Global Well-Posedness and Finite-Dimensional Global Attractor for a 3-D Planetary Geostrophic Viscous Model

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Dedicated to the memory of Professor Jacques-Louis Lions

Abstract

In this paper we consider a three-dimensional planetary geostrophic viscous model of the gyre-scale mid-latitude ocean. We show the global existence and uniqueness of the weak and strong solutions to this model. Moreover, we establish the existence of a finite-dimensional global attractor to this dissipative evolution system. © 2003 Wiley Periodicals, Inc.

1 Introduction

The starting point for many models in ocean and atmospheric dynamics uses the so-called Boussinesq approximation equations with rotation. These are, roughly speaking, the Navier-Stokes equations with rotation, forced by the heat buoyancy and coupled with the heat transport equation. Since the heat transport equation satisfies some sort of maximum principle, one is able to show that the temperature remains bounded in various L^p-norms for all time. Therefore, in studying the question of global regularity and well-posedness for this system of equations, one faces the same mathematical difficulties as in the case of three-dimensional Navier-Stokes equations with rotation. This is because, in both cases, the momentum equations are the same, and the subtlety in dealing with the nonlinearity, the advection terms, is similar. However, due to the rotation and other geophysical situations, such as the shallowness of the oceans and the atmosphere, geophysicists take advantage of certain geophysical balances, such as geostrophic balance or hydrostatic balance, to derive reasonable, yet simplified, balanced models [30]. Averaging in the vertical directions, justified by shallowness, some of these models are two-dimensional, such as the Charney-Stommel ocean circulation model

Communications on Pure and Applied Mathematics, Vol. LVI, 0001–0036 (2003)

(cf. [4, 30, 37]). Some of these models are, therefore, treated using the same mathematical tools used for the two-dimensional Euler and Navier-Stokes equations (see, e.g., [2, 9, 10, 18, 19, 25, 26, 29, 42] and references therein).

In this article we consider the following planetary geostrophic viscous model (see [32, 34, 35]):

$$\nabla p + f\vec{k} \times v + \epsilon L_1 v = 0\,, \tag{1.1}$$

$$\partial_z p + T = 0\,, \tag{1.2}$$

$$\nabla \cdot v + \partial_z w = 0\,, \tag{1.3}$$

$$\partial_t T + v \cdot \nabla T + w\partial_z T + L_2 T = Q\,, \tag{1.4}$$

in the domain

$$\Omega = M \times (-h, 0) \subset \mathbb{R}^3\,,$$

where M is a bounded smooth domain in $\mathbb{R}^2$, or the square $M = (0, 1) \times (0, 1)$. Here $v = (v_1, v_2)$, (v_1, v_2, w) is the velocity field, T is the temperature, and p is the pressure. $f = f_0(\beta + y)$ is the Coriolis parameter, Q is a heat source, and ϵ is a positive, dimensionless constant. The operators L_1 and L_2 are given by

$$L_1 = -A_h\Delta - A_v\partial_z^2\,, \tag{1.5}$$

$$L_2 = -K_h\Delta - K_v\partial_z^2\,, \tag{1.6}$$

where A_h and A_v are positive molecular viscosities, and K_h and K_v are positive conductivity constants. We set $\nabla p = (\partial_x p, \partial_y p)$, $\nabla \cdot v = \partial_x v_1 + \partial_y v_2$, and $\Delta = \partial_x^2 + \partial_y^2$. We denote the different parts of the boundary of Ω by

$$\Gamma_u = \{(x, y, z) \in \Omega : z = 0\}\,, \tag{1.7}$$

$$\Gamma_b = \{(x, y, z) \in \Omega : z = -h\}\,, \tag{1.8}$$

$$\Gamma_s = \{(x, y, z) \in \Omega : (x, y) \in \partial M\}\,. \tag{1.9}$$

We equip the system (1.1)–(1.4) with the following boundary conditions, with wind-driven on the top surface and nonslip and nonflux on the side walls and bottom (see, e.g., [30, 31, 34, 35, 36]):

$$\text{on } \Gamma_u : A_v\frac{\partial v}{\partial z} = \tau,\ \ w = 0\,,\ \ -K_v\frac{\partial T}{\partial z} = \alpha(T - T^*)\,, \tag{1.10}$$

$$\text{on } \Gamma_b : \frac{\partial v}{\partial z} = 0,\ \ w = 0\,,\ \ \frac{\partial T}{\partial z} = 0\,, \tag{1.11}$$

$$\text{on } \Gamma_s : v \cdot \vec{n} = 0\,,\ \ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0\,,\ \ \frac{\partial T}{\partial \vec{n}} = 0\,, \tag{1.12}$$

where $\tau(x, y)$ is the wind stress, $\vec{n}$ is the normal vector of Γ_s, $T^*(x, y)$ is the typical temperature of the top (upper) surface, and $\alpha > 0$ is a positive constant.

Due to the boundary conditions (1.10)–(1.12), it is natural to assume that T^* satisfies the compatibility boundary condition

$$\frac{\partial T^*}{\partial \vec{n}} = 0 \quad \text{on } \partial M\,. \tag{1.13}$$

In addition, we supply the system with the initial condition

$$T(x, y, z, 0) = T_0(x, y, z) . \tag{1.14}$$

We became familiar with this model (1.1)–(1.14) from [32], where the authors established global existence (without uniqueness) of the weak solutions and short-time existence of strong solutions. These are similar to the well-known results for the case of the three-dimensional Bénard convection problem and the three-dimensional Navier-Stokes equations. However, the main difference between this model and the three-dimensional Bénard convection with rotation lies in the momentum equations. In this model the momentum equations are linear in the velocity field. In particular, the main obstacle to proving the global regularity for the three-dimensional Bénard problem (as well as the Navier-Stokes equations), the nonlinear advection term $(u \cdot \nabla)u$, is absent in this case. In this paper we take advantage of this very fact, in addition to the "maximum principle" for the temperature, to establish the global regularity of the strong solution and the uniqueness of the weak solution for this geophysical model.

It is worth noting the similarity between this model and the model of Bénard convection in a porous medium. In the latter the Darcy law is used to model the balance of momentum. As a matter of fact, the global regularity of the three-dimensional Bénard convection in a porous medium is proven in [12] and, by using a different approach, also in [28]. Indeed, the main idea used here is inspired by [12, 13, 28]. We would like to emphasize that our proofs rely on certain technical elliptic regularity results for nonlocal, Stokes-type, second-order elliptic systems in domains with corners. Such results are readily available in the classical literature for the case of smooth domains (see, for example, [23, 41]). However, due to the fact that our physical domain Ω has corners, one can use similar techniques to those developed in [21, 44] to obtain the needed elliptic regularity. Such elliptic regularity results have also been used recently in [20] for studying the primitive equations in domains with corners.

In a recent work [33], which was brought to our attention by S. Wang after we completed an earlier draft of this paper, the authors proved the global existence of strong solutions to the system (1.1)–(1.14). However, they established their result under a stronger restriction on the initial data, namely, that either $T_0 \in L^\infty(\Omega)$ or $T_0 \in H^2(\Omega)$, which is needed for the proof of the maximum principle on T. Here, we show the global existence of strong solutions under a milder condition, namely, $T_0 \in H^1(\Omega)$, thus avoiding the direct use of the maximum principle for T and thereby answering one of the three open problems posed in [33]. Moreover, we establish the uniqueness of weak solutions and the existence of a finite-dimensional global attractor, the remaining two open problems posed in [33]. Our results for this three-dimensional model are in a sense similar to the well-known results for the two-dimensional Bénard convection and Navier-Stokes equations.

In the inviscid case, i.e., $\epsilon = 0$, the natural physical boundary condition for the velocity field is no-normal flow, i.e., $v \cdot \vec{n} = 0$. The natural physical boundary condition for the temperature on the literal boundary is no-heat flux (see, e.g., [34]). Due to the rotation, i.e., $f \neq 0$, the no-normal flow dictates an additional independent boundary condition on the temperature that makes the heat parabolic PDE overdetermined with boundary conditions and hence ill-posed. For further details about the physical and numerical background and history of this problem, the reader is referred to [34] and references therein.

To remedy this situation, the authors of [34] introduced, in an ad hoc fashion, a fourth-order diffusion term in the temperature equation that compensates for the additional boundary condition. Samelson, Temam, and Wang [32] studied the well-posedness of the inviscid system by adding a bi-Laplacian to the heat equation. To take advantage of the dissipation nature of the bi-Laplacian, the authors of [32] supplemented this operator by yet another boundary condition that is independent of the previous two boundary conditions imposed on the temperature, thereby making the hyperdiffusion problem overdetermined. In a subsequent paper [3], we introduce a proper fourth-order diffusion operator, which is dissipative when it is subject to the two natural physical boundary conditions for the velocity field and temperature, namely, no-normal flow for the velocity field and no-heat flux at the literal boundary. Moreover, we prove the global well-posedness and regularity of both the weak and strong solutions to the above-mentioned inviscid hyperdiffusion, thermocline planetary geostrophic model. Furthermore, we also show that this model possesses a finite-dimensional global attractor.

The rest of this paper is organized as follows: In Section 2 we introduce the functional setting. In Section 3 we establish the existence and uniqueness of the global weak solution. In Section 4 we prove the global existence and uniqueness of a strong solution under the assumption that $T_0 \in H^1(\Omega)$. In Section 5 we show the existence of a global attractor for the system (1.1)–(1.14) and provide an upper bound estimating its dimension.

2 Preliminaries and Functional Setting

2.1 New Formulation

Let us follow [32] to derive an equivalent formulation for the system (1.1)–(1.14). By integrating equation (1.3) in the z-direction, we obtain

$$w(x,y,z,t) = w(x,y,-h,t) - \int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)\,d\xi\,. \tag{2.1}$$

Since $w(x,y,z,t) = 0$ at $z = -h$ and $z = 0$ (see (1.10) and (1.11)), we have

$$w(x,y,z,t) = -\int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)\,d\xi \tag{2.2}$$

and

$$\int_{-h}^{0} \nabla \cdot v(x,y,\xi,t)\,d\xi = \nabla \cdot \int_{-h}^{0} v(x,y,\xi,t)\,d\xi = 0\,. \tag{2.3}$$

By integrating equation (1.2) with respect to z, we obtain

$$p(x,y,z,t) = -\int_{-h}^{z} T(x,y,\xi,t)d\xi + p_s(x,y,t)\,, \tag{2.4}$$

where $p_s(x,y,t)$ is a free function to be determined. Moreover, notice that by setting

$$T = T^* + \widetilde{T}\,, \tag{2.5}$$

we convert the boundary condition (1.10) to be homogeneous; namely, $\widetilde{T}$ satisfies the following homogeneous boundary conditions:

$$\frac{\partial \widetilde{T}}{\partial z}\bigg|_{z=-h} = 0\,, \quad \left(\frac{\partial \widetilde{T}}{\partial z} + \frac{\alpha}{K_v}\widetilde{T}\right)\bigg|_{z=0} = 0\,, \quad \frac{\partial \widetilde{T}}{\partial \vec{n}}\bigg|_{\Gamma_s} = 0 \tag{2.6}$$

(notice that we have also used the compatibility condition (1.13)). Based on all the above, we get the following new formulation for the system (1.1)–(1.14):

$$\nabla\left[p_s(x,y,t) - \int_{-h}^{z} \widetilde{T}(x,y,\xi,t)d\xi - (z+h)T^*(x,y,t)\right] + f\vec{k}\times v + \epsilon L_1 v = 0\,, \tag{2.7}$$

$$\nabla\cdot\int_{-h}^{0} v(x,y,z,t)dz = 0\,, \tag{2.8}$$

$$\partial_t\widetilde{T} + L_2\widetilde{T} + v\cdot\nabla\widetilde{T} - \left(\nabla\cdot\int_{-h}^{z} v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T} + v\cdot\nabla T^* = Q^*\,, \tag{2.9}$$

$$\frac{\partial v}{\partial z}\bigg|_{z=0} = \tau\,, \quad \frac{\partial v}{\partial z}\bigg|_{z=-h} = 0\,, \quad v\cdot\vec{n}\bigg|_{\Gamma_s} = 0\,, \quad \frac{\partial v}{\partial \vec{n}}\times\vec{n}\bigg|_{\Gamma_s} = 0\,, \tag{2.10}$$

$$\left(\partial_z\widetilde{T} + \frac{\alpha}{K_v}\widetilde{T}\right)\bigg|_{z=0} = 0\,, \quad \partial_z\widetilde{T}\bigg|_{z=-h} = 0\,, \quad \frac{\partial}{\partial\vec{n}}\widetilde{T}\bigg|_{\Gamma_s} = 0\,, \tag{2.11}$$

$$\widetilde{T}(x,y,z,0) = T_0(x,y,z) - T^*(x,y)\,, \tag{2.12}$$

where

$$Q^* = Q + K_h\Delta T^*\,.$$

In the above system, the unknowns are the vector field $v(x,y,z,t)$ and the scalar functions $p_s(x,y,t)$ and $\widetilde{T}(x,y,z,t)$, while T^*, τ, Q^*, and $\widetilde{T}_0$ are given.

It is clear that once we determine $v(x,y,z,t)$, $p_s(x,y,t)$, and $\widetilde{T}(x,y,z,t)$, we can easily recover, thanks to (2.1), (2.4), and (2.5), the original unknowns of the system (1.1)–(1.14), that is, (v,w), T, and p, which makes the new formulation equivalent to the original system (1.1)–(1.14).

