j\cdot\nabla\varphi\,dx\\
&\leqslant -\nu\sum_{j=1}^m \|\nabla\theta_j\|^2 + \int_T \left(\sum_{j=1}^m \theta_j^2\right)^{1/2}\left(\sum_{j=1}^m |v_j|^2\right)^{1/2} |\nabla\varphi|\,dx\\
&\leqslant -\nu\sum_{j=1}^m \|\nabla\theta_j\|^2 + \|\rho\|_\infty^{1/2}\left(\sum_{j=1}^m \|\theta_j\|^2\right)^{1/2}\|\nabla\varphi\|\\
&= -\nu\sum_{j=1}^m \|\nabla\theta_j\|^2 + \|\rho\|_\infty^{1/2}\, m^{1/2}\,\|\nabla\varphi\|,
\end{aligned}$$

where

$$\rho(x)=\sum_{j=1}^m |v_j(x)|^2.$$

We now use the estimate for ρ from Lemma 1 below (see (17)), the lower bound for the eigenvalues of the Laplacian

$$\sum_{j=1}^m \|\nabla\theta_j\|^2 \geqslant \sum_{j=1}^m \lambda_j \geqslant c_1\lambda_1 \sum_{j=1}^m j \geqslant \frac{c_1}{2}\,\lambda_1 m^2,$$

and the estimate (12). We obtain

$$q(m) \leqslant g(m) = -\frac{c_1}{2}\lambda_1 \nu m^2 + \sqrt{2}\, c(\gamma)\, m^{1/2} \frac{\|f\|}{\nu}.$$

It is well known that the following estimate holds for the Hausdorff dimension of $\mathscr{A}$ (see, for instance, [3], [13], [14], [37]):

$$\dim_H \mathscr{A} \leqslant d_*,$$

where d_* is the positive root of the equation $g(d) = 0$. It was shown in [9] that the same upper estimate also holds for the fractal dimension:

$$\dim_F \mathscr{A} \leqslant d_*.$$

Therefore

$$\begin{aligned} \dim_F \mathscr{A} &\leqslant (2\sqrt{2}\, c(\gamma)/c_1)^{2/3}\, G^{2/3} = (8\sqrt{2}/(\sqrt{2}-1)^2)^{2/3}\, c(\gamma)^{2/3}\, G^{2/3} \\ &= 16.33 c(\gamma)^{2/3}\, G^{2/3}. \end{aligned} \tag{16}$$

We observe that in view of (18) this estimate can be written (as $\gamma \to 0$) in the form

$$\dim_F \mathscr{A} \leqslant 16.33 c_2^{1/3} (\log(1/\gamma))^{1/3}\, G^{2/3}.$$

Finally, since the right hand side of the estimate (12) is independent of α we can use the estimate for ρ from the classical theory [14], [15], [37]

$$\|\rho\|_\infty \leqslant c_3 \left(1 + \log\left(\lambda_1^{-1} \sum_{j=1}^{m} \|\nabla \theta_j\|^2\right)\right),$$

which is also independent of α. We then immediately infer the classical estimate

$$\dim_F \mathscr{A} \leqslant c_4 (\log(1+G))^{1/3}\, G^{2/3}.$$

Combining these two estimates for $\dim_F \mathscr{A}$ we obtain (14) and complete the proof of the theorem. □

Lemma 1. *Suppose that the family* $\{\theta_j\}_{j=1}^m$ *is orthonormal in* H: $\int \theta_i \theta_j \, dx = \delta_{ij}$. *For* $v_j = n \times \nabla(\Delta - \alpha^2 \Delta^2)^{-1} \theta_j$ *we define the function* ρ:

$$\rho(x) = \sum_{j=1}^{m} |v_j(x)|^2 = \sum_{j=1}^{m} |\nabla(\Delta - \alpha^2 \Delta^2)^{-1} \theta_j(x)|^2.$$

Then ρ satisfies the estimate

$$\|\rho\|_\infty \leqslant 2c(\gamma)^2, \tag{17}$$

where $c(\gamma)$ is given in (19) and for an absolute constant c_2 (and $c_1 = (\sqrt{2}-1)^2/4$)

$$\begin{aligned} c(\gamma)^2 &\leqslant \frac{1}{2\pi c_1}\left[\log\left(\frac{1+c_1\gamma^2}{c_1\gamma^2}\right)-\frac{c_1\gamma^2}{(1+c_1\gamma^2)^2}\right] \qquad \textit{for all} \quad \gamma>0, \\ c(\gamma)^2 &\leqslant c_2 \log(1/\gamma) \qquad \textit{as} \quad \gamma \to 0. \end{aligned} \tag{18}$$

Proof. For $\theta = \theta_j \in H \subset L_2$ we have $v = v_j \in H^3$ and hence by the Sobolev embedding theorem

$$\|v\|_\infty = \|\nabla(\Delta - \alpha^2\Delta^2)^{-1}\theta\|_\infty \leqslant c(\gamma)\,\|\theta\|$$

for some dimensionless constant $c(\gamma)$ to be evaluated later. Next, suppose that $\xi_1,\ldots,\xi_m \in \mathbb{R}$ and $\sum_{j=1}^m \xi_j^2 = 1$. Then using the orthonormality of the θ_j and the above inequality we obtain

$$\begin{aligned} \left|\sum_{j=1}^m \xi_j v_j(x)\right| &\leqslant c(\gamma)\left\|\sum_{j=1}^m \xi_j\theta_j\right\| = c(\gamma)\left(\int\left(\sum_{j=1}^m \xi_j\theta_j(x)\right)^2 dx\right)^{1/2} \\ &= c(\gamma)\left(\sum_{i,j=1}^m \xi_j\xi_i\delta_{ij}\right)^{1/2} = c(\gamma)\left(\sum_{j=1}^m \xi_j^2\right)^{1/2} = c(\gamma). \end{aligned}$$

Using the representation $v_j(x) = v_j^1(x)\cdot e_1 + v_j^2(x)\cdot e_2$ we find that

$$\left(\sum_{j=1}^m \xi_j v_j^1(x)\right)^2 + \left(\sum_{j=1}^m \xi_j v_j^2(x)\right)^2 \leqslant c(\gamma)^2.$$

First, we set $\xi_j = v_j^1(x)/(\sum_{j=1}^m (v_j^1(x))^2)^{1/2}$ and later we set $\xi_j = v_j^2(x)/(\sum_{j=1}^m (v_j^2(x))^2)^{1/2}$. We obtain

$$\rho(x) = \sum_{j=1}^m |v_j(x)|^2 = \sum_{j=1}^m (v_j^1(x))^2 + \sum_{j=1}^m (v_j^2(x))^2 \leqslant 2c(\gamma)^2,$$

which proves (17).