2.2 Functional Spaces and Inequalities

Let us denote by $L^2(\Omega)$ and $H^1(\Omega)$, $H^2(\Omega)$, ..., the usual L^2-Lebesgue and Sobolev spaces, respectively. We denote by

$$|T| = \left(\int_\Omega |T(x,y,z)|^2 \, dx\, dy\, dz \right)^{1/2} \tag{2.13}$$

for every $T \in L^2(\Omega)$, and by

$$\|T\| = \left(\alpha \int_{\Gamma_u} |T(x,y,0)|^2 \, dx\, dy + \int_\Omega \left[K_h |\nabla T(x,y,z)|^2 + K_v |\partial_z T(x,y,z)|^2 \right] dx\, dy\, dz \right)^{1/2} \tag{2.14}$$

for every $T \in H^1(\Omega)$. Let

$$\widetilde{\mathcal{V}} = \left\{ \widetilde{T} \in C^\infty(\overline{\Omega}) : \left.\frac{\partial \widetilde{T}}{\partial z}\right|_{z=-h} = 0,\ \left.\left(\frac{\partial \widetilde{T}}{\partial z} + \frac{\alpha}{K_v} \widetilde{T} \right)\right|_{z=0} = 0,\ \left.\frac{\partial \widetilde{T}}{\partial \vec{n}}\right|_{\Gamma_s} = 0 \right\}.$$

We also denote by H' the dual space of $H^1(\Omega)$ with the dual action $\langle \cdot, \cdot \rangle$.

Next, we recall the following version of the Poincaré inequality (cf. [1, 43]):

PROPOSITION 2.1 *The norm defined as in* (2.14) *is equivalent to the* $H^1(\Omega)$ *norm*; *namely, there is a constant* K_1 *such that*

$$\frac{1}{K_1} \|T\|^2 \le \|T\|^2_{H^1(\Omega)} \le K_1 \|T\|^2 \tag{2.15}$$

for every $T \in H^1(\Omega)$. *Moreover, we have*

$$|T|^2 \le \widetilde{K} \|T\|^2 \quad \text{for all } T \in H^1(\Omega) \tag{2.16}$$

where

$$\widetilde{K} = \max\left\{ \frac{2h}{\alpha}, \frac{2h^2}{K_v} \right\}. \tag{2.17}$$

For convenience, we state the following version of the Sobolev interpolation inequalities (cf. [1]):

$$\begin{cases} \|h(x,y)\|_{L^4(M)} \le C_4 \|h(x,y)\|^{1/2}_{L^2(M)} \|h(x,y)\|^{1/2}_{H^1(M)} \\ \|h(x,y)\|_{L^6(M)} \le C_4 \|h(x,y)\|^{1/3}_{L^2(M)} \|h(x,y)\|^{2/3}_{H^1(M)} \end{cases} \tag{2.18}$$

and

$$\begin{cases} \|g(x,y,z)\|_{L^3(\Omega)} \le C_5 |g(x,y,z)|^{1/2} \|g(x,y,z)\|^{1/2}_{H^1(\Omega)} \\ \|g(x,y,z)\|_{L^6(\Omega)} \le C_5 \|g(x,y,z)\|_{H^1(\Omega)} \end{cases} \tag{2.19}$$

for all $h \in H^1(M)$ and $g \in H^1(\Omega)$, respectively. Also, we recall the integral version of the Minkowski inequality for the L^p-spaces, $p \ge 1$. Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are two positive integers.

Suppose that $f(\xi, \eta)$ is measurable over $\Omega_1 \times \Omega_2$. Then

$$\left[\int_{\Omega_1} \left(\int_{\Omega_2} |f(\xi,\eta)| d\eta\right)^p d\xi\right]^{1/p} \le \int_{\Omega_2} \left(\int_{\Omega_1} |f(\xi,\eta)|^p \, d\xi\right)^{1/p} d\eta\,. \tag{2.20}$$

Hereafter, C, which may depend on the domain Ω and the constant parameters ϵ, $f_0, \beta, \alpha, A_h, A_v, K_h$, and K_v in the system (1.1)–(1.14), will denote a constant that may change from line to line.

Finally, we state the following proposition, which plays an important role in our proof of the well-posedness to the model (1.1)–(1.14). The proof of this proposition will be given in the appendix.

PROPOSITION 2.2 *Let $u = (u_1, u_2)$ be a smooth vector field, and let f and g be smooth scalar functions. Then*

$$\left|\int_\Omega \left(\nabla \cdot \int_{-h}^z u(x,y,\xi,t) d\xi\right) fg \, dx\, dy\, dz\right| \le C|f| \|u\|^{1/2}_{H^1(\Omega)} \|u\|^{1/2}_{H^2(\Omega)} \|g\|^{1/2}_{H^1(\Omega)} |g|^{1/2}\,.$$

2.3 Regularity Results

In this subsection we recall some regularity results that will be used in our proofs. First, by following the techniques developed in [21, 44] (see also [23, p. 89] for the case of smooth domains), one can show the following regularity results for solutions to the boundary value problem:

$$\begin{cases} L_2^o \widetilde{T} =: -K_h \Delta \widetilde{T} - K_v \partial_z^2 \widetilde{T} = G(x,y,z) \text{ in } \Omega \\ (\partial_z \widetilde{T} + \frac{\alpha}{K_v} \widetilde{T})\big|_{z=0} = 0,\ \partial_z \widetilde{T}\big|_{z=-h} = 0,\ \frac{\partial}{\partial \vec{n}} \widetilde{T}\big|_{\Gamma_s} = 0. \end{cases} \tag{2.21}$$

PROPOSITION 2.3 *Let $G(x,y,z) \in L^2(\Omega)$ be given, and let $\widetilde{T}$ be the solution of the boundary value problem* (2.21). *Then, for every $\widetilde{T} \in \widetilde{\mathcal{V}}$,*

$$\|\widetilde{T}\|_{H^2(\Omega)} \le \frac{C_1}{\min\{K_h, K_v\}} |G| = \frac{C_1}{\min\{K_h, K_v\}} |L_2^o \widetilde{T}|\,. \tag{2.22}$$

Let $\widetilde{T} \in \widetilde{\mathcal{V}}$ be an arbitrary function; notice that by integrating by parts and using the boundary conditions (2.11), for all $\widetilde{T} \in \widetilde{\mathcal{V}}$ we have

$$\begin{aligned}
&\int_\Omega \widetilde{T} L_2^o \widetilde{T}\, dx\, dy\, dz \\
&\quad = -\int_\Omega \widetilde{T}\big(K_h \Delta \widetilde{T} + K_v \partial_z^2 \widetilde{T}\big) dx\, dy\, dz \\
&\quad = \int_\Omega \big[K_h |\nabla \widetilde{T}|^2 + K_v |\partial_z \widetilde{T}|^2\big] dx\, dy\, dz - \int_{\Gamma_u} K_v \widetilde{T} \partial_z \widetilde{T}\, dx\, dy \\
&\quad = \int_\Omega \big[K_h |\nabla \widetilde{T}|^2 + K_v |\partial_z \widetilde{T}|^2\big] dx\, dy\, dz + \alpha \int_{\Gamma_u} |\widetilde{T}|^2\, dx\, dy \\
&\quad = \|\widetilde{T}\|^2 .
\end{aligned} \tag{2.23}$$

As a result of (2.16), (2.22), (2.23), and the Rellich lemma [1], one can show that the operator L_2^o with domain

$$\mathcal{D}(L_2^o) = \text{closure of } \widetilde{\mathcal{V}} \text{ with respect to the } H^2(\Omega) \text{ topology},$$

is a positive self-adjoint operator with compact inverse. Therefore, the space $L^2(\Omega)$ possesses an orthonormal basis $\{\phi_k(x, y, z)\}_{k=1}^\infty$ of eigenfunctions of the operator L_2^o such that

$$L_2^o \phi_k(x, y, z) = \lambda_k \phi_k(x, y, z) \tag{2.24}$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lim_{k\to\infty} \lambda_k = \infty$. Moreover, we have the following Weyl's asymptotic formula (cf. [5]): There is a constant $C_0 > 0$ such that

$$\frac{k^{2/3}}{C_0} \le \frac{\lambda_k}{\lambda_1} \le C_0 k^{2/3} . \tag{2.25}$$

We will let $H_m = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $P_m : L^2(\Omega) \to H_m$ be the $L^2(\Omega)$ orthogonal projection onto H_m.

Next, we recall the following regularity results, which will be used to show the regularity of solutions of equations (2.7) and (2.8) with the boundary conditions (2.10). First, being motivated by the two-dimensional Stokes problem, we consider the following system:

$$\nabla q_s(x, y) + f \vec{k} \times u(x, y) - \epsilon A_h \Delta u(x, y) = \frac{\epsilon}{h} \tau(x, y) \quad \text{in } M , \tag{2.26}$$

$$\nabla \cdot u = 0 \quad \text{in } M , \tag{2.27}$$

$$u \cdot \vec{n} = 0 , \quad \frac{\partial u}{\partial \vec{n}} \times \vec{n} = 0 , \quad \text{on } \partial M . \tag{2.28}$$

Here, the vector field $u(x, y)$ and the scalar function $q_s(x, y)$ are the unknowns, while $\tau(x, y)$ is given. One can follow the proof of the existence and uniqueness theorem of solutions for the Stokes problem and the techniques developed in [21, 44] to show the following results (cf. [8, 17, 38, 39]):

PROPOSITION 2.4 *Let* $\tau \in H_0^1(M)$ *be given. Then there is a unique solution* $(q_s(x, y), u(x, y))$ (q_s *is unique up to a constant*) *of the system* (2.26)–(2.28) *such that*

(i) $u \in (H^2(M))^2$ *and* $q_s \in H^1(M)$, *and*
(ii) *for* $\gamma = 0$ *or* 1,

$$\epsilon A_h \|u\|^2_{H^{\gamma+1}(M)} + \|q_s\|^2_{H^\gamma(M)} \leq \frac{C\epsilon}{h} \|\tau\|^2_{H^1(M)} . \tag{2.29}$$

Furthermore, and again by following the techniques developed in [21, 44] (for the case of smooth domains, see [23, p. 89] and [41]), we have the following regularity results for solutions of the following three-dimensional second-order elliptic boundary value problem:

$$\nabla\eta(x, y, z) + f\vec{k} \times v + \epsilon L_1 v = 0 \quad \text{in } \Omega, \tag{2.30}$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \tau , \quad \left.\frac{\partial v}{\partial z}\right|_{z=-h} = 0 , \quad \left. v \cdot \vec{n}\right|_{\Gamma_s} = 0 , \quad \left.\frac{\partial v}{\partial \vec{n}} \times \vec{n}\right|_{\Gamma_s} = 0 . \tag{2.31}$$

Here $v(x, y, z)$ is the unknown, while $\eta(x, y, z)$ and $\tau(x, y)$ are given.

PROPOSITION 2.5 *Let* $\eta \in H^\gamma(\Omega)$, $\gamma = 0$ *or* 1, *and* $\tau \in H_0^1(M)$ *be given. Then there is a unique solution* $v \in H^{\gamma+1}(\Omega)$ *of equations* (2.30)–(2.31) *such that*

$$\|v\|_{H^{\gamma+1}(\Omega)} \leq \frac{C_2}{\epsilon \widetilde{A}} \big(\|\eta\|_{H^\gamma(\Omega)} + \epsilon \|\tau\|_{H^1(M)}\big) \tag{2.32}$$

where

$$\widetilde{A} = \min\{A_h, A_v\} . \tag{2.33}$$

Finally, let us present our weak formulation of the system (2.7)–(2.12) and state the definition of weak solutions. Again, as we mentioned before, the role of the fundamental functional spaces in problem (2.7)–(2.12) will be played by $H^1(\Omega)$ and H', and not by $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, as in the Dirichlet boundary value problem (cf. [23, pp. 61, 121]).