We now set $L=1$ so that $T=[0,2\pi]^2$ and $\alpha=\gamma$. Since $c(\gamma)$ is dimensionless, this involves no loss of generality. To find the constant $c(\gamma)$ we use the Fourier series

$$\theta(x) = \sum_{k\in\mathbb{Z}_0^2} a_k e^{ikx}, \qquad \mathbb{Z}_0^2 = \mathbb{Z}^2\setminus\{0\}, \qquad \|\theta\|^2 = 4\pi^2 \sum_{k\in\mathbb{Z}_0^2} |a_k|^2.$$

The eigenvalues of the Laplacian are known explicitly, hence,

$$(\Delta-\alpha^2\Delta^2)^{-1}\,\theta(x)=-\sum_{k\in\mathbb{Z}_0^2}\frac{a_k}{k^2+\gamma^2k^4}\,e^{ikx},$$

$$\begin{aligned}|\nabla(\Delta-\alpha^2\Delta^2)^{-1}\,\theta(x)|&\leqslant\sum_{k\in\mathbb{Z}_0^2}\frac{|a_k|\,|k|}{k^2+\gamma^2k^4}\\&\leqslant\left(\sum_{k\in\mathbb{Z}_0^2}\frac{k^2}{(k^2+\gamma^2k^4)^2}\right)^{1/2}\left(\sum_{k\in\mathbb{Z}_0^2}|a_k|^2\right)^{1/2}=c(\gamma)\,\|\theta\|,\end{aligned}$$

where

$$c(\gamma)^2=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}_0^2}\frac{1}{k^2(1+\gamma^2k^2)^2}. \tag{19}$$

We now order the eigenvalues of $-\Delta$ according to magnitude and multiplicity:

$$1=\lambda_1\leqslant\lambda_2\leqslant\cdots,$$

where

$$\{\lambda_p,\,p=1,...\}=\{k^2=k_1^2+k_2^2,\,k=(k_1,k_2)\in\mathbb{Z}_0^2\}.$$

Then

$$c(\gamma)^2=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}_0^2}\frac{1}{k^2(1+\gamma^2k^2)^2}=\frac{1}{2\pi}\sum_{p=1}^{\infty}\frac{1}{\lambda_p(1+\gamma^2\lambda_p)^2}.$$

It is well known that in two dimensions

$$\lambda_p\geqslant c_1\lambda_1\,p=c_1\,p,\qquad(\lambda_1=1)$$

for an absolute constant c_1. For instance, we can take $c_1=(\sqrt{2}-1)^2/4$ (see [20], where it is shown that $\lambda_p\geqslant\psi(p)/4,\psi(p)=(\sqrt{1+p}-1)^2$ and, consequently, since the function $\psi(p)/p$ is increasing, $\psi(p)\geqslant\psi(1)\,p=c_1\,p$). Therefore

$$\begin{aligned}2\pi c(\gamma)^2&\leqslant\sum_{p=1}^{\infty}\frac{1}{c_1\,p(1+\gamma^2c_1\,p)^2}=\frac{1}{c_1(1+\gamma^2c_1)^2}+\sum_{p=2}^{\infty}\frac{1}{c_1\,p(1+\gamma^2c_1\,p)^2}\\&\leqslant\frac{1}{c_1(1+\gamma^2c_1)^2}+\int_1^{\infty}\frac{dx}{c_1x(1+\gamma^2c_1x)^2}=\frac{1}{c_1}\left[\log\left(\frac{1+c_1\gamma^2}{c_1\gamma^2}\right)-\frac{c_1\gamma^2}{(1+c_1\gamma^2)^2}\right],\end{aligned}$$

which proves (18) and the lemma. □

Remark 1. Inequalities for the L_∞-norm of the function ρ defined by an orthonormal family $\{\theta_j\}$ appeared for the first time in [30]. Foias (see [37] and the first edition of this book) then proposed a simple direct proof that we followed in our proof above.

3. NAVIER–STOKES-α MODEL ON THE TWO-DIMENSIONAL SPHERE

Let S^2 be the unit sphere with spherical coordinates λ, $0 \leqslant \lambda \leqslant 2\pi$ (the longitude) and ϕ, $-\pi/2 \leqslant \phi \leqslant \pi/2$ (the latitude). We assume below for simplicity that the radius $R = 1$, but as in the periodic case all our results and estimates mentioned below are written in such a form that they hold for any $R > 0$. We also assume that the sphere rotates around the axis through the poles $\phi = \pm\pi/2$ with constant angular velocity ω. By analogy with (4) we write the equations of the Navier–Stokes-α model on the rotating sphere as follows:

$$\begin{aligned} \partial_t(u-\gamma^2 \Delta u) - \nu\Delta(u-\gamma^2 \Delta u) - u \times \operatorname{rot}(u-\gamma^2 \Delta u) + nl \times u &= -\nabla p + f, \\ \operatorname{div} u &= 0, \\ u(0) &= u_0, \end{aligned} \tag{20}$$

where $\gamma = \alpha/R$. Here u is the tangent velocity vector, p is the modified pressure, f is the forcing term, $l = 2\omega \sin\phi$ is the Coriolis parameter, n is a unit outward normal vector. We observe that for $\alpha = 0$ we obtain the classical Navier–Stokes equations on the two-dimensional rotating sphere [5], [26], [27].