DEFINITION 2.6 Let $\widetilde{T}_0 \in L^2(\Omega)$, and let S be any fixed positive time. The vector field $v(x, y, z, t)$ and the scalar functions $p_s(x, y, t)$ and $\widetilde{T}(x, y, z, t)$ are called a *weak solution of* (2.7)–(2.12) *on time interval* $[0, S]$ if

$$\begin{aligned} p_s(x, y, t) &\in C([0, S], L^2(M)) \cap L^2([0, S], H^1(M)) , \\ v(x, y, z, t) &\in C([0, S], H^1(\Omega)) \cap L^2([0, S], H^2(\Omega)) , \\ \widetilde{T}(x, y, z, t) &\in C([0, S], L^2(\Omega)) \cap L^2([0, S], H^1(\Omega)) , \\ \partial_t \widetilde{T}(x, y, z, t) &\in L^1([0, S], H') \end{aligned}$$

(recall that H' is the dual space of $H^1(\Omega)$), and if they satisfy

$$\begin{aligned}
&\int_\Omega \nabla\Big[p_s(x,y,t) - \int_{-h}^{z}(\widetilde{T}(x,y,\xi,t)+T^*)d\xi\Big]\phi\,dx\,dy\,dz \\
&\quad + \int_\Omega (f\vec{k}\times v)\phi\,dx\,dy\,dz + \epsilon\int_\Omega (A_h\nabla v\cdot\nabla\phi + A_v\partial_z v\partial_z\phi)dx\,dy\,dz \\
&\quad = \int_{\Gamma_u} A_v\tau\phi\,dx\,dy\,dz
\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}
&\int_\Omega \widetilde{T}(t)\psi\,dx\,dy\,dz + \int_{t_0}^{t}\int_\Omega \big(K_h\nabla\widetilde{T}\cdot\nabla\psi + K_v\partial_z\widetilde{T}\partial_z\psi\big)dx\,dy\,dz \\
&\quad + \alpha\int_{t_0}^{t}\int_{\Gamma_u}\widetilde{T}\psi\,dx\,dy + \int_{t_0}^{t}\int_\Omega (v\cdot\nabla T^*)\psi\,dx\,dy\,dz \\
&\quad + \int_{t_0}^{t}\int_\Omega \Big[(v\cdot\nabla\widetilde{T})\psi - \Big(\nabla\cdot\int_{-h}^{z} v(x,y,\xi,t)d\xi\Big)\partial_z\widetilde{T}\psi\Big]dx\,dy\,dz \\
&\quad = \int_\Omega \widetilde{T}(t_0)\psi\,dx\,dy\,dz + \int_{t_0}^{t}\int_\Omega Q^*\psi\,dx\,dy\,dz
\end{aligned} \tag{2.35}$$

for every $\phi \in (C^\infty(\overline{\Omega}))^2$ and $\psi \in C^\infty(\overline{\Omega})$, and almost every $t, t_0 \in [0, S]$.

Moreover, a weak solution is called a *strong solution of* (2.7)–(2.12) *on* $[0, S]$ if, in addition, it satisfies

$$\begin{aligned}
p_s(x,y,t) &\in C([0,S], H^1(M)) \cap L^2([0,S], H^2(M))\,, \\
v(x,y,z,t) &\in C([0,S], H^1(\Omega)) \cap L^2([0,S], H^2(\Omega))\,, \\
\widetilde{T}(x,y,z,t) &\in C([0,S], H^1(\Omega)) \cap L^2([0,S], H^2(\Omega))\,.
\end{aligned}$$

(Notice that for strong solutions we require $\widetilde{T}_0 \in H^1(\Omega)$).

Observe that the difference between our definition of weak and strong solution is in the regularity of the temperature $\widetilde{T}$.

3 Global Existence, Uniqueness, and Well-Posedness of Weak Solutions

Now we are ready to show the global existence and uniqueness of weak solutions to the system (2.7)–(2.12).

THEOREM 3.1 *Suppose that* $\tau \in H_0^1(M)$, $T^* \in H^2(M)$, *and* $Q \in L^2(\Omega)$. *Then for every* $\widetilde{T}_0 = T_0 - T^* \in L^2(\Omega)$ *and* $S > 0$, *there is a unique weak solution*

$(p_s, v, \widetilde{T})$ *(p_s is unique up to a constant) of the system* (2.7)–(2.12) *on the interval* $[0, S]$. *Moreover,* $\widetilde{T}$ *satisfies*

$$\partial_t \widetilde{T} \in L^2(0, S; H'), \quad |\widetilde{T}|^2 \le K_2(S, Q, |T_0|, T^*, \tau), \tag{3.1}$$

$$\int_0^S \|\widetilde{T}(s)\|^2 \, ds \le K_3(S, Q, |T_0|, T^*, \tau), \tag{3.2}$$

where $K_2(S, Q, |T_0|, T^*, \tau)$ *and* $K_3(S, Q, |T_0|, T^*)$ *are as specified in* (3.17) *and* (3.19)*, respectively.*

PROOF: The existence of weak solutions was proven in [32]. For the sake of completeness we present the proof again. In particular, we establish certain estimates that will be used later in the proof of uniqueness. We will use a Galerkin-like procedure based on the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ to show the existence. Let $m \in \mathbb{Z}^+$ be fixed. The Galerkin approximating system of order m that we use for (2.7)–(2.12) reads

$$\nabla\left[p_s(x, y, t) - \int_{-h}^{z} \widetilde{T}_m(x, y, \xi, t) d\xi - (z + h) T^*(x, y)\right] + f\vec{k} \times v + \epsilon L_1 v = 0, \tag{3.3}$$

$$\nabla \cdot \int_{-h}^{0} v(x, y, z, t) dz = 0, \tag{3.4}$$

$$\frac{\partial}{\partial t}\widetilde{T}_m + L_2^o \widetilde{T}_m + P_m\big[v \cdot \nabla \widetilde{T}_m\big] + P_m\left[-\left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi\right)\frac{\partial \widetilde{T}_m}{\partial z} + (v \cdot \nabla) T^*\right] = P_m Q^*, \tag{3.5}$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \tau, \quad \left.\frac{\partial v}{\partial z}\right|_{z=-h} = 0, \quad \left. v \cdot \vec{n}\right|_{\Gamma_s} = 0, \quad \left.\frac{\partial v}{\partial \vec{n}} \times \vec{n}\right|_{\Gamma_s} = 0, \tag{3.6}$$

$$\widetilde{T}_m(x, y, z, 0) = P_m[T_0(x, y, z) - T^*(x, y)], \tag{3.7}$$

where $\widetilde{T}_m = \sum_{k=1}^m a_k(t)\phi_k(x, y, z)$. In the above system the unknowns are the vector field $v(x, y, z, t)$ and the scalar functions $p_s(x, y, t)$ and $\widetilde{T}_m(x, y, z, t)$ (i.e., the coefficients $\{a_k(t)\}_{k=1}^m$), while T^*, τ, Q, and $\widetilde{T}_0$ are given. Observe that v and p_s depend on m.

First, by applying Proposition 2.4 and Proposition 2.5, we have that, for every fixed and given $\widetilde{T}_m$, there is a unique $(p_s, v) = (p_s(\widetilde{T}_m), v(\widetilde{T}_m))$ (p_s is unique up to a constant) such that, for $\gamma = 0$ or 1,

$$v \in H^{\gamma+1}(\Omega) \quad \text{and} \quad p_s \in H^{\gamma}(M).$$

Furthermore,

$$\|v(\cdot, t)\|^2_{H^{\gamma+1}(\Omega)} + \|p_s\|^2_{H^{\gamma}(M)} \le C\big(\|\widetilde{T}_m\|^2_{H^{\gamma}(\Omega)} + \|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)}\big). \tag{3.8}$$

By replacing $v = v(\widetilde{T}_m)$ in equation (3.5), we get an ODE system with the unknown $\widetilde{T}_m$. That is, equation (3.5) is an ODE system with the unknown $a_k(t)$, $k = 1, 2, \dots, m$. Furthermore, since equation (3.3) is linear, and due to the estimate (3.8), it is easy to check that each term of equation (3.5) is locally Lipschitz in $\widetilde{T}_m$. Therefore, there is a unique solution $a_k(t)$, $k = 1, 2, \dots, m$, to the equation (3.5) for a short interval of time $[0, S^*)$. As a result, we also have the existence and uniqueness of $p_s(x, y, t)$ and $v(x, y, z, t)$ for a short interval of time $[0, S^*)$. Moreover, since $\widetilde{T}_m \in \widetilde{\mathcal{V}}$, by (3.8),

$$v(x, y, z, t) \in (H^2(\Omega))^2 \quad \text{and} \quad p_s(x, y, t) \in H^1(M) \quad \text{for all } t \in [0, S^*)\,.$$

By taking the $L^2(\Omega)$ inner product of equation (3.5) with $\widetilde{T}_m$ and using (2.23), we reach

$$\begin{aligned} &\frac{1}{2}\frac{d|\widetilde{T}_m|^2}{dt} + \|\widetilde{T}_m\|^2 + \int_\Omega (v \cdot \nabla T^*)\widetilde{T}_m \, dx\, dy\, dz \\ &\quad + \int_\Omega \left[v \cdot \nabla \widetilde{T}_m - \left(\nabla \cdot \int_{-h}^z v(x, y, \xi, t) d\xi \right) \partial_z \widetilde{T}_m \right] \widetilde{T}_m \, dx\, dy\, dz \\ &\quad = \int_\Omega \widetilde{T}_m Q^* \, dx\, dy\, dz\,. \end{aligned} \tag{3.9}$$

It is easy to show by integrating by parts and by using the boundary conditions (2.10) and (2.11) that

$$\int_\Omega \left[v \cdot \nabla \widetilde{T}_m - \left(\nabla \cdot \int_{-h}^z v(x, y, \xi, t) d\xi \right) \partial_z \widetilde{T}_m \right] \widetilde{T}_m \, dx\, dy\, dz = 0\,. \tag{3.10}$$

Furthermore, by the Hölder inequality we have

$$\left| \int_\Omega (v \cdot \nabla T^*)\widetilde{T}_m \, dx\, dy\, dz \right| \le C\|v\|_{L^6(\Omega)} \|\nabla T^*\|_{L^3(M)} |\widetilde{T}_m|\,.$$

By (2.19) and (3.8), we have

$$\|v\|_{L^6(\Omega)} \le C\|v\|_{H^1(\Omega)} \le C\big[|\widetilde{T}_m| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)}\big]\,. \tag{3.11}$$

By (2.18), we obtain $\|\nabla T^*\|_{L^3(M)} \le C\|T^*\|_{H^2(M)}$. As a result of the above estimates, we obtain

$$\begin{aligned} &\left| \int_\Omega (v \cdot \nabla T^*)\widetilde{T}_m \, dx\, dy\, dz \right| \le \\ &\qquad C\|T^*\|_{H^2(M)} |\widetilde{T}_m|^2 + C\|T^*\|_{H^2(M)} \big[\|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)}\big]\,. \end{aligned} \tag{3.12}$$

Applying the Cauchy-Schwarz inequality and the definition of Q^*, we obtain

$$\left| \int_\Omega Q^* \widetilde{T}_m \, dx\, dy\, dz \right| \le C\big[|Q| + \|T^*\|_{H^2(M)}\big] |\widetilde{T}_m|\,. \tag{3.13}$$

Therefore, from estimates (3.10)–(3.13) above and (2.23), (3.9) gives

$$\frac{1}{2}\frac{d|\widetilde{T}_m|^2}{dt} + \|\widetilde{T}_m\|^2 \leq \big[|Q| + \|T^*\|_{H^2(M)}\big]|\widetilde{T}_m| + C\|T^*\|_{H^2(M)}|\widetilde{T}_m|^2 + C\|T^*\|_{H^2(M)}\big[\|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)}\big].$$

By Cauchy-Schwarz inequality, we get

$$\frac{d|\widetilde{T}_m|^2}{dt} + 2\|\widetilde{T}_m\|^2 \leq C\|T^*\|_{H^2(M)}|\widetilde{T}_m|^2 + C\big[|Q|^2 + \|T^*\|_{H^2(M)}\big(\|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)}\big)\big]. \tag{3.14}$$

Thanks to the Gronwall inequality, we conclude

$$|\widetilde{T}_m(t)|^2 \leq e^{C\|T^*\|_{H^2(M)}t}\big[|T_0|^2 + C\big(|Q|^2 + \|T^*\|_{H^2(M)}(\|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)})\big)\big], \tag{3.15}$$

when $0 \leq t < S^*$. But since the right-hand side is bounded as t goes to S^*, we conclude that $\widetilde{T}_m(t)$ must exist globally, i.e., $S^* = +\infty$. Therefore, for any given $S > 0$ and any $t \in [0, S]$, we have

$$|\widetilde{T}_m(t)|^2 \leq K_2(S, Q, |T_0|, T^*, \tau), \tag{3.16}$$

where

$$K_2(S, Q, |T_0|, T^*, \tau) = e^{C\|T^*\|_{H^2(M)}S}\big[|T_0|^2 + C\big(|Q|^2 + \|T^*\|_{H^2(M)}(\|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)})\big)\big]. \tag{3.17}$$

By integrating (3.14) with respect to t over $[0, S]$ and by (3.16), we get

$$\int_0^S \|\widetilde{T}_m\|^2\,ds \leq K_3(S, Q, |T_0|, T^*, \tau), \tag{3.18}$$

where

$$K_3(S, Q, |T_0|, T^*, \tau) = |T_0|^2 + C|Q|^2 S + C\|T^*\|_{H^2(M)}\big(\|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)} + K_2(S, Q, |T_0|, T^*, \tau)\big)S \tag{3.19}$$

and $K_2(S, Q, |T_0|, T^*, \tau)$ is as in (3.17). Notice that estimate (3.16) is unbounded in time (i.e., as $S \to \infty$), but it is uniformly bounded in m. However, in Section 5 we will present a sharper estimate that is asymptotically bounded in time.

As a result of all the above, we have that $\widetilde{T}_m$ exists globally in time and is uniformly bounded, in m, in the $L^\infty([0, S]; L^2(\Omega))$ and $L^2([0, S]; H^1(\Omega))$ norms.

Next, let us show that $\partial_t \widetilde{T}_m$ is uniformly bounded, in m, in the $L^2([0, S]; H')$ norm. From (3.5), we have, for every $\psi \in C^\infty(\Omega)$,

$$\left\langle \frac{\partial}{\partial t} \widetilde{T}_m, \psi \right\rangle = \left\langle P_m Q^* - L_2 \widetilde{T}_m, \psi \right\rangle - \left\langle P_m \left[v \cdot \nabla \widetilde{T}_m - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t \right) d\xi) \frac{\partial \widetilde{T}_m}{\partial z} + (v \cdot \nabla) T^* \right], \psi \right\rangle .$$

Recall that $\langle \cdot, \cdot \rangle$ is the dual action of H', the dual space of $H^1(\Omega)$. It is clear from (2.16) that

$$\begin{aligned} \left| \langle P_m Q^*, \psi \rangle \right| &\le C \left(|Q| + h^{1/2} \|T^*\|_{H^2(M)} \right) |\psi| \\ &\le C \widetilde{K} \left(|Q| + h^{1/2} \|T^*\|_{H^2(M)} \right) \|\psi\| , \end{aligned} \tag{3.20}$$

and by integration by parts we have

$$\left| \langle L_2 \widetilde{T}_m, \psi \rangle \right| \le C \|\widetilde{T}_m\| \|\psi\| . \tag{3.21}$$

Next, let us get an estimate for

$$\left| \left\langle P_m \left[v \cdot \nabla \left(\widetilde{T}_m + T^* \right) - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \frac{\partial \widetilde{T}_m}{\partial z} \right], \psi \right\rangle \right| = \left| \int_\Omega \left[v \cdot \nabla \left(\widetilde{T}_m + T^* \right) - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \frac{\partial \widetilde{T}_m}{\partial z} \right] \psi_m \, dx \, dy \, dz \right|$$

where $\psi_m = P_m \psi$. Thus, by integration by parts, we obtain

$$\left| \left\langle P_m \left[v \cdot \nabla \left(\widetilde{T}_m + T^* \right) - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \frac{\partial \widetilde{T}_m}{\partial z} \right], \psi \right\rangle \right| = \left| \int_\Omega \left[v \cdot \nabla \psi_m - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \frac{\partial \psi_m}{\partial z} \right] \left(\widetilde{T}_m + T^* \right) dx \, dy \, dz \right| .$$

Next, we estimate

$$\left| \int_\Omega (v \cdot \nabla \psi_m) \left(\widetilde{T}_m + T^* \right) dx \, dy \, dz \right| \le \|\psi_m\|_{H^1(\Omega)} \|v\|_{L^6(\Omega)} \|\widetilde{T}_m + T^*\|_{L^3(\Omega)} .$$

Applying (2.19) and (3.11), we have

$$\begin{aligned} \left| \int_\Omega (v \cdot \nabla \psi_m) \left(\widetilde{T}_m + T^* \right) \right| &\le C \left[\|\widetilde{T}_m\|_{H^1(\Omega)} + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \right] \\ &\quad \times \left[|\widetilde{T}_m|^{1/2} \|\widetilde{T}_m\|_{H^1(\Omega)}^{1/2} + \|T^*\|_{H^2(M)} \right] \|\psi_m\|_{H^1(\Omega)} . \end{aligned} \tag{3.22}$$

Applying Proposition 2.2 by setting $u = v$, $f = \partial_z \psi_m$, and $g = \widetilde{T}_m + T^*$, respectively, and since $\partial_z T^* = 0$, we have

$$\left| \int_\Omega \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \partial_z \widetilde{T}_m \psi_m \, dx \, dy \, dz \right| \tag{3.23}$$

$$\leq C \|\psi_m\| \, \|v\|_{H^1(\Omega)}^{1/2} \, \|v\|_{H^2(\Omega)}^{1/2} \left\| \widetilde{T}_m + T^* \right\|_{H^1(\Omega)}^{1/2} \left| \widetilde{T}_m + T^* \right|^{1/2}$$

$$\leq C \big(|\widetilde{T}_m| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \big)$$

$$\times \big(\|\widetilde{T}_m\| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \big) \|\psi_m\| \, .$$

By (3.22) and (3.23), we have

$$\left| \left\langle P_m \left[v \cdot \nabla \big(\widetilde{T}_m + T^* \big) - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \frac{\partial \widetilde{T}_m}{\partial z} \right], \psi \right\rangle \right| \leq$$

$$C \big(|\widetilde{T}_m| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \big)$$

$$\times \big(\|\widetilde{T}_m\| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \big) \|\psi_m\|_{H^1(\Omega)} \, .$$

Since $\psi \in H^1(\Omega)$, the Fourier series

$$\sum_{k=1}^{\infty} \left(\int_\Omega \psi \phi_k \, dx \, dy \, dz \right) \phi_k = \psi_m + \sum_{k=m+1}^{\infty} \left(\int_\Omega \psi \phi_k \, dx \, dy \, dz \right) \phi_k$$

converges to ψ in $H^1(\Omega)$ (cf. [23, p. 54]). As a result, we get

$$\|\psi_m\|_{H^1(\Omega)} \leq C \|\psi\|_{H^1(\Omega)} \, ;$$

therefore,

$$\left| \left\langle P_m \left[v \cdot \nabla \big(\widetilde{T}_m + T^* \big) - \left(\nabla \cdot \int_{-h}^{z} v(x, y, \xi, t) d\xi \right) \frac{\partial \widetilde{T}_m}{\partial z} \right], \psi \right\rangle \right| \leq$$

$$C \big(|\widetilde{T}_m| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \big) \tag{3.24}$$

$$\times \big(\|\widetilde{T}_m\| + \|T^*\|_{H^2(M)} + \|\tau\|_{H^1(M)} \big) \|\psi\|_{H^1(\Omega)} \, .$$

By estimates (3.20)–(3.24), (3.16), and (3.18), we have

$$\left| \langle \partial_t \widetilde{T}_m, \psi \rangle \right| \leq$$

$$C \widetilde{K} \big(|Q| + \|T^*\|_{H^2(M)} \big) \|\psi\| + C \|\widetilde{T}_m\| \|\psi\|$$

$$+ C \big(|\widetilde{T}_m| + \|\tau\|_{H^1(\Omega)} + \|T^*\|_{H^2(\Omega)} \big) \big(\|\widetilde{T}_m\| + \|\tau\|_{H^1(\Omega)} + \|T^*\|_{H^2(\Omega)} \big) \|\psi\| \, .$$

Thus, due to (3.16) and (3.18), we have

$$\int_0^S \|\partial_t \widetilde{T}_m(t)\|_{H'}^2 \, dt \leq K_4(S, Q, |T_0|, T^*, \tau) \, , \tag{3.25}$$

where

$$
\begin{aligned}
&K_4(S, Q, |T_0|, T^*, \tau) = \\
&\quad C K_3(S, Q, |T_0|, T^*, \tau) + C\big[|Q|^2 + \|T^*\|^2_{H^2(M)}\big] S \\
&\quad\quad + C\big[K_2(S, Q, |T_0|, T^*, \tau) + \|\tau\|^2_{H^1(\Omega)} + \|T^*\|^2_{H^2(\Omega)}\big] \\
&\quad\quad\quad \times \big[K_3(S, Q, |T_0|, T^*, \tau) + \|T^*\|^2_{H^2(M)} + \|\tau\|^2_{H^1(M)}\big] S \,.
\end{aligned} \tag{3.26}
$$

Therefore, $\partial_t \widetilde{T}_m$ is uniformly bounded in m in the $L^2([0, S]; H')$ norm. Thanks to (3.8), (3.16), (3.18), and (3.25), one can apply the Aubin compactness theorem (cf. [8, 27, 38]) and extract a subsequence $\{\widetilde{T}_{m_j}\}$ of $\{\widetilde{T}_m\}$, a subsequence $\{v_{m_j}\}$ of $\{v_m = v(\widetilde{T}_m)\}$, a subsequence $\{p_{s_{m_j}}\}$ of $\{p_{s_m} = p_s(\widetilde{T}_m)\}$, and a subsequence $\{\partial_t \widetilde{T}_{m_j}\}$ of $\{\partial_t \widetilde{T}_m\}$, which converge to

$$
\begin{aligned}
\widetilde{T} &\in L^\infty([0, S]; L^2(\Omega)) \cap L^2([0, S]; H^1(\Omega)) \,, \\
v &\in L^\infty([0, S]; H^1(\Omega)) \cap L^2([0, S]; H^2(\Omega)) \,, \\
p_s &\in L^\infty([0, S]; L^2(\Omega)) \cap L^2([0, S]; H^1(\Omega)) \,, \\
\partial_t \widetilde{T} &\in L^2([0, S]; H') \,,
\end{aligned}
$$

respectively, in the following sense:

$$
\begin{cases}
\widetilde{T}_{m_j} \to \widetilde{T} & \text{in } L^\infty([0, S]; L^2(\Omega)) \\
\widetilde{T}_{m_j} \to \widetilde{T} & \text{in } L^2([0, S]; H^1(\Omega)) \text{ weakly} \\
p_{s_{m_j}} \to p_s & \text{in } L^\infty([0, S]; L^2(M)) \\
p_{s_{m_j}} \to p_s & \text{in } L^2([0, S]; H^1(M)) \text{ weakly} \\
v_{m_j} \to v & \text{in } L^\infty([0, S]; H^1(\Omega)) \\
v_{m_j} \to v & \text{in } L^2([0, S]; H^2(\Omega)) \text{ weakly} \\
\partial_t \widetilde{T}_{m_j} \to \partial_t \widetilde{T} & \text{in } L^2([0, S]; H') \text{ weakly.}
\end{cases}
$$

Notice that since $\widetilde{T}_{m_j} \in \widetilde{\mathcal{V}}$, by (3.8) and integration by parts it is clear that

$$
\begin{aligned}
&\int_\Omega \nabla\Big[p_{s_{m_j}}(x, y, t) - \int_{-h}^{z} \widetilde{T}_{m_j}(x, y, \xi, t) d\xi - (z + h) T^*(x, y, t)\Big] \phi \, dx\, dy\, dz \\
&\quad + \int_\Omega \big(f \vec{k} \times v_{m_j}\big) \phi \, dx\, dy\, dz + \epsilon \int_\Omega \big(A_h \nabla v_{m_j} \cdot \nabla \phi + A_v \partial_z v_{m_j} \partial_z \phi\big) dx\, dy\, dz \\
&\quad = \int_{\Gamma_u} A_v \tau \phi \, dx\, dy\, dz
\end{aligned}
$$

and

$$\int_\Omega \widetilde{T}_{m_j}(x,y,z,t)\psi\,dx\,dy\,dz - \int_\Omega \widetilde{T}_{m_j}(x,y,z,t_0)\psi\,dx\,dy\,dz$$
$$+\int_{t_0}^{t}\int_\Omega \big(K_h\nabla\widetilde{T}_{m_j}\cdot\nabla\psi + \partial_z\widetilde{T}_{m_j}\partial_z\psi\big)dx\,dy\,dz + \alpha\int_{t_0}^{t}\int_{\Gamma_u}\widetilde{T}_{m_j}\psi\,dx\,dy$$
$$+\int_{t_0}^{t}\int_\Omega \bigg[\big(v_{m_j}\cdot\nabla\widetilde{T}_{m_j}\big) - \bigg(\nabla\cdot\int_{-h}^{z} v_{m_j}(x,y,\xi,t)d\xi\bigg)\partial_z\widetilde{T}_{m_j}\bigg]\psi\,dx\,dy\,dz$$
$$+\int_{t_0}^{t}\int_\Omega \big(v_{m_j}\cdot\nabla T^*\big)\psi\,dx\,dy\,dz$$
$$=\int_\Omega Q^*\psi\,dx\,dy\,dz$$

for every $\phi\in(C^\infty(\overline{\Omega}))^2$ and $\psi\in C^\infty(\overline{\Omega})$, and for every $t, t_0\in[0,S]$. By passing to the limit, one can show as in the case of Navier-Stokes equations (see [8, 38]) that $\widetilde{T}$ also satisfies (2.34) and (2.35). In other words, $(p_s, v, \widetilde{T})$ is a weak solution of the system (2.7)–(2.12).