The operators div and $\nabla = \operatorname{grad}$ have the conventional meaning: for a scalar ψ and a vector u

$$\nabla\psi = \frac{1}{\cos\phi}\,\partial_\lambda\psi \cdot e_\lambda + \partial_\phi\psi \cdot e_\phi, \qquad \operatorname{div} u = \frac{1}{\cos\phi}\,(\partial_\lambda u^\lambda + \partial_\phi(\cos\phi\, u^\phi))$$

Next, Δu is the vector Laplacian of u (the Laplace–de Rham operator)

$$\Delta u = \nabla u - \operatorname{rot}\operatorname{rot} u = \nabla \operatorname{div} u - n \times \nabla \operatorname{div}(n \times u).$$

Here for a vector u and a scalar ψ

$$\operatorname{rot} u = -n \cdot \operatorname{div}\,(n \times u), \qquad \operatorname{rot} \psi = -n \times \nabla\psi,$$

respectively. In addition, as in the planar case

$$\operatorname{rot}\operatorname{rot} \psi = -n \cdot \operatorname{div} \nabla\psi = -n \cdot \Delta\psi,$$

where Δ is the scalar Laplacian and in spherical coordinates

$$\Delta\psi = \operatorname{div}\operatorname{grad}\psi = \frac{1}{\cos\phi}\,\partial_\phi(\cos\phi\,\partial_\phi\psi) + \frac{1}{\cos^2\phi}\,\partial_\lambda^2\psi.$$

Using the condition $\operatorname{div} u = 0$ we introduce the stream function ψ: $u = n \times \nabla\psi$. Substituting this in the first equation in (20) and applying the operator rot (to vectors) we obtain, similarly to the planar case, the scalar vorticity equation

$$\partial_t(\Delta\psi - \gamma^2\,\Delta^2\psi) - \nu\Delta(\Delta\psi - \gamma^2\,\Delta^2\psi) + J(\psi, \Delta\psi - \gamma^2\,\Delta^2\psi + l) = \operatorname{rot} f \equiv F, \tag{21}$$

or setting $\varphi = \Delta\psi - \gamma^2\,\Delta^2\psi$ we write this equation in the form

$$\partial_t\varphi - \nu\,\Delta\varphi + J((\Delta - \gamma^2\Delta^2)^{-1}\,\varphi, \varphi + l) = F,$$
$$\varphi(0) = \operatorname{rot}(u_0 - \gamma^2\,\Delta u_0), \tag{22}$$

where for the Jacobian operator we have

$$J(a, b) = (n \times \nabla a)\cdot\nabla b = \operatorname{div}(a \operatorname{rot} b) = \frac{1}{\cos\phi}\,(\partial_\lambda a\,\partial_\phi b - \partial_\phi a\,\partial_\lambda b),$$

and, in addition to (11) we have

$$\int_{S^2} J((\Delta - \gamma^2\Delta^2)^{-1}\,\varphi, l)\,\varphi\,dS = \int_{S^2} J(\psi, l)(\Delta\psi - \gamma^2\Delta^2\psi)\,dS = 0. \tag{23}$$

To obtain an upper bound for the dimension of the attractor we proceed exactly as in Section 1 preserving the notation used there. In view of (23) the estimate (12) holds for the solutions φ of (22).

As in Lemma 2 we have for the function ρ

$$\rho(s) = \sum_{j=1}^m |v_j(s)|^2 = \sum_{j=1}^m |\nabla(\Delta - \gamma^2\Delta^2)^{-1}\,\theta_j(s)|^2$$

the estimate

$$\|\rho\|_\infty \leqslant 2c_{S^2}(\gamma)^2, \tag{24}$$

where $c_{S^2}(\gamma)$ is the dimensionless constant in the embedding inequality

$$\|\nabla(\Delta - \gamma^2\Delta^2)^{-1}\,\theta\|_\infty \leqslant c_{S^2}(\gamma)\,\|\theta\|. \tag{25}$$

Lemma 2. *The constant $c_{S^2}(\gamma)$ satisfies the following explicit estimate:*

$$c_{S^2}(\gamma) \leqslant \left(\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(1+\gamma^2 n(n+1))^2} \right)^{1/2}. \tag{26}$$

Proof. We recall the structure of the spectrum of the Laplacian on the unit sphere (see, for instance, [22], [38]):

$$-\Delta Y_{mn} = n(n+1)\, Y_{mn},$$

where $Y_{mn} = Y_{mn}(\lambda, \phi)$, $n = 1, 2,\ldots, m = -n,\ldots, n$ are the orthonormal spherical harmonics and the eigenvalue $n(n+1)$ has multiplicity $2n+1$.

We also recall the following important identity [26]. At each point $s \in S^2$

$$\sum_{m=-n}^{n} |\nabla Y_{mn}(s)|^2 = \frac{1}{4\pi}(2n+1)\, n(n+1).$$

Using the Fourier series in spherical harmonics we write

$$\theta(s) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \theta_{mn} Y_{mn}(s), \qquad \|\theta\|^2 = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} |\theta_{mn}|^2$$

and obtain:

$$\begin{aligned}
&|\nabla(\Delta - \gamma^2 \Delta^2)^{-1}\, \theta(s)| \\
&= \left| \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{\theta_{mn}}{n(n+1)+\gamma^2(n(n+1))^2} \nabla Y_{mn}(s) \right| \\
&= \left| \sum_{n=1}^{\infty} \frac{1}{n(n+1)+\gamma^2(n(n+1))^2} \sum_{m=-n}^{n} \theta_{mn} \nabla Y_{mn}(s) \right| \\
&\leqslant \sum_{n=1}^{\infty} \frac{1}{n(n+1)+\gamma^2(n(n+1))^2} \left(\sum_{m=-n}^{n} |\theta_{mn}|^2 \right)^{1/2} \left(\sum_{m=-n}^{n} |\nabla Y_{mn}(s)|^2 \right)^{1/2} \\
&= \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{(n(n+1)(2n+1))^{1/2}}{n(n+1)+\gamma^2(n(n+1))^2} \left(\sum_{m=-n}^{n} |\theta_{mn}|^2 \right)^{1/2} \\
&\leqslant \frac{1}{\sqrt{4\pi}} \left(\sum_{n=1}^{\infty} \frac{(n(n+1)(2n+1))}{(n(n+1)+\gamma^2(n(n+1)^2))^2} \right)^{1/2} \|\theta\| \\
&= \frac{1}{\sqrt{4\pi}} \left(\sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)(1+\gamma^2 n(n+1))^2} \right)^{1/2} \|\theta\|,
\end{aligned}$$

which proves (26). Also, as in Lemma 1 we have $c_{S^2}(\gamma)^2 \leqslant c_5 \log(1/\gamma)$ as $\gamma \to 0$. □

Before we write the estimate for the global Lyapunov exponents we need another elementary inequality, which we prove for the sake of completeness. Denoting by λ_j the eigenvalues of the Laplacian on the unit sphere counting multiplicities we have

$$\{\lambda_j\ j=1,...\} = \{2, 2, 2, 6, 6, 6, 6, 6,..., \underbrace{k(k+1),..., k(k+1)}_{2k+1 \text{ times}},...\}. \quad (27)$$

In other words, $\lambda_j = [j^{1/2}]([j^{1/2}]+1)$.