Next, we show the uniqueness. Let $(p_s', v', \widetilde{T}_1)$ and $(p_s'', v'', \widetilde{T}_2)$ be two weak solutions of the system (2.7)–(2.12) with initial values $\widetilde{T}_0'(x,y,z)$ and $\widetilde{T}_0''(x,y,z)$, respectively. Let $u = v'' - v'$, $\chi = \widetilde{T}_2 - \widetilde{T}_1$, and $q_s = p_s'' - p_s'$. It is clear from (2.7)–(2.12) that q_s, u, and χ satisfy

$$\nabla\bigg[q_s(x,y,t) - \int_{-h}^{z}\chi(x,y,\xi,t)d\xi\bigg] + f\vec{k}\times u + \epsilon L_1 u = 0\,, \tag{3.27}$$

$$\nabla\cdot\int_{-h}^{0} u(x,y,z,t)dz = 0\,, \tag{3.28}$$

$$\begin{aligned}&\partial_t\chi + L_2^o\chi + u\cdot\nabla\widetilde{T}_1 + v_2\cdot\nabla\chi + u\cdot\nabla T^*\\ &\quad-\bigg(\nabla\cdot\int_{-h}^{z}u(x,y,\xi,t)d\xi\bigg)\frac{\partial\widetilde{T}_1}{\partial z} - \bigg(\nabla\cdot\int_{-h}^{z}v''(x,y,\xi,t)d\xi\bigg)\partial_z\chi\\ &\quad= 0\,,\end{aligned} \tag{3.29}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=0} = 0\,,\quad \frac{\partial u}{\partial z}\bigg|_{z=-h} = 0\,,\quad u\cdot\vec{n}\Big|_{\Gamma_s} = 0\,,\quad \frac{\partial u}{\partial\vec{n}}\times\vec{n}\bigg|_{\Gamma_s} = 0\,, \tag{3.30}$$

$$\bigg(\frac{\partial\chi}{\partial z} + \frac{\alpha}{K_v}\chi\bigg)\bigg|_{z=0} = 0\,,\quad \frac{\partial\chi}{\partial z}\bigg|_{z=-h} = 0\,,\quad \frac{\partial\chi}{\partial\vec{n}}\bigg|_{\partial M} = 0\,, \tag{3.31}$$

$$\chi(x,y,z,0) = \widetilde{T}_0''(x,y,z) - \widetilde{T}_0'(x,y,z)\,. \tag{3.32}$$

By averaging (3.27) and (3.29) with respect to z and using (3.28), we get

$$\nabla\left[q_s(x,y,t)+\frac{1}{h}\int_{-h}^{0}\xi\chi(x,y,\xi,t)d\xi\right]+f\vec{k}\times\overline{u}-\epsilon A_h\Delta\overline{u}=0\,, \tag{3.33}$$

$$\nabla\cdot\overline{u}=0\,, \tag{3.34}$$

$$\overline{u}\cdot\vec{n}=0\,,\quad \frac{\partial\overline{u}}{\partial\vec{n}}\times\vec{n}=0\quad \text{on } \partial M\,, \tag{3.35}$$

where

$$\overline{u}(x,y,t)=\frac{1}{h}\int_{-h}^{0}u(x,y,z,t)dz\,.$$

By taking the $L^2(\Omega)$ inner product to equation (3.33) with $\overline{u}$, we obtain

$$\int_\Omega\left[\nabla\left(q_s(x,y,t)+\frac{1}{h}\int_{-h}^{0}\xi\chi(x,y,\xi,t)d\xi\right)-\epsilon A_h\Delta\overline{u}\right]\overline{u}\,dx\,dy\,dz=0\,.$$

By using integration by parts and applying (3.34) and (3.35), we get

$$\int_\Omega|\nabla\overline{u}|^2\,dx\,dy\,dz=0\,.$$

Thus, $\overline{u}$ is a constant function. By (3.35), we reach $\overline{u}=0$. As a result, we have

$$q_s(x,y,t)=-\frac{1}{h}\int_{-h}^{0}\xi\chi(x,y,\xi,t)d\xi \tag{3.36}$$

(q_s is unique up to a constant); therefore, (3.27) can be written as

$$-\nabla\left[\frac{1}{h}\int_{-h}^{0}\xi\chi(x,y,\xi,t)d\xi+\int_{-h}^{z}\chi(x,y,\xi,t)d\xi\right]+f\vec{k}\times u+\epsilon L_1u=0\,. \tag{3.37}$$

Notice that u satisfies the boundary condition (3.30). For this second-order elliptic boundary value problem we have the following regularity results (by following similar techniques to those developed in [21, 44] (for the case of smooth domains, see [23, p. 89] and [41]):

$$\|u\|_{H^1(\Omega)}\le\frac{C_2}{\epsilon\widetilde{A}}|\chi|\quad\text{and}\quad\|u\|_{H^2(\Omega)}\le\frac{C_2}{\epsilon\widetilde{A}}\|\chi\|\,. \tag{3.38}$$

By taking the H' dual action to equation (3.29) with χ, we obtain

$$\begin{aligned}&\left\langle\partial_t\chi+L_2^o\chi,\chi\right\rangle+\left\langle u\cdot\nabla\widetilde{T}_1+v''\cdot\nabla\chi+u\cdot\nabla T^*,\chi\right\rangle\\&-\left\langle\left(\nabla\cdot\int_{-h}^{z}u(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}_1-\left(\nabla\cdot\int_{-h}^{z}v''(x,y,\xi,t)d\xi\right)\partial_z\chi,\chi\right\rangle=0\,.\end{aligned}$$

Since $\partial_t\chi\in L^2([0,S],H')$, we apply Lions' lemma [38, lemma 1.2, p. 260] and (2.23) to reach

$$\langle\partial_t\chi,\chi\rangle=\frac{1}{2}\frac{d|\chi|^2}{dt}\quad\text{and}\quad\langle L_2^o\chi,\chi\rangle=\|\chi\|^2\,.$$

Moreover, we have

$$\left\langle u\cdot\nabla\widetilde{T}_1+v''\cdot\nabla\chi+u\cdot\nabla T^*,\,\chi\right\rangle=\int\limits_\Omega\left[u\cdot\nabla\widetilde{T}_1+v''\cdot\nabla\chi+u\cdot\nabla T^*\right]\chi$$

and

$$\left\langle\big(\nabla\cdot\int_{-h}^z u(x,y,\xi,t)d\xi\big)\partial_z\widetilde{T}_1+\left(\nabla\cdot\int_{-h}^z v''(x,y,\xi,t)d\xi\right)\partial_z\chi,\,\chi\right\rangle=$$
$$\int\limits_\Omega\left[\left(\nabla\cdot\int_{-h}^z u(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}_1+\left(\nabla\cdot\int_{-h}^z v''(x,y,\xi,t)d\xi\right)\partial_z\chi\right]\chi$$

as long as the integrals make sense. Therefore, we have

$$\begin{aligned}\frac{1}{2}\frac{d|\chi|^2}{dt}+\|\chi\|^2=\int\limits_\Omega\Bigg[&-u\cdot\nabla\widetilde{T}_1-v''\cdot\nabla\chi-u\cdot\nabla T^*\\&+\left(\nabla\cdot\int_{-h}^z u(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}_1\\&+\left(\nabla\cdot\int_{-h}^z v''(x,y,\xi,t)d\xi\right)\partial_z\chi\Bigg]\chi\,.\end{aligned}$$

Next we estimate the above equation term by term.

(1) By integrating by parts and (3.30), we reach

$$\int\limits_\Omega\left[v''\cdot\nabla\chi-\left(\nabla\cdot\int_{-h}^z v''(x,y,\xi,t)d\xi\right)\partial_z\chi\right]\chi=0\,. \tag{3.39}$$

(2) We have

$$\left|\int\limits_\Omega u\cdot\nabla\big(\widetilde{T}_1+T^*\big)\chi\right|\le\left\|\widetilde{T}_1+T^*\right\|_{H^1(\Omega)}\|u\|_{L^6(\Omega)}\|\chi\|_{L^3(\Omega)}\,.$$

By applying (3.38) and (2.19), we obtain

$$\|u\|_{L^6(\Omega)}\le\frac{C}{\epsilon\widetilde{A}}|\chi|\quad\text{and}\quad\|\chi\|_{L^3(\Omega)}\le C|\chi|^{1/2}\|\chi\|^{1/2}\,.$$

Thus,

$$\left|\int\limits_\Omega u\cdot\nabla\big(\widetilde{T}_1+T^*\big)\chi\right|\le C\big[\|\widetilde{T}_1\|+\|T^*\|_{H^1(M)}\big]|\chi|^{3/2}\|\chi\|^{1/2}\,. \tag{3.40}$$

(3) Applying Proposition 2.2 by setting $u=u$, $f=\partial_z\widetilde{T}_1$, and $g=\chi$, respectively, we have

$$\left|\int\limits_\Omega\left(\nabla\cdot\int_{-h}^z u(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}_1\chi\right|\le$$
$$C\left\|\widetilde{T}_1\right\|_{H^1(\Omega)}\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{H^2(\Omega)}\|\chi\||\chi|^{1/2}\,.$$

Applying (3.38) to the above estimate, we get

$$\left|\int_\Omega \left(\nabla\cdot\int_{-h}^{z} u(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T}_1\chi\,dx\,dy\,dz\right| \le C\|\widetilde{T}_1\||\chi|\|\chi\|\,. \tag{3.41}$$

Therefore, from the above estimates (3.39)–(3.41), we get

$$\frac{1}{2}\frac{d|\chi|^2}{dt} + \|\chi\|^2 \le C\big(\|\widetilde{T}_1\| + \|T^*\|_{H^1(M)}\big)|\chi|^{3/2}\,\|\chi\|^{1/2} + C\|\widetilde{T}_1\||\chi|\|\chi\|\,.$$

By Young's inequality, we obtain

$$\frac{d|\chi|^2}{dt} + \|\chi\|^2 \le C\big[1 + \|\widetilde{T}_1\|^2 + \|T^*\|^{4/3}_{H^1(M)}\big]|\chi|^2\,.$$

Thanks to the Gronwall inequality, we get

$$|\chi|^2(t) \le |\chi|^2(t_0)e^{C[1+\|T^*\|^{4/3}_{H^1(M)}](t-t_0)+\int_{t_0}^{t}\|\widetilde{T}_1\|^2\,ds}\,. \tag{3.42}$$

In particular, when we start with the same initial data, that is, $\widetilde{T}_0''(x,y,z) = \widetilde{T}_0'(x,y,z)$, let $t_0 = 0$ in (3.42), and recall that $\widetilde{T}_1 \in L^2([0,S]; H^1(\Omega))$, we conclude that

$$\chi(t) = 0\,.$$

In other words, the weak solution is unique. □

COROLLARY 3.2 *The weak solution of the system* (2.7)–(2.12) *depends continuously on the initial data*; *that is, the problem is well-posed.*

PROOF: The proof is an immediate consequence of inequality (3.42). □

4 Global Existence, Uniqueness, and Well-Posedness of Strong Solutions

In previous sections we have reformulated the system (1.1)–(1.14) and established the appropriate elliptic regularity results for the velocity field (v, w), and we have proved the existence, uniqueness, and well-posedness of the weak solution for the reformulated system (2.7)–(2.12). In this section we show the global existence, uniqueness, and well-posedness of strong solutions for the system (2.7)–(2.12).

THEOREM 4.1 *Suppose that* $\tau \in H_0^1(M)$, $Q \in H^1(\Omega)$, *and* $T^* \in H^2(M)$. *Then for every* $\widetilde{T}_0 = T_0 - T^* \in H^1(\Omega)$ *and* $S > 0$, *there is a unique strong solution* $\widetilde{T}$ *of the system* (2.7)–(2.12) *such that*

$$\|\widetilde{T}\|^2_{H^1(\Omega)} + \int_0^S \|\widetilde{T}(t)\|^2_{H^2(\Omega)}\,dt \le K_s(S, T_0, \tau, Q, T^*, \tau) \tag{4.1}$$

where $K_s(S, T_0, \tau, Q, T^*, \tau)$ *will be specified in* (4.7).

Remark. The steps of the following proof are formal in the sense that they can be made rigorous by proving their corresponding counterpart estimates first for the Galerkin approximation system (3.3)–(3.7). Then the estimates for the exact solution can be established by passing to the limit in the Galerkin procedure by using the appropriate "compactness theorems" and using the uniqueness of the strong solution.

PROOF: Suppose that $(p_s, v, \widetilde{T})$ is the weak solution with initial value $\widetilde{T}_0$. By taking the $L^2(\Omega)$ inner product of equation (2.9) with $L_2^o\widetilde{T}$, we reach

$$\begin{aligned}
&\frac{1}{2}\frac{d\|\widetilde{T}\|^2}{dt} + |L_2^o\widetilde{T}|^2 \\
&\quad = \int_\Omega \big[Q + K_h\Delta T^*\big]L_2^o\widetilde{T}\,dx\,dy\,dz \\
&\qquad + \int_\Omega \left[-v\cdot\nabla(\widetilde{T}+T^*) + \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}\right]L_2^o\widetilde{T}\,dx\,dy\,dz \\
&\quad \le \big[|Q| + h^{1/2}K_h\|T^*\|_{H^2(M)} + \|v\|_{L^6(\Omega)}\|\nabla(\widetilde{T}+T^*)\|_{L^3(\Omega)}\big]\big|L_2^o\widetilde{T}\big| \\
&\qquad + \left|\int_\Omega \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}L_2^o\widetilde{T}\,dx\,dy\,dz\right|.
\end{aligned}$$

Let us consider the above inequality term by term.