The following inequality holds:

$$\sum_{j=1}^{m} \lambda_j \geqslant \frac{\lambda_1}{4} m^2. \quad (28)$$

Since on the unit sphere $\lambda_1 = 2$, (28) is equivalent to the inequality $\sum_{j=1}^{m} \lambda_j \geqslant m^2/2$. We represent m in the form

$$m = n^2 + p, \quad 0 \leqslant p \leqslant 2n$$

so that $n^2 \leqslant m < (n+1)^2$ for some n. Then, in view of (27),

$$\sum_{j=1}^{m} \lambda_j = \sum_{k=1}^{n-1} k(k+1)(2k+1) + (p+1)\, n(n+1)$$
$$= \frac{n^2(n^2-1)}{2} + (p+1)\, n(n+1) \geqslant \frac{(n^2+p)^2}{2} = \frac{m^2}{2}$$

for $0 \leqslant p \leqslant 2n$.

In exactly the same way as in the planar case we obtain for the sphere the following explicit estimate for the sums of the first m global Lyapunov exponents $q(m)$:

$$q(m) \leqslant g(m) = -\frac{\lambda_1 \nu}{4} m^2 + \sqrt{2}\, c_{S^2}(\gamma)\, m^{1/2} \frac{\|f\|}{\nu}.$$

Hence, we obtain for the dimension of the attractor $\mathscr{A} = \mathscr{A}_{NS-\alpha}$ of the Eq. (20) the following estimate in terms of the Grashof number $G = \|f\|/(\lambda_1 \nu^2)$ (where $\lambda_1 = 2/R^2$):

$$\dim_F \mathscr{A} \leqslant (4\sqrt{2}\, c_{S^2}(\gamma))^{2/3}\, G^{2/3} \leqslant 2 \left(\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(1+\gamma^2 n(n+1))^2} \right)^{1/3} G^{2/3},$$

where $\gamma = \alpha/R$.

In conclusion, we observe that as in the planar case the estimate for the classical Navier–Stokes equations in [27] holds for the Eq. (20) uniformly with respect to γ:

$$\dim_F \mathscr{A} \leqslant 4(3\pi)^{-1/3}\, G^{2/3}(\log G)^{1/3}.$$

We finally obtain

$$\dim_F \mathscr{A} \leqslant G^{2/3} \cdot \min(32 c_{S^2}(\gamma)^2, 64/(3\pi) \log G)^{1/3}.$$

4. LOWER BOUNDS FOR THE SPACE PERIODIC NAVIER–STOKES-α MODEL

We now return to the Eq. (4) and obtain in this section lower bounds for the dimension of the attractor $\mathscr{A} = \mathscr{A}_{NS-\alpha}$ constructed in Section 1. This is in order to check how sharp are our upper bounds. Since a global attractor is a maximal strictly invariant compact set, it follows that the attractor contains the unstable manifolds of stationary points, that is, the invariant manifolds along which the solutions tend exponentially to the stationary point as $t \to -\infty$ [3].

The family of stationary solutions that our analysis is based on is the well-known family of Kolmogorov flows [3], [31], [35], [39], [40].

We set $L = 1$ (so that $\alpha = \gamma$) and consider the following family of forces (see [31]) depending on the integer parameter s:

$$f = f_s = \begin{cases} f_1 = \dfrac{1}{\sqrt{2\pi}}\, \nu^2 \lambda s^2 \sin s x_2, \\ f_2 = 0, \end{cases} \tag{29}$$

where $\lambda = \lambda(s)$ is a parameter to be chosen later. Then

$$\|f\| = \nu^2 \lambda s^2, \qquad G = \lambda s^2 \tag{30}$$

and

$$\operatorname{rot} f_s = F_s = -\frac{1}{\sqrt{2\,\pi}}\, \nu^2 \lambda s^3 \cos s x_2. \tag{31}$$

Corresponding to (31) is the stationary solution

$$\varphi = \varphi_s = -\frac{1}{\sqrt{2\,\pi}}\, \nu \lambda s \cos s x_2. \tag{32}$$

of the equation

$$\partial_t \varphi - \nu \,\Delta\varphi + J((\Delta - \gamma^2\Delta^2)^{-1}\,\varphi, \varphi) = F_s. \tag{33}$$

In fact, since φ_s depends only on x_2, it follows that $(\Delta-\gamma^2\Delta^2)^{-1}\,\varphi_s$ also depends only on x_2 and therefore $J((\Delta-\gamma^2\Delta^2)^{-1}\,\varphi_s, \varphi_s) \equiv 0$; the equality $-\nu\,\Delta\varphi_s = F_s$ is verified directly. We linearize (33) about the stationary solution φ_s and consider the eigenvalue problem

$$\mathscr{L}_{\varphi_s} = J((\Delta-\gamma^2\Delta^2)^{-1}\,\varphi_s, \varphi) + J((\Delta-\gamma^2\Delta^2)^{-1}\,\varphi, \varphi_s) - \nu\,\Delta\varphi = -\sigma\varphi. \tag{34}$$

We look for unstable eigenmodes φ with $\operatorname{Re}\sigma > 0$.

We use the orthonormal basis of trigonometric functions, which are the eigenfunctions of the Laplacian,

$$\left\{\frac{1}{\sqrt{2}\,\pi}\sin kx, \frac{1}{\sqrt{2}\,\pi}\cos kx\right\}, \qquad kx = k_1x_1 + k_2x_2,$$

$$k \in \mathbb{Z}^2_+ = \{k \in \mathbb{Z}^2_0,\ k_1 \geqslant 0,\ k_2 \geqslant 0\} \cup \{k \in \mathbb{Z}^2_0,\ k_1 \geqslant 1,\ k_2 \geqslant 1\}$$

and represent φ as a Fourier series

$$\varphi = \frac{1}{\sqrt{2}\,\pi}\sum_{k\in\mathbb{Z}^2_+} a_k \cos kx + b_k \sin kx.$$

Substituting this into (34) and using the equality $J(a, b) = -J(b, a)$ we obtain

$$\frac{\lambda s}{\sqrt{2}\pi}\sum_{k\in\mathbb{Z}^2_+}\left(\frac{1}{s^2+\gamma^2 s^4} - \frac{1}{k^2+\gamma^2 k^4}\right) J(\cos sx_2, a_k\cos kx + b_k \sin kx)$$
$$+ \sum_{k\in\mathbb{Z}^2_+}(k^2+\hat\sigma)(a_k\cos kx + b_k\sin kx) = 0, \tag{35}$$

where $\hat\sigma = \sigma/\nu$. Next, we have the following two similar formulas

$$\begin{aligned}
&J(\cos sx_2, \cos(k_1x_1+k_2x_2))\\
&\quad = -k_1 s \sin sx_2 \sin(k_1x_1+k_2x_2)\\
&\quad = \frac{k_1 s}{2}(\cos(k_1x_1+(k_2+s)\,x_2) - \cos(k_1x_1+(k_2-s)\,x_2),\\
&J(\cos sx_2, \sin(k_1x_1+k_2x_2))\\
&\quad = k_1 s \sin sx_2 \cos(k_1x_1+k_2x_2)\\
&\quad = \frac{k_1 s}{2}(\sin(k_1x_1+(k_2+s)\,x_2) - \sin(k_1x_1+(k_2-s)\,x_2).
\end{aligned} \tag{36}$$

We substitute formulas (36) into (35) and take the scalar product of the result with $(\sqrt{2}\,\pi)^{-1}\cos(k_1'x_1+k_2'x_2)$. We obtain the following equation for the coefficients $a_{k_1',k_2'}$, which we write omitting the prime (the equation for $b_{k_1',k_2'}$ is exactly the same):

$$-\frac{\lambda k_1 s^2}{2\sqrt{2}\,\pi}\left(\frac{k_1^2+(k_2+s)^2+\gamma^2(k_1^2+(k_2+s)^2)^2-(s^2+\gamma^2 s^4)}{(s^2+\gamma^2 s^4)(k_1^2+(k_2+s)^2+\gamma^2(k_1^2+(k_2+s)^2)^2)}\right)a_{k_1 k_2+s}$$
$$+\frac{\lambda k_1 s^2}{2\sqrt{2}\,\pi}\left(\frac{k_1^2+(k_2-s)^2+\gamma^2(k_1^2+(k_2-s)^2)^2-(s^2+\gamma^2 s^4)}{(s^2+\gamma^2 s^4)(k_1^2+(k_2-s)^2+\gamma^2(k_1^2+(k_2-s)^2)^2)}\right)a_{k_1 k_2-s}$$
$$+(k_1^2+k_2^2+\hat{\sigma})\,a_{k_1 k_2}=0. \tag{37}$$

We set here

$$a_{k_1 k_2}\left(\frac{k^2+\gamma^2 k^4-s^2-\gamma^2 s^4}{(1+\gamma^2 s^2)(k^2+\gamma^2 k^4)}\right)=:c_{k_1 k_2}.$$

Setting here as in [31]

$$k_1=t,\quad k_2=sn+r,\qquad \text{and}\qquad c_{tsn+r}=e_n,$$
$$t=1,2,\ldots,\quad r\in\mathbb{Z},\quad r_{\min}<r<r_{\max},$$

where the numbers $r_{\min}, r_{\max}$ satisfy $r_{\max}-r_{\min}<s$ and will be specified below (see Fig. 1), and introducing

$$P_\gamma(a,b)=a^2+b^2+\gamma^2(a^2+b^2)^2 \tag{38}$$

we obtain for each t and r the following recurrence relation:

$$d_n e_n+e_{n-1}-e_{n+1}=0,\qquad n=0,\pm 1,\pm 2,\ldots, \tag{39}$$

where

$$d_n=\frac{2\sqrt{2}\,\pi(1+\gamma^2 s^2)\,P_\gamma(t,sn+r)(t^2+(sn+r)^2+\hat{\sigma})}{(P_\gamma(t,sn+r)-P_\gamma(s,0))\,\lambda t}. \tag{40}$$

We note that for $\gamma=0$ the recurrence relation (39), (40) agrees exactly with the recurrence relation from [31].

We look for non-trivial decaying solutions $\{e_n\}$ of (39), (40). Each non-trivial decaying solution with $\operatorname{Re}\hat{\sigma}>0$ produces an unstable eigenfunction φ of the eigenvalue problem (34).

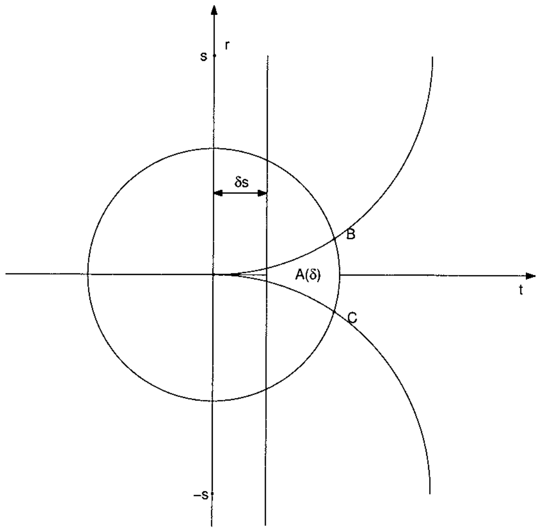

Fig. 1. The region $A(\delta)$.