- By (2.19) and Proposition 2.5, we obtain

$$\|v\|_{L^6(\Omega)} \le \frac{C}{\epsilon\widetilde{A}}\big[|\widetilde{T}| + \|T^*\|_{H^2(M)} + \epsilon\|\tau\|_{H^1(M)}\big]. \tag{4.2}$$

- By (2.19) and Proposition 2.3, we get

$$\begin{aligned}
\|\nabla\widetilde{T}\|_{L^3(\Omega)} + h^{1/3}\|\nabla T^*\|_{L^3(M)} &\le C\|\widetilde{T}\|^{1/2}\|\widetilde{T}\|_{H^2(\Omega)}^{1/2} + C\|T^*\|_{H^2(M)} \\
&\le \frac{C}{\widetilde{K}^{1/2}}\|\widetilde{T}\|^{1/2}|L_2^o\widetilde{T}|^{1/2} + C\|T^*\|_{H^2(M)}.
\end{aligned} \tag{4.3}$$

- Applying Proposition 2.2 by setting $u = v$, $f = L_2^o\widetilde{T}$, and $g = \partial_z\widetilde{T}$, respectively, we have

$$\begin{aligned}
&\left|\int_\Omega \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}L_2^o\widetilde{T}\,dx\,dy\,dz\right| \\
&\quad \le C\big|L_2^o\widetilde{T}\big|\|v\|_{H^1(\Omega)}^{1/2}\,\|v\|_{H^2(\Omega)}^{1/2}\,\big\|\partial_z\widetilde{T}\big\|_{H^1(\Omega)}^{1/2}\,\big|\partial_z\widetilde{T}\big|^{1/2} \\
&\quad \le \frac{C}{\epsilon\widetilde{A}\widetilde{K}^{1/2}}\big[|\widetilde{T}|^{1/2}\|\widetilde{T}\|^{1/2} + \|T^*\|_{H^2(M)} + \epsilon\|\tau\|_{H^1(M)}\big]\|\widetilde{T}\|^{1/2}\,\big|L_2^o\widetilde{T}\big|^{3/2}.
\end{aligned} \tag{4.4}$$

Therefore, from the above estimates (4.2)–(4.4) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d\|\widetilde{T}\|^2}{dt} + |L_2^o\widetilde{T}|^2 \le \\
&\quad C\big[\|T^*\|_{H^2(\Omega)} + |Q|\big]\big|L_2^o\widetilde{T}\big| \\
&\quad + \frac{C}{\epsilon\widetilde{A}\widetilde{K}^{1/2}}\big[|\widetilde{T}| + \|T^*\|_{H^2(M)} + \epsilon\|\tau\|_{H^1(M)}\big] \\
&\quad \times \big[\|\widetilde{T}\|^{1/2}|L_2^o\widetilde{T}|^{1/2} + \|T^*\|_{H^2(M)}\big]\big|L_2^o\widetilde{T}\big| \\
&\quad + \frac{C}{\epsilon\widetilde{A}\widetilde{K}^{1/2}}\big[|\widetilde{T}|^{1/2}\|\widetilde{T}\|^{1/2} + \|T^*\|_{H^2(M)} + \epsilon\|\tau\|_{H^1(M)}\big]\|\widetilde{T}\|^{1/2}\big|L_2^o\widetilde{T}\big|^{3/2}.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and Young's inequality, we get

$$\begin{aligned}
&\frac{d\|\widetilde{T}\|^2}{dt} + \big|L_2^o\widetilde{T}\big|^2 \le \\
&\quad C\big[1 + \|T^*\|^4_{H^2(\Omega)} + |Q|^2 + \|\tau\|^4_{H^1(M)} + |\widetilde{T}|^4\big] \\
&\quad\quad + C\big(\|T^*\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + |\widetilde{T}|^4 + |\widetilde{T}|^2\|\widetilde{T}\|^2\big)\|\widetilde{T}\|^2.
\end{aligned} \tag{4.5}$$

Again, by the Gronwall inequality and Theorem 3.1, for every $0 \le t \le S$, we obtain

$$\|\widetilde{T}(t)\|^2 + \int_0^t \big|L_2^o\widetilde{T}(s)\big|^2\,ds \le K_s(S, T_0, \tau, Q, T^*, \tau), \tag{4.6}$$

where

$$\begin{aligned}
&K_s(S, T_0, \tau, Q, T^*, \tau) = \\
&\quad\Bigg[2\|T_0\|^2 + 2h\|T^*\|^2_{H^1(M)} \\
&\quad\quad + C\frac{1 + \|T^*\|^4_{H^2(\Omega)} + |Q|^2 + \|\tau\|^4_{H^1(M)} + (K_2(S, Q, |T_0|, T^*, \tau))^2}{1 + \|T^*\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + (K_2(S, Q, |T_0|, T^*, \tau))^2}\Bigg] \\
&\quad \times \exp\Big(CS\big[\|T^*\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + (K_2(S, Q, |T_0|, T^*, \tau))^2\big] \\
&\quad\quad + K_2(S, Q, |T_0|, T^*, \tau)K_3(S, Q, |T_0|, T^*, \tau)\Big),
\end{aligned} \tag{4.7}$$

and $K_2(S, Q, |T_0|, T^*, \tau)$ and $K_3(S, Q, |T_0|, T^*, \tau)$ are as in (3.17) and (3.19), respectively. In addition, by using similar steps that led to (4.2)–(4.4), one can show that

$$\partial_t\widetilde{T} \in L^2([0, S], L^2(\Omega)).$$

Therefore, $\widetilde{T} \in C([0,S], H^1(\Omega)) \cap L^2([0,S], H^2(\Omega))$. Moreover, by Proposition 2.5, we have

$$v \in C([0,S], H^1(\Omega)) \cap L^2([0,S], H^2(\Omega))\,.$$

In other words, $(v, \widetilde{T})$ is a strong solution. Since the strong solution must be a weak solution and, by Theorem 3.1, there is only one weak solution, the strong solution is unique. □

5 Global Attractor

In previous sections we proved the existence, uniqueness, and well-posedness of the weak and strong solution of the system (1.1)–(1.14). In this section we show the existence of the global attractor. Moreover, we give an upper bound for its Hausdorff dimension. To establish this, we first present sharper estimates for various norms of the solution, which are asymptotically uniform in time. Namely, we demonstrate the existence of absorbing balls for the dynamical system introduced by the solution operator of the system (1.1)–(1.14).

Denote by $S(t)T_0 = T(t)$ the solution operator of the system (1.1)–(1.14) with initial data T_0. Under the conditions of Theorem 3.1 and Theorem 4.1, one can show that

$$T(t) = S(t)T_0 \in L^2(\Omega) \quad \text{for all } T_0 \in L^2(\Omega),\ t \ge 0\,,$$

and

$$T(t) = S(t)T_0 \in H^1(\Omega) \quad \text{for all } T_0 \in H^1(\Omega),\ t \ge 0\,.$$

THEOREM 5.1 *Suppose that $\tau \in H_0^1(M)$, $Q \in L^2(\Omega)$, and $T^* \in H^2(M)$. Then there is a global compact attractor $\mathcal{A} \subset L^2(\Omega)$ for the system* (1.1)–(1.14); *moreover, $\mathcal{A}$ has finite Hausdorff dimension.*

PROOF: First, let us show that there is an absorbing ball in $L^2(\Omega)$ and $H^1(\Omega)$. Let T be the solution of the system (1.1)–(1.14) with initial data $T_0 \in L^2(\Omega)$. In other words, $\widetilde{T} = T - T^*$ is the solution of the system (2.7)–(2.12) with initial data $\widetilde{T}_0 = T_0 - T^* \in L^2(\Omega)$. By taking the H' dual action to equation (2.9) with $\widetilde{T}$, we obtain

$$\begin{aligned}\left\langle \partial_t \widetilde{T} + L_2^o \widetilde{T}, \widetilde{T}\right\rangle + \left\langle v \cdot \nabla \widetilde{T}, \widetilde{T}\right\rangle \\ + \left\langle -\left(\nabla \cdot \int_{-h}^{z} v(x,y,\xi,t)\,d\xi\right)\partial_z \widetilde{T} + v \cdot \nabla T^*, \widetilde{T}\right\rangle = \langle Q^*, \widetilde{T}\rangle\,.\end{aligned}$$

Since $\partial_t \widetilde{T} \in L^2([0,S], H')$, we apply Lions' lemma [38, lemma 1.2, p. 260] and reach $\langle \partial_t \widetilde{T}, \widetilde{T} \rangle = \frac{1}{2}(d|\widetilde{T}|^2/dt)$. By (2.23), we have

$$\begin{aligned} &\frac{1}{2}\frac{d|\widetilde{T}|^2}{dt} + \|\widetilde{T}\|^2 = \\ &\int_\Omega Q^* \widetilde{T}\, dx\, dy\, dz - \int_\Omega \Big[v \cdot \nabla \widetilde{T} - \Big(\nabla \cdot \int_{-h}^z v(x,y,\xi,t) d\xi \Big) \partial_z \widetilde{T} \\ &\qquad + v \cdot \nabla T^* \Big] \widetilde{T}\, dx\, dy\, dz\,. \end{aligned} \tag{5.1}$$

By taking $\psi = T^*$ in the weak formulation (2.35), we get

$$\begin{aligned} &\int_\Omega \widetilde{T}(t) T^*\, dx\, dy\, dz + \int_{t_0}^t \Big[\int_\Omega K_h \nabla \widetilde{T} \cdot \nabla T^*\, dx\, dy\, dz + \alpha \int_{\Gamma_u} \widetilde{T}\, T^*\, dx\, dy \Big] \\ &\quad + \int_{t_0}^t \int_\Omega \Big[v \cdot \nabla \widetilde{T} - \Big(\nabla \cdot \int_{-h}^z v(x,y,\xi,t) d\xi \Big) \partial_z \widetilde{T} + v \cdot \nabla T^* \Big] T^*\, dx\, dy\, dz \\ &\quad = \int_\Omega \widetilde{T}(t_0) T^*\, dx\, dy\, dz + \int_{t_0}^t \int_\Omega Q^* T^*\, dx\, dy\, dz\,. \end{aligned}$$

It is equivalent to

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \widetilde{T}(t) T^*\, dx\, dy\, dz + \int_\Omega K_h \nabla \widetilde{T} \cdot \nabla T^*\, dx\, dy\, dz + \alpha \int_{\Gamma_u} \widetilde{T}\, T^*\, dx\, dy \\ &\quad + \int_\Omega \Big[v \cdot \nabla \widetilde{T} - \Big(\nabla \cdot \int_{-h}^z v(x,y,\xi,t) d\xi \Big) \partial_z \widetilde{T} + v \cdot \nabla T^* \Big] T^*\, dx\, dy\, dz \\ &\quad = \int_\Omega Q^* T^*\, dx\, dy\, dz\,. \end{aligned} \tag{5.2}$$

By adding (5.1) and (5.2), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(|\widetilde{T}|^2 + 2\int_\Omega \widetilde{T}(t) T^*\, dx\, dy\, dz \Big) + \|\widetilde{T}\|^2 + \alpha \int_{\Gamma_u} \widetilde{T} T^*\, dx\, dy \\ &\quad + \int_\Omega \Big[v \cdot \nabla \widetilde{T} - \Big(\nabla \cdot \int_{-h}^z v(x,y,\xi,t) d\xi \Big) \partial_z \widetilde{T} + v \cdot \nabla T^* \Big] (\widetilde{T} + T^*) dx\, dy\, dz \\ &\quad + \int_\Omega K_h \nabla \widetilde{T} \cdot \nabla T^*\, dx\, dy\, dz \\ &\quad = \int_\Omega Q^* (\widetilde{T} + T^*) dx\, dy\, dz\,. \end{aligned}$$

Notice that the following equalities hold:

$$|\widetilde{T}|^2 + \int\limits_\Omega 2\widetilde{T}T^*\,dx\,dy\,dz = |T|^2 - \int\limits_\Omega |T^*|^2\,dx\,dy\,dz\,,$$

$$\|\widetilde{T}\|^2 + \int\limits_\Omega K_h\nabla\widetilde{T}\cdot\nabla T^*\,dx\,dy\,dz + \alpha\int\limits_{\Gamma_u}\widetilde{T}T^*\,dx\,dy =$$
$$\|T\|^2 - \|T^*\|^2 - \int\limits_\Omega K_h\nabla\widetilde{T}\cdot\nabla T^*\,dx\,dy\,dz - \alpha\int\limits_{\Gamma_u}\widetilde{T}T^*\,dx\,dy\,,$$

$$\int\limits_\Omega\left[v\cdot\nabla\widetilde{T} - \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T} + v\cdot\nabla T^*\right](\widetilde{T}+T^*)\,dx\,dy\,dz =$$
$$\int\limits_\Omega\left[v\cdot\nabla T - \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z T\right]T\,dx\,dy\,dz\,,$$

$$\int\limits_\Omega Q^*\big(\widetilde{T}+T^*\big)dx\,dy\,dz = \int\limits_\Omega\big[QT + K_hT\Delta T^*\big]dx\,dy\,dz\,.$$

Therefore, we get

$$\frac{1}{2}\frac{d|T|^2}{dt} + \|T\|^2 + \int\limits_\Omega\left[v\cdot\nabla T - \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z T\right]T\,dx\,dy\,dz$$
$$= \|T^*\|^2 + \int\limits_\Omega K_h\nabla\widetilde{T}\cdot\nabla T^*\,dx\,dy\,dz + \alpha\int\limits_{\Gamma_u}\widetilde{T}T^*\,dx\,dy$$
$$+ \int\limits_\Omega\big[QT + K_hT\Delta T^*\big]dx\,dy\,dz\,.$$

By integration by parts and (1.3), we obtain

$$\int\limits_\Omega\left[v\cdot\nabla T - \left(\nabla\cdot\int_{-h}^z v(x,y,\xi,t)d\xi\right)\partial_z T\right]dx\,dy\,dz = 0\,. \tag{5.3}$$

Since $\partial_z T^* = 0$, by integration by parts and (1.13), we get

$$\Bigg|\|T^*\|^2 + \int\limits_\Omega K_h\nabla\widetilde{T}\cdot\nabla T^*\,dx\,dy\,dz + \alpha\int\limits_{\Gamma_u}\widetilde{T}T^*\,dx\,dy$$
$$+ \int\limits_\Omega K_hT\Delta T^*\,dx\,dy\,dz\Bigg| = \alpha\Bigg|\int\limits_{\Gamma_u}TT^*\,dx\,dy\Bigg| \le \alpha^{1/2}\|T^*\|_{L^2(M)}\|T\|\,.$$

By applying the Cauchy-Schwarz inequality to the above estimate, we reach

$$\begin{aligned}&\Bigg| \|T^*\|^2 + \int_\Omega K_h \nabla \widetilde{T} \cdot \nabla T^* \, dx\, dy\, dz \\ &+ \alpha \int_{\Gamma_u} \widetilde{T} T^* \, dx\, dy + \int_\Omega K_h T \Delta T^* \, dx\, dy\, dz \Bigg| \\ &\le \frac{\alpha}{2} \|T^*\|^2_{L^2(M)} + \frac{1}{2} \|T\|^2 .\end{aligned} \tag{5.4}$$

Therefore, by estimates (5.3) and (5.4), we obtain

$$\frac{d|T|^2}{dt} + \|T\|^2 \le \alpha \|T^*\|^2_{L^2(M)} + 2|Q||T| .$$

By the Cauchy-Schwarz inequality and the Poincaré inequality (2.16), we have

$$\frac{d|T|^2}{dt} + \frac{1}{2}\|T\|^2 \le \alpha \|T^*\|^2_{L^2(M)} + 2\widetilde{K}|Q|^2 , \tag{5.5}$$

where $\widetilde{K}$ is as in (2.17). Thus, again by the Poincaré inequality (2.16), we obtain

$$\frac{d|T|^2}{dt} + \frac{1}{2\widetilde{K}}|T|^2 \le \alpha \|T^*\|^2_{L^2(M)} + 2\widetilde{K}|Q|^2 .$$

By the Gronwall lemma, we get

$$|T(t)|^2 \le |T_0|^2 e^{-\frac{1}{2\widetilde{K}}t} + 2\alpha\widetilde{K}\|T^*\|^2_{L^2(M)} + 4\widetilde{K}^2|Q|^2.$$

As a result of the above, when t is large enough such that

$$|T_0|^2 e^{-\frac{1}{\widetilde{K}}t} \le 2\alpha\widetilde{K}\|T^*\|^2_{L^2(M)} + 4\widetilde{K}^2|Q|^2 ,$$

we have

$$|T(t)|^2 \le \widetilde{R}_a(T^*, Q) =: 4\alpha\widetilde{K}\|T^*\|^2_{L^2(M)} + 8\widetilde{K}^2|Q|^2 ; \tag{5.6}$$

in particular,

$$\limsup_{t\to\infty} |T(t)|^2 \le 2\alpha\widetilde{K}\|T^*\|^2_{L^2(M)} + 4\widetilde{K}^2|Q|^2 .$$

In other words, when t is large enough, we have

$$|\widetilde{T}(t)|^2 \le R_a(T^*, Q) =: 2\widetilde{R}_a(T^*, Q) + 2\|T^*\|^2_{L^2(M)} , \tag{5.7}$$

where $\widetilde{R}_a(T^*, Q)$ is as in (5.6). Therefore, there is an absorbing ball in $L^2(\Omega)$ with radius $\widetilde{R}_a(T^*, Q)$ for system (1.1)–(1.14) and with radius $R_a(T^*, Q)$ for system (2.8)–(2.12), respectively.

Next, we show that there is an absorbing ball in $H^1(\Omega)$. First, notice that from (5.5), we have

$$\int_t^{t+r} \|T(s)\|^2 ds \le 2|T(t)| + \left[4\alpha\widetilde{K}\|T^*\|^2_{L^2(M)} + 8\widetilde{K}^2|Q|^2\right] r .$$

Therefore, by (5.7), when t is large enough, we get

$$\int_t^{t+r} \|T(s)\|^2\, ds \le K_r(r, Q, T^*)\,, \tag{5.8}$$

where

$$K_r(r, Q, T^*) = 2R_a(T^*, Q) + \big[4\alpha \widetilde{K} \|T^*\|^2_{L^2(M)} + 8\widetilde{K}^2 |Q|^2\big] r \tag{5.9}$$

and $R_a(T^*, Q)$ is as in (5.7).

From the proof of Theorem 4.1, we recall the inequality (4.5)

$$\begin{aligned}
&\frac{d\|\widetilde{T}\|^2}{dt} + \big|L_2^o \widetilde{T}\big|^2 \\
&\quad \le C\big[1 + \|T^*\|^4_{H^2(\Omega)} + |Q|^2 + \|\tau\|^4_{H^1(M)} + |\widetilde{T}|^4\big] \\
&\qquad + C\big(\|T^*\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + |\widetilde{T}|^4 + |\widetilde{T}|^2 \|\widetilde{T}\|^2\big) \|\widetilde{T}\|^2\,.
\end{aligned}$$

From the above, (2.24), and (5.7), we obtain for t large enough

$$\begin{aligned}
&\frac{d\|\widetilde{T}\|^2}{dt} + \frac{\|\widetilde{T}\|^2}{\lambda_1} \\
&\quad \le C\big[1 + \|T^*\|^4_{H^2(\Omega)} + |Q|^2 + \|\tau\|^4_{H^1(M)} + R_a^4(T^*, Q)\big] \\
&\qquad + C\big(\|T^*\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + R_a^4(T^*, Q) + R_a^2(T^*, Q)\|\widetilde{T}\|^2\big) \|\widetilde{T}\|^2\,,
\end{aligned}$$

where λ_1 is the first eigenfunction of operator L_2^o. Using the uniform Gronwall inequality (cf. [40, p. 89]), we obtain, when t is large enough,

$$\|\widetilde{T}(t)\| \le R_v(r, T^*, Q, \tau) \tag{5.10}$$

where $r > 0$ is fixed and

$$\begin{aligned}
&R_v(r, T^*, Q, \tau) \\
&\quad = C\bigg[\frac{R_a(T^*, Q)}{r^{1/2}} + \|T^*\|_{H^1(M)} + |Q| \\
&\qquad\quad + \frac{C}{\lambda_1^{1/2}}\big(1 + \|T^*\|^2_{H^2(\Omega)} + |Q| + \|\tau\|^2_{H^1(M)} + R_a^2(T^*, Q)\big)\bigg] \\
&\qquad\quad \times e^{C\left[(R_a(T^*,Q))^4 + \left(\|T^*\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + (R_a(T^*,Q))^4\right) r\right]}\,.
\end{aligned} \tag{5.11}$$

Therefore, we have shown that there is an absorbing ball $\mathcal{B}$ in $H^1(\Omega)$ with radius $R_v(r, T^*, Q, \tau)$. From the proofs of Theorems 3.1 and 4.1, we conclude that the operator $S(t)$ is a compact operator. Following the standard procedure (cf. [7, 8, 11, 24, 40] for details), one can prove that there is a global attractor

$$\mathcal{A} = \bigcap_{t>0} S(t)\mathcal{B} \subset H^1(\Omega)\,.$$

Moreover, $\mathcal{A}$ is compact in $L^2(\Omega)$.

In addition to the compactness of the semigroup $S(t)$, one can show its differentiability on $\mathcal{A}$ with respect to the initial data. Therefore, one can use the trace formula (cf. [7, 8, 40]) to get an upper bound for the dimension of the global attractor $\mathcal{A}$.

Let $(p_s, v, \widetilde{T})$ be a given solution of the system (2.7)–(2.12) with $\widetilde{T} \in \mathcal{A}$. Since $\widetilde{T}$ is on the attractor $\mathcal{A}$, $(p_s, v, \widetilde{T})$ is a strong solution to the system (2.7)–(2.12). It is clear that the first variation equations of the system (2.7)–(2.12) around $(p_s, v, \widetilde{T})$ read as follows:

$$\nabla\left[q_s(x,y,t) - \int_{-h}^{z} \chi(x,y,\xi,t)\,d\xi\right] + f\vec{k}\times u + \epsilon L_1 u = 0\,, \tag{5.12}$$

$$\nabla\cdot\int_{-h}^{0} u(x,y,z,t)\,dz = 0\,, \tag{5.13}$$

$$\partial_t\chi = F'(\widetilde{T})\chi\,, \tag{5.14}$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0\,,\quad \left.\frac{\partial u}{\partial z}\right|_{z=-h} = 0\,,\quad \left. u\cdot\vec{n}\right|_{\Gamma_s} = 0\,,\quad \left.\frac{\partial u}{\partial \vec{n}}\times\vec{n}\right|_{\Gamma_s} = 0\,, \tag{5.15}$$

$$\left.\left(\partial_z\chi + \frac{\alpha}{K_v}\chi\right)\right|_{z=0} = 0\,,\quad \left.\partial_z\chi\right|_{z=-h} = 0\,,\quad \left.\partial_n\chi\right|_{\partial M} = 0\,, \tag{5.16}$$

$$\chi(x,y,z,0) = \zeta\,, \tag{5.17}$$

where q_s, u, and χ are the unknown perturbations about p_s, v, and $\widetilde{T}$, respectively, with a given initial perturbation $\zeta \in L^2(\Omega)$. Moreover, here

$$\begin{aligned} F'(\widetilde{T})\chi = -\Bigg[& L_2^o\chi + u\cdot\nabla(\widetilde{T}+T^*) + v\cdot\nabla\chi - \left(\nabla\cdot\int_{-h}^{z} u(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T} \\ & - \left(\nabla\cdot\int_{-h}^{z} v(x,y,\xi,t)\,d\xi\right)\partial_z\chi\Bigg]. \end{aligned}$$

It is not difficult to show that the above coupled second-order elliptic and linear parabolic system has a unique solution ($q_s(t)$, $u(t)$, and $\chi(t)$). Moreover, by using techniques similar to the ones developed in [21, 44], one can show that this solution satisfies, for $t > 0$,

$$\chi(t) \in H^1(\Omega)\,,\quad u(t)\in H^2(\Omega)\,,\quad q_s(t)\in H^1(\Omega)\,.$$

For any positive integer m we consider the volume element

$$\left|\chi_1(t)\wedge\chi_2(t)\wedge\cdots\wedge\chi_m(t)\right|_{\wedge^m L^2(\Omega)}\,.$$

We have the following trace formula (cf. [7, 8, 40]):

$$\frac{1}{2}\frac{d}{dt}\Big|\chi_1(t)\wedge\chi_2(t)\wedge\cdots\wedge\chi_m(t)\Big|^2_{\wedge^m L^2(\Omega)} = \\ \operatorname{Tr}\big(\widetilde{P}_m(t)\circ F'(\widetilde{T}(t))\circ\widetilde{P}_m(t)\big)\big|\chi_1(t)\wedge\chi_2(t)\wedge\cdots\wedge\chi_m(t)\big|^2_{\wedge^m L^2(\Omega)},$$

which gives

$$\tag{5.18} \big|\chi_1(t)\wedge\chi_2(t)\wedge\cdots\wedge\chi_m(t)\big|^2_{\wedge^m L^2(\Omega)} = \\ \big|\zeta_1\wedge\zeta_2\wedge\cdots\wedge\zeta_m\big|^2_{\wedge^m L^2(\Omega)}\exp\left(\int_0^t \operatorname{Tr}\big(\widetilde{P}_m(s)\circ F'(\widetilde{T}(s))\circ\widetilde{P}_m(s)\big)ds\right),$$

where $\chi_1(s), \chi_2(s), \dots, \chi_m(s)$ are the solutions of (5.12)–(5.17) corresponding to the initial data $\zeta_1, \zeta_2, \dots, \zeta_m$, respectively. The trace of the linear operator $(\widetilde{P}_m(s)\circ F'(\widetilde{T})\circ\widetilde{P}_m(s))$ is $\operatorname{Tr}(\widetilde{P}_m(s)\circ F'(\widetilde{T}(s))\circ\widetilde{P}_m(s))$, and the $L^2(\Omega)$ orthogonal projector onto the space spanned by $\{\chi_1(s), \chi_2(s), \dots, \chi_m(s)\}$ is $\widetilde{P}_m(s)$. Thanks to (5.18), we conclude that $\{\chi_1(s), \chi_2(s), \dots, \chi_m(s)\}$ are linearly independent for every $s \geq 0$ if and only if $\{\zeta_1, \zeta_2, \dots, \zeta_m\}$ are linearly independent. Hence, from now on we assume that $\{\zeta_1, \zeta_2, \dots, \zeta_m\}$ are linearly independent.