Theorem 2. *Given an integer $s>0$ let a pair of integers t, r satisfying the conditions*

$$t^2+r^2<s^2/3, \qquad t^2+(-s+r)^2>s^2, \qquad t^2+(s+r)^2>s^2, \quad t\geqslant \delta s,$$
$$r_{\min}<r<r_{\max}, \qquad r_{\min}=-\frac{s}{6}, \qquad r_{\max}=\frac{s}{6}, \qquad 0<\delta<1/\sqrt{3} \tag{41}$$

be fixed (see Fig. 1). For any $\lambda>0$ there exists a unique real eigenvalue $\hat{\sigma}=\hat{\sigma}(\lambda)$, which increases monotonically as $\lambda\to\infty$ and satisfies the inequality

$$c_1(\gamma, t, r, s)\,\lambda\leqslant\hat{\sigma}(\lambda)\leqslant c_2(\gamma, t, r, s)\,\lambda. \tag{42}$$

The unique $\lambda_0=\lambda_0(s)$ solving the equation

$$\hat{\sigma}(\lambda_0)=0$$

satisfies the inequality

$$2\pi s(1+\gamma^2 s^2)\,\delta^2<\lambda_0<\frac{20\sqrt{5}\,\pi}{9\sqrt{2}}\,\delta^{-2}s(1+\gamma^2 s^2) \qquad \textit{for} \quad \gamma\geqslant 0,$$
$$2\pi\,\delta^2 s<\lambda_0<\frac{20\pi}{3\sqrt{6}}\,\delta^{-2}s \qquad \textit{for} \quad \gamma=0. \tag{43}$$

Proof. We first observe that the following inequalities hold for any (t, r) satisfying (41):

$$
\begin{aligned}
&s^2 \leqslant t^2+(-s+r)^2=\operatorname{dist}((0, s),(t, r))^2 \leqslant \operatorname{dist}((0, s), C)^2=(5/3)\, s^2 \\
&s^2 \leqslant t^2+(s+r)^2=\operatorname{dist}((0,-s),(t, r))^2 \leqslant \operatorname{dist}((0,-s), B)^2=(5/3)\, s^2
\end{aligned} \tag{44}
$$

see Fig. 1, where $B=(\sqrt{11}\, s/6, s/6)$ and $C=(\sqrt{11}\, s/6, -s/6)$. Next, in view of (41) for any real $\hat{\sigma}$ satisfying $\hat{\sigma} > -t^2-r^2$ we have in the recurrence relation (39), (40)

$$
d_n > 0 \quad \text{for} \quad n \neq 0 \quad \text{and} \quad \lim_{|n| \to \infty} d_n = \infty. \tag{45}
$$

The main tool in the analysis of (39) are continued fractions and a variant of Pincherle's theorem (see [28], [31], [35], [39]) saying that under condition (45) the recurrence relation (39) has a decaying solution $\{e_n\}$ with $\lim_{|n| \to \infty} e_n = 0$ if and only if

$$
-d_0 = \cfrac{1}{d_{-1} + \cfrac{1}{d_{-2} + \cdots}} + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cdots}}. \tag{46}
$$

Singling out the dependence on $\hat{\sigma}$ we denote left-hand side of this equation by $f(\hat{\sigma})$ and the right-hand side by $g(\hat{\sigma})$

$$
f(\hat{\sigma}) = -d_0 = \frac{2\sqrt{2}\,\pi(1+\gamma^2 s^2)\, P_\gamma(t, r)(t^2+r^2+\hat{\sigma})}{(P_\gamma(s, 0)-P_\gamma(t, r))\,\lambda t}, \tag{47}
$$

$$
g(\hat{\sigma}) = \cfrac{1}{d_{-1} + \cfrac{1}{d_{-2} + \cdots}} + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cdots}}. \tag{48}
$$

We deduce from (47) that

$$
f(-t^2-r^2)=0, \quad \text{and} \quad f(\hat{\sigma}) \to \infty \quad \text{as} \quad \hat{\sigma} \to \infty
$$

and from (48) and (40) that

$$
g(\hat{\sigma}) < \frac{1}{d_{-1}} + \frac{1}{d_1}, \qquad g(\hat{\sigma}) \to 0 \quad \text{as} \quad \hat{\sigma} \to \infty.
$$

Hence, there exists a $\hat{\sigma} > -t^2-r^2$ such that

$$
f(\hat{\sigma}) = g(\hat{\sigma}). \tag{49}
$$

From elementary properties of continued fractions we deduce as in [31], [39] that the $\hat{\sigma}$ so obtained is unique and increases monotonically with λ.

To establish (42) we deduce from (49) and (48) that

$$\frac{1}{d_{-1}+\frac{1}{d_{-2}}}+\frac{1}{d_1+\frac{1}{d_2}}<f(\hat{\sigma})<\frac{1}{d_{-1}}+\frac{1}{d_1}. \tag{50}$$

Taking into account the conditions $t^2+(-s+r)^2>s^2$ and $t^2+(s+r)^2>s^2$ we obtain from the right-hand inequality in (50)

$$\begin{aligned}
&\frac{(2\sqrt{2}\,\pi(1+\gamma^2s^2))^2\,P_\gamma(t,r)(t^2+r^2+\hat{\sigma})}{(P_\gamma(s,0)-P_\gamma(t,r))\,\lambda t}\\
&\quad<\frac{(P_\gamma(t,-s+r)-P_\gamma(s,0))\,\lambda t}{P_\gamma(t,-s+r)(t^2+(-s+r)^2+\hat{\sigma})}+\frac{(P_\gamma(t,s+r)-P_\gamma(s,0))\,\lambda t}{P_\gamma(t,s+r)(t^2+(s+r)^2+\hat{\sigma})}\\
&\quad<\frac{\lambda t}{s^2+\hat{\sigma}}\left[\frac{(P_\gamma(t,-s+r)-P_\gamma(s,0))}{P_\gamma(t,-s+r)}+\frac{(P_\gamma(t,s+r)-P_\gamma(s,0))}{P_\gamma(t,s+r)}\right]<\frac{2\lambda t}{s^2+\hat{\sigma}}.
\end{aligned} \tag{51}$$

Hence,

$$\frac{(\hat{\sigma}+t^2+r^2)(\hat{\sigma}+s^2)}{(\lambda t)^2}<\frac{2(P_\gamma(s,0)-P_\gamma(t,r))}{(2\sqrt{2}\,\pi(1+\gamma^2s^2))^2\,P_\gamma(t,r)} \tag{52}$$

and therefore

$$\hat{\sigma}(\lambda)\leqslant c_2(\gamma,t,r,s)\,\lambda \qquad \text{as} \quad \lambda\to\infty.$$

From the left-hand side inequality in (50), where $d_{-1}, d_1, d_{-2}, d_2, f>0$, we see that

$$fd_1+\frac{f}{d_2}>1 \qquad \text{and} \qquad fd_{-1}+\frac{f}{d_{-2}}>1. \tag{53}$$