Let $\{\psi_1(s), \psi_2(s), \dots, \psi_m(s)\}$ be an $L^2(\Omega)$ orthonormal basis of the space spanned by $\{\chi_1(s), \chi_2(s), \dots, \chi_m(s)\}$. Notice that $\{\psi_1(s), \psi_2(s), \dots, \psi_m(s)\}$ are in $H^1(\Omega)$ for $s > 0$. Thus we have

$$\operatorname{Tr}\big(\widetilde{P}_m(s)\circ F'(\widetilde{T}(s))\circ\widetilde{P}_m(s)\big) = \sum_{j=1}^m \big(F'(\widetilde{T}(s))\psi_j(s), \psi_j(s)\big).$$

Notice that

$$\big(F'(\widetilde{T}(s))\psi_j(s), \psi_j(s)\big) = -\|\psi_j(s\|^2 + \int_\Omega \big[u_j\cdot\nabla(\widetilde{T}+T^*)\big]\psi_j(s)dx\,dy\,dz \\ - \int_\Omega \left[\left(\nabla\cdot\int_{-h}^z u_j(x,y,\xi,s)d\xi\right)\partial_z\widetilde{T}\right]\psi_j(s)dx\,dy\,dz,$$

where, for $j = 1, 2, \dots, m$, $u_j(x, y, z, s)$ is the solution of the following linear system:

$$\nabla\left[(q_s)_j(x,y,s) - \int_{-h}^z \psi_j(x,y,\xi,s)d\xi\right] + f\vec{k}\times u_j + \epsilon L_1 u_j = 0,$$

$$\nabla\cdot\int_{-h}^0 u_j(x,y,z,s)dz = 0,$$

$$\left.\frac{\partial u_j}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial u_j}{\partial z}\right|_{z=-h} = 0, \quad \left.u_j\cdot\vec{n}\right|_{\Gamma_s} = 0, \quad \left.\frac{\partial u_j}{\partial\vec{n}}\times\vec{n}\right|_{\Gamma_s} = 0.$$

Here $(q_s)_j$ and u_j are the unknowns, while ψ_j is given and fixed. Following the same steps that led to the estimates (3.40) and (3.41), we have

$$\left|\int_\Omega u_j\cdot\nabla(\widetilde{T}+T^*)\psi_j(s)\right| \le \frac{C}{\epsilon\widetilde{A}}\big[\|\widetilde{T}\|+h^{1/2}\|T^*\|_{H^1(M)}\big]|\psi_j(s)|^{3/2}\|\psi_j(s)\|^{1/2}$$
$$\le \frac{C}{\epsilon\widetilde{A}\lambda_1^{1/4}}\big[\|\widetilde{T}\|+h^{1/2}\|T^*\|_{H^1(M)}\big]|\psi_j(s)|\|\psi_j(s)\|$$

and

$$\left|\left(\nabla\cdot\int_{-h}^z u_j(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}\psi_j(s)\right| \le \frac{C}{\epsilon\widetilde{A}\lambda_1^{1/4}}\|\widetilde{T}\||\psi_j(s)|\|\psi_j(s)\|\,.$$

Here we also use (2.24). Recall that $|\psi_j|=1$ for $j=1,2,\dots,m$. Thus,

$$\left|\int_\Omega\left[u_j\cdot\nabla(\widetilde{T}+T^*)-\left(\nabla\cdot\int_{-h}^z u_j(x,y,\xi,s)d\xi\right)\partial_z\widetilde{T}\right]\psi_j(s)dx\,dy\,dz\right| \le$$
$$\frac{C}{\epsilon\widetilde{A}\lambda_1^{1/4}}\big[\|\widetilde{T}(s)\|+h^{1/2}\|T^*\|_{H^1(M)}\big]\|\psi_j(s)\|\,.$$

By using the Cauchy-Schwarz inequality and the above estimate, we have

$$\big(F'(\widetilde{T}(s))\psi_j(s),\psi_j(s)\big) \le -\frac{1}{2}\|\psi_j(s\|^2+\frac{C}{\epsilon^2\widetilde{A}^2\lambda_1^{1/2}}\big[\|\widetilde{T}(s)\|^2+h\|T^*\|^2_{H^1(M)}\big]\,.$$

By (2.25), we have

$$\sum_{j=1}^m\|\psi_j(s\|^2 \ge \lambda_1+\lambda_2+\cdots+\lambda_m \ge C\lambda_1 m^{5/3}\,.$$

As a result, we obtain

$$\mathrm{Tr}\left(\widetilde{P}_m(s)\circ F'(\widetilde{T}(s))\circ\widetilde{P}_m(s)\right) \le$$
$$-C\lambda_1 m^{5/3}+\frac{C}{\epsilon^2\widetilde{A}^2\lambda_1^{1/2}}\big[\|\widetilde{T}(s)\|^2+h\|T^*\|^2_{H^1(\Omega)}\big];$$

hence,

$$\frac{1}{t}\int_0^t\mathrm{Tr}\left(\widetilde{P}_m(s)\circ F'(\widetilde{T}(s))\circ\widetilde{P}_m(s)\right)ds \le$$
$$-C\lambda_1 m^{5/3}+\frac{C}{\epsilon^2\widetilde{A}^2\lambda_1^{1/2}}\frac{1}{t}\int_0^t\big[\|\widetilde{T}(s)\|^2+h\|T^*\|^2_{H^1(\Omega)}\big]ds\,.$$

Therefore, by applying (3.2), we get

$$\limsup_{t\to\infty}\sup_{|\widetilde{T}_0|\le R_a(T^*,Q)}\sup_{\substack{\zeta_j\in L^2(\Omega)\\|\zeta_j|\le 1\\j=1,2,\dots,m}}\frac{1}{t}\int_0^t\mathrm{Tr}\left(\widetilde{P}_m(s)\circ F'(\widetilde{T}(s))\right)\circ\widetilde{P}_m(s)ds \le$$
$$-C\lambda_1 m^{5/3}+K_5(\epsilon,\widetilde{A},\widetilde{K},T^*,Q)\,,$$

where

$$K_5(\epsilon, \widetilde{A}, \widetilde{K}, T^*, Q) = \frac{C}{\epsilon^2 \widetilde{A}^2 \lambda_1^{1/2}} \left[h \|T^*\|^2_{H^1(M)} + |Q|^2 \right]. \tag{5.19}$$

In order to guarantee $-C\lambda_1 m^{5/3} + K_5(\epsilon, \widetilde{A}, \widetilde{K}, T^*, Q) \le 0$, we need to choose m large enough such that

$$m > C\left(\frac{K_5(\epsilon, \widetilde{A}, \widetilde{K}, T^*, Q)}{\lambda_1} \right)^{3/5}.$$

Therefore, the Hausdorff and fractal dimensions of the attractor $\mathcal{A}$ can be estimated by (cf. [11])

$$d_H(\mathcal{A}) \le d_F(\mathcal{A}) \le C\left(\frac{K_5(\epsilon, \widetilde{A}, \widetilde{K}, T^*, Q)}{\lambda_1} \right)^{3/5}.$$

□

Remark. The bound for the dimension of the global attractor presented here is not sharp. We stress that our focus here is rather on the existence of a finite-dimensional global attractor. As far as we know, there are no educational heuristic physical arguments for the number of degrees of freedom for models of the type studied here. Therefore, trying to make our estimate for the dimension of the global attractor sharper without having a target bound is not necessarily the most efficient thing to do at the moment. Furthermore, one can follow the usual procedure (see, e.g., [6, 14, 15, 16, 22] and the references therein) to show that the system (1.1)–(1.14) has a finite number of determining modes and nodes and determining functionals and projections. All this indicates that the system has a finite number of asymptotic degrees of freedom. Whether this system possesses a global invariant inertial manifold [16] remains a challenging open problem.

Appendix: Proof of Proposition 2.2

Let $u = (u_1, u_2)$ be a smooth vector field, and let f and g be smooth scalar functions. Then

$$\left| \int_\Omega \left(\nabla \cdot \int_{-h}^{z} u(x, y, \xi, t) d\xi \right) f g \, dx \, dy \, dz \right| \le \int_M \left(\int_{-h}^{0} |\nabla u| dz \right) \left(\int_{-h}^{0} |fg| dz \right) dx \, dy .$$

Using the Cauchy-Schwarz inequality, we obtain

$$\int_{-h}^{0} |fg| dz \le \left(\int_{-h}^{0} |f|^2 dz \right)^{1/2} \left(\int_{-h}^{0} |g|^2 \, dz \right)^{1/2};$$

thus,

$$\left|\int_\Omega \left(\nabla\cdot\int_{-h}^z u(x,y,\xi,t)d\xi\right) fg\,dx\,dy\,dz\right| \le$$
$$\int_M \left(\int_{-h}^0 |\nabla u| dz\right)\left(\int_{-h}^0 |f|^2\,dz\right)^{1/2} \times \left(\int_{-h}^0 |g|^2\,dz\right)^{1/2} dx\,dy\,.$$

Applying the Hölder inequality, we reach

$$\left|\int_\Omega \left(\nabla\cdot\int_{-h}^z u(x,y,\xi,t)d\xi\right) fg\,dx\,dy\,dz\right| \le$$
$$\left(\int_M \int_{-h}^0 |f|^2\,dz\,dx\,dy\right)^{1/2}$$
$$\times \left[\int_M \left(\int_{-h}^0 |g|^2\,dz\right)^2 dx\,dy\right]^{1/4} \left[\int_M \left(\int_{-h}^0 |\nabla u| dz\right)^4 dx\,dy\right]^{1/4}.$$

By using the Minkowski inequality (2.20), we get

$$\left[\int_M \left(\int_{-h}^0 |g|^2\,dz\right)^2 dx\,dy\right]^{1/2} \le \int_{-h}^0 \left(\int_M |g|^4\,dx\,dy\right)^{1/2} dz\,.$$

Thanks to (2.18), for every fixed z we have

$$\left(\int_M |g(x,y,z,t)|^4\,dx\,dy\right)^{1/4} \le C_4 \|g(\cdot,\cdot,z,t)\|_{L^2(M)}^{1/2} \|g(\cdot,\cdot,z,t)\|_{H^1(M)}^{1/2}\,.$$

As a result of the above and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_{-h}^0 \left(\int_M |g(x,y,z,t)|^4\,dx\,dy\right)^{1/2} dz \\
&\le C \int_{-h}^0 \|g(\cdot,\cdot,z,t)\|_{L^2(M)} \|g(\cdot,\cdot,z,t)\|_{H^1(M)}\,dz \\
&\le C\left(\int_{-h}^0 \|g(\cdot,\cdot,z,t)\|_{L^2(M)}^2\,dz\right)^{1/2} \left(\int_{-h}^0 \|g(\cdot,\cdot,z,t)\|_{H^1(M)}\,dz\right)^{1/2} \\
&\le C\|g\|_{H^1(\Omega)}|g|\,;
\end{aligned}$$

therefore,

$$\left[\int_M \left(\int_{-h}^0 |g(x,y,z,t)|^2\,dz\right)^2 dx\,dy\right]^{1/4} \le C\|g\|_{H^1(\Omega)}^{1/2}|g|^{1/2}. \tag{A.1}$$

By using (2.18) we have

$$\left[\int_M \left(\int_{-h}^0 |\nabla u(x,y,z)|dz\right)^4 dx\,dy\right]^{1/4} \le$$
$$C_4 \left\| \int_{-h}^0 |\nabla u(\cdot,\cdot,z)|d\xi \right\|_{L^2(M)}^{1/2} \left\| \int_{-h}^0 |\nabla u(\cdot,\cdot,z)|dz \right\|_{H^1(M)}^{1/2}.$$

Notice that

$$\begin{aligned}\left\| \int_{-h}^0 |\nabla u(\cdot,\cdot,z)|d\xi \right\|_{L^2(M)}^{1/2} &= \left[\int_M \left(\int_{-h}^0 |\nabla u|dz\right)^2 dx\,dy\right]^{1/4} \\ &\le h^{1/4}\left[\int_M \int_{-h}^0 |\nabla u(x,y,z)|^2\,dz\,dx\,dy\right]^{1/4} \\ &= h^{1/4}\|u\|_{H^1(\Omega)}^{1/2}.\end{aligned}$$

On the other hand,

$$\left[\int_M \left|\nabla \int_{-h}^0 |\nabla u(x,y,z)|\,dz\right|^2 dx\,dy\right]^{1/4} \le$$
$$\left[\int_M \left(\int_{-h}^0 |\nabla(\nabla u(x,y,z))|dz\right)^2 dx\,dy\right]^{1/4}.$$

Again, by using the Minkowski inequality (2.20), we get

$$\begin{aligned}&\left[\int_M \left(\int_{-h}^0 \left|\nabla(\nabla u(x,y,z))\right|dz\right)^2 dx\,dy\right]^{1/2} \\ &\le \int_{-h}^0 \left(\int_M \left|\nabla(\nabla u(x,y,z))\right|^2 dx\,dy\right)^{1/2} dz \\ &\le h^{1/2}\left(\int_{-h}^0 \int_M \left|\nabla(\nabla u(x,y,z))\right|^2 dx\,dy\,dz\right)^{1/2} \\ &\le C\|u\|_{H^2(\Omega)};\end{aligned}$$

thus,

$$\left[\int_M \left(\int_{-h}^0 |\nabla u(x,y,z)|dz\right)^4 dx\,dy\right]^{1/4} \le C\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{1/2}. \tag{A.2}$$

As a result of (A.1) and (A.2), we have

$$\text{(A.3)}\quad \left|\iint_{\Omega}\left(\nabla\cdot\int_{-h}^{z}u(x,y,\xi,t)d\xi\right)fg\,dx\,dy\,dz\right|\leq C|f|\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{1/2}\|g\|_{H^1(\Omega)}^{1/2}|g|^{1/2}.$$

Acknowledgments. E.S.T. thanks the Courant Institute of Mathematical Sciences, New York University, and Princeton University, where part of this work was completed, for their kind hospitality. This work was supported in part by National Science Foundation grants DMS-9706964 and DMS-9704632 and by the Department of Energy under contract W-7405-ENG-36.

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Received October 2001.
Revised May 2002.