Next,

$$\begin{aligned}
d_{-1}f&=\frac{(\hat{\sigma}+t^2+r^2)(\hat{\sigma}+t^2+(-s+r)^2)}{(\lambda t)^2}\\
&\quad\cdot\frac{(2\sqrt{2}\,\pi(1+\gamma^2s^2))^2\,P_\gamma(t,r)\,P_\gamma(t,-s+r)}{(P_\gamma(s,0)-P_\gamma(t,r))(P_\gamma(t,-s+r)-P_\gamma(s,0))},\\
d_1f&=\frac{(\hat{\sigma}+t^2+r^2)(\hat{\sigma}+t^2+(s+r)^2)}{(\lambda t)^2}\\
&\quad\cdot\frac{(2\sqrt{2}\,\pi(1+\gamma^2s^2))^2\,P_\gamma(t,r)\,P_\gamma(t,s+r)}{(P_\gamma(s,0)-P_\gamma(t,r))(P_\gamma(t,s+r)-P_\gamma(s,0))}
\end{aligned} \tag{54}$$

and

$$\frac{f}{d_{-2}} = \frac{\hat{\sigma}+t^2+r^2}{\hat{\sigma}+t^2+(-2s+r)^2} \cdot \frac{P_\gamma(t, r)}{P_\gamma(s, 0)-P_\gamma(t, r)} \cdot \frac{P_\gamma(t, -2s+r)-P_\gamma(s, 0)}{P_\gamma(t, -2s+r)},$$
$$\frac{f}{d_2} = \frac{\hat{\sigma}+t^2+r^2}{\hat{\sigma}+t^2+(2s+r)^2} \cdot \frac{P_\gamma(t, r)}{P_\gamma(s, 0)-P_\gamma(t, r)} \cdot \frac{P_\gamma(t, 2s+r)-P_\gamma(s, 0)}{P_\gamma(t, 2s+r)}. \tag{55}$$

The first and third factors in (55) are clearly less than one. Therefore, since

$$\frac{f}{d_{-2}}, \frac{f}{d_2} < \frac{P_\gamma(t, r)}{P_\gamma(s, 0)-P_\gamma(t, r)} = \frac{t^2+r^2+\gamma^2(t^2+r^2)^2}{s^2+\gamma^2 s^4-t^2-r^2-\gamma^2(t^2+r^2)^2} < \frac{1}{2}, \tag{56}$$

uniformly with respect to γ, provided that $t^2+r^2 < s^2/3$, we obtain from (53)

$$f d_{-1} > \tfrac{1}{2}, \qquad f d_1 > \tfrac{1}{2}. \tag{57}$$

Combining this with the first or second equality in (54) we obtain

$$\hat{\sigma} \geqslant c_1(\gamma, t, r, s)\, \lambda,$$

which proves (42).

The estimate (42) shows that for (t, r) satisfying (41) there exists a unique $\hat{\sigma}$, which increases monotonically with λ and, hence $\hat{\sigma} = 0$ for some $\lambda = \lambda_0 : \hat{\sigma}(\lambda_0) = 0$. To estimate this λ_0 from below we set $\hat{\sigma} = 0$ in (52) and obtain

$$\lambda_0 > 2\pi s(1+\gamma^2 s^2)(1+r^2/t^2)^{1/2} \left(\frac{P_\gamma(t, r)}{P_\gamma(s, 0)-P_\gamma(t, r)} \right)^{1/2}$$
$$> 2\pi s(1+\gamma^2 s^2) \left(\frac{P_\gamma(t, r)}{P_\gamma(s, 0)} \right)^{1/2} > 2\pi s(1+\gamma^2 s^2)\, \delta^2,$$

where we used the fact that in view of the condition $t^2+r^2 > \delta^2 s^2$, $0 < \delta < 1$ (see (41)), we have $P_\gamma(t, r)/P_\gamma(s, 0) > (s^2\delta^2+\gamma^2 s^4 \delta^4)/(s^2+\gamma^2 s^4) > \delta^4$.

To get an upper bound for λ_0 we consider two cases: $r \geqslant 0$ and $r < 0$.

If $r \geqslant 0$, we set $\hat{\sigma} = 0$ in the expression for $d_1 f$ in (54) and use (57):

$$\lambda_0 < \frac{4\pi(1+\gamma^2 s^2)(t^2+r^2)^{1/2}\,(t^2+(s+r)^2)^{1/2}}{t}$$
$$\times \left(\frac{P_\gamma(t, r)}{P_\gamma(s, 0)-P_\gamma(t, r)} \right)^{1/2} \left(\frac{P_\gamma(t, s+r)}{P_\gamma(t, s+r)-P_\gamma(s, 0)} \right)^{1/2}. \tag{58}$$

By (44) we find that

$$P_\gamma(t, s+r) < (25/9)(s^2+\gamma^2 s^4) \quad \text{for all } \gamma \geqslant 0 \quad \text{and} \quad P_0(t, s+r) < (5/3)\, s^2.$$

Next, since $r \geqslant 0$ we have for $\gamma \geqslant 0$

$$P_\gamma(t, s+r) - P_\gamma(s, 0) \geqslant (t^2+r^2)(1+\gamma^2 s^2) \geqslant \delta^2 s^2 (1+\gamma^2 s^2).$$

Combining all these inequalities and $t \geqslant \delta s$ and taking into account (56) for the second factor in (58) we find

$$\lambda_0 < \frac{20\sqrt{5}\,\pi}{9\sqrt{2}}\,\delta^{-2} s(1+\gamma^2 s^2), \quad \text{for } \gamma \geqslant 0; \qquad \lambda_0 < \frac{20\,\pi}{3\sqrt{6}}\,\delta^{-2} s \quad \text{for } \gamma = 0. \tag{59}$$

If $r < 0$, we set $\hat{\sigma} = 0$ in the expression for $d_{-1} f$ in (54) and again use (57) and the estimate

$$P_\gamma(t, -s+r) - P_\gamma(s, 0) \geqslant (t^2+r^2)(1+\gamma^2 s^2) \geqslant \delta^2 s^2 (1+\gamma^2 s^2),$$

which holds for $r \leqslant 0$ and $\gamma \geqslant 0$. As a result we obtain the same estimate (59) and complete the proof of the theorem. □

The region in the (t, r)-plane satisfying (41) is shown in Fig. 1. Denoting by $d(s)$ the number of points of the integer lattice inside the region $A(\delta)$ we obviously have

$$d(s) := \#\{(t, r) \in D(s) = \mathbb{Z}^2 \cap A(\delta)\} \backsimeq a(\delta) \cdot s^2 \qquad \text{as} \quad s \to \infty, \tag{60}$$

where $a(\delta) \cdot s^2 = |A(\delta)|$ is the area of the region $A(\delta)$.

Next, taking into account the remark after (36) we see by Theorem 2 that for each pair $(t, r) \in D(s)$ and the parameter λ in (29), (30) chosen as follows

$$\lambda = \lambda_{\{\gamma \geqslant 0\}} = \frac{20\sqrt{5}\,\pi}{9\sqrt{2}}\,\delta^{-2} s(1+\gamma^2 s^2), \quad \text{and} \quad \lambda = \lambda_{\{\gamma = 0\}} = \frac{20\,\pi}{3\sqrt{6}}\,\delta^{-2} s \tag{61}$$

there exists a unique real positive eigenvalue $\hat{\sigma} > 0$ of multiplicity two. Hence, the dimension of the unstable manifold near the stationary solution φ_s is at least $2d(s)$ and we obtain as a result

$$\dim \mathscr{A} \geqslant 2d(s) \backsimeq 2a(\delta) \cdot s^2. \tag{62}$$

It is reasonable to consider three cases.

The case $\gamma=0$.

Setting $\gamma = 0$ we recover the results of [31] with improved values of the constants. In view of (30) and (61) we have

$$G = \frac{20\pi}{3\sqrt{6}}\,\delta^{-2}s^3$$

and writing the estimate (62) in terms of the Grashof number G we obtain

$$\dim \mathscr{A} \geqslant 2a(\delta)\cdot s^2 \simeq 2\left(\frac{3\sqrt{6}}{20\pi}\right)^{2/3} a(\delta)\,\delta^{4/3}G^{2/3},$$

$$\dim \mathscr{A} \gtrsim 2\left(\frac{3\sqrt{6}}{20\pi}\right)^{2/3}\Big(\max_{0<\delta<1/\sqrt{3}} a(\delta)\,\delta^{4/3}\Big)\,G^{2/3} = 0.006\cdot G^{2/3},$$

where the value of $\max_{0<\delta<1/\sqrt{3}} a(\delta)\,\delta^{4/3} = 0.012$.

The case $\gamma \ll 1$.

Here we can obtain the following lower bound for $G \sim (1/\gamma)^3$. Let $0 < s < 1/\gamma$. Then

$$G \leqslant 2\,\frac{20\sqrt{5}\,\pi}{9\sqrt{2}}\,\delta^{-2}s^3$$

and arguing as before we obtain

$$\dim \mathscr{A} \gtrsim 2\left(\frac{9\sqrt{2}}{40\sqrt{5}\,\pi}\right)^{2/3}\Big(\max_{0<\delta<1/\sqrt{3}} a(\delta)\,\delta^{4/3}\Big)\,G^{2/3} = 0.0032\cdot G^{2/3}.$$

In particular, setting $s \backsimeq 1/\gamma$ and recalling the estimate (14) we obtain the following estimate (which we write in terms of γ since it holds for any $L > 0$):

$$C_1\,\frac{1}{\gamma^2} \leqslant \dim \mathscr{A} \leqslant C_2\,\frac{1}{\gamma^2}\left(\log\frac{1}{\gamma}\right)^{1/3}.$$

The case $\gamma=\infty$.

We first observe that for any fixed forcing f and $\gamma \to \infty$ the dynamics eventually becomes trivial: for $\gamma > \gamma_f$ all the solutions are attracted exponentially to the unique stationary solution. Therefore, to get non-trivial dynamics we replace the forcing f by $\gamma^2 f$ in the system (4) and let $\gamma \to \infty$ to reach the system

$$\begin{aligned} \partial_t(\Delta u) - \nu\Delta(\Delta u) - u \times \operatorname{rot}(\Delta u) &= \nabla p + L^{-2} f, \\ \operatorname{div} u &= 0, \\ u(0) &= u_0. \end{aligned} \tag{63}$$

Repeating the proof of Theorem 1 we easily obtain the following estimate for the dimension of the attractor $\mathscr{A} = \mathscr{A}_{NS-\infty}$

$$\dim_F \mathscr{A} \leqslant (2\sqrt{2}\, c_\infty / c_1)^{2/3}\, G^{2/3} = 14.8 \cdot G^{2/3}, \tag{64}$$

where

$$c_\infty^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}_0^2} \frac{1}{k^6} = 0.741.$$

For the lower bound we have

$$G = \frac{20\sqrt{5}\,\pi}{9\sqrt{2}}\, \delta^{-2} s^5$$

and we find as before that

$$\dim \mathscr{A} \gtrsim 2 \left(\frac{9\sqrt{2}}{20\sqrt{5}\,\pi} \right)^{2/5} \left(\max_{0 < \delta < 1/\sqrt{3}} a(\delta)\, \delta^{4/5} \right) G^{2/5} = 0.016 \cdot G^{2/5}.$$

5. CONCLUSION

We consider here the two-dimensional NS-α model on the torus and the sphere and we provide estimates to the dimension of its global attractor.

Our estimates indicate that as $\alpha \to 0$ we recover the usual NSE estimates, from below and above, for the dimension of the global attractor.

The case when α is comparable with the wave length of the Kolmogorov forcing term we observe that the dimension of the global attractor grows like $(L/\alpha)^2$ as $\alpha \to 0^+$.

For $\alpha \to \infty$ we have trivial dynamics since in this case we have effectively a milder forcing term. However, if we magnify the forcing term by $\gamma^2 = (\alpha/L)^2$ and let $\alpha \to \infty$, we get a gap between the upper bound for the dimension of the global attractor, which is like $G^{2/3}$, and the lower bound which is like $G^{2/5}$. In view of the lower bound for λ in (43) (which for $\alpha = \infty$ takes the form $\lambda_0 > \text{const}\ s^3$) we cannot get anything better than $G^{2/5}$ using the Kolmogorov flows as base flows (29). Hence, to seal the gap between the upper and lower estimates for $\alpha = \infty$ we have to consider stationary flows other than Kolmogorov flows or to improve the upper bound (this is unless the gap is real); a subject of future research.

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