Numerical Criterion for the Stabilization of Steady States of the Navier-Stokes Equations

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Dedicated to Professors Ciprian Foias and Roger Temam

ABSTRACT. This paper introduces an explicit numerical criterion for the stabilization of steady state solutions of the Navier-Stokes equations (NSE) with linear feedback control. Given a linear feedback controller that stabilizes a steady state solution to the closed-loop standard Galerkin (or nonlinear Galerkin) NSE discretization, it is shown that, if the number of modes involved in the computation is large enough, this controller stabilizes a nearby steady state of the closed-loop NSE. We provide an explicit estimate, in terms of the physical parameters, for the number of modes required in order for this criterion to hold. Moreover, we provide an estimate for the distance between the stabilized numerical steady state and the actually stabilized steady state of the closed-loop Navier-Stokes equations. More accurate approximation procedures, based on the concept of postprocessing the Galerkin results, are also presented. All the criterion conditions are imposed on the *computed numerical solution*, and no *a priori* knowledge is required about the steady state solution of the full PDE. This criterion holds for a large class of unbounded linear feedback operators and can be slightly modified to include certain nonlinear parabolic systems such as reaction-diffusion systems.

1. Introduction

There is, recently, a reinforced interest in the important subject of flow control (see for example the monograph [36] or the proceedings of the recent NSF workshop on Control of Flows [56]). This is certainly due to an array of emerging technologies, such as the microfabrication of minute, programmable (and cheap) sensors and actuators. Part of the excitement is certainly due to cross-fertilization at the intersection between physical fluid mechanics and nonlinear partial differential equations. Indeed, the last few years have seen extensive applications of systems theory approaches and methods to the control and stabilization of flow models (see for example [1, 9, 13, 14, 41, 55]).

For physical problems modeled by partial differential equations (as fluid flow is governed by the Navier-Stokes Equations (NSE),) the *state* in the corresponding control problem is infinite dimensional. Practical considerations (finite precision, memory capacity of computers that would implement control action) dictate the discretization of the distributed parameter system (DPS) to a finite dimensional dynamical system ([3, 22, 68]). How well this (lumped) discretized system captures the dynamics of the original PDE is the subject of ongoing research (see, e.g., [42, 49, 50, 66, 74] and references therein). The *dissipative* nature of many evolutionary PDEs forms the basis of arguments for the qualitative validity of such approximations.

Using existing control methodologies, it is conceptually straightforward to synthesize finite-dimensional controllers for the finite dynamical systems resulting from such (finite difference, finite element, spectral) discretizations. A definitive sequence of articles by Balas, Curtain, Sakawa and others over the last 20 years ([3, 4, 5, 6, 7, 22, 68]) states that such finite dimensional controllers can be used to stabilize *linear* infinite dimensional systems from which the discretization originates. The controllers should of course be based on a sufficiently converged discretization of the PDE (see [5] and also later [7]). One of our goals in this article is to prove that such finite dimensional controllers can indeed stabilize unstable steady states of the *closed-loop* NSE, and provide estimates of the necessary discretization dimension.

High order and ill-conditioning, however, make the routine design of controllers for such problems a difficult issue on its own ([71]); recourse to nontrivial computational methods is required in order to assess basic properties of the linearized state space models, such as controllability, stabilizability, observability and stability ([10, 25]). A second level of model reduction then becomes necessary: after the reduction of the infinite-dimensional system to a ("large") finite dimensional one, we seek to exploit the dissipativity of the original PDE to construct (or approximate) accurate, dynamic, "small" finite dimensional models that can be used in controller design. Such "further reduced" models are often based on modal representations of the dynamics, the modes coming from the leading part of the linearized problem ([8]), from their Krylov subspace approximations ([44]), from empirically determined eigenfunctions ([43], and references therein), or from an

appropriate (dissipative) part of the linear problem operator, such as the eigen-functions of the Stokes operator in the case of the Navier-Stokes equations ([17], [73], [74]). Beyond the qualitative similarity with singular perturbation methods that construct invariant manifolds and exploit them in controller design for finite dimensional systems ([57]), there is an extensive interest in the use of inertial manifolds and approximate inertial manifolds for closed loop dynamics analysis and controller design (see, for example, [12, 15, 69, 70]). Motivated by the theory of Inertial Manifolds and Approximate Inertial Manifolds, we will also explore a second (Nonlinear Galerkin, NLG) model reduction step.

In this paper we demonstrate our results and tools on a specific system, the Navier-Stokes equations of viscous incompressible fluids. We use the NSE as an illustrative paradigm. However, our results and technique hold for a larger class of *nonlinear dissipative evolution* systems of equations.

The Navier-Stokes equations

$$(1.1) \qquad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = g, \qquad\qquad \text{in } \Omega,$$

$$(1.2) \qquad \nabla \cdot u = 0, \qquad\qquad \text{in } \Omega,$$

$$(1.3) \qquad u = 0, \qquad\qquad \text{on } \partial\Omega,$$

$$(1.4) \qquad u(x,0) = u_{\text{in}}(x)$$

are the equations that govern the motion of viscous incompressible flows in a domain Ω, of $\mathbb{R}^2$ or $\mathbb{R}^3$, subject to the no-slip Dirichlet boundary conditions. Here, u represents the unknown velocity vector, and p is the unknown pressure scalar; where $\nu > 0$, the constant kinematic viscosity, and g, the body forcing term, are given.

It is well known that in both the two-dimensional and three-dimensional cases the system (1.1)-(1.4) has a steady state solution ([17], [37], [72], [73]). More-over, for small Reynolds numbers (for example, large viscosity or small forcing) the system has a unique steady state solution which is globally stable ([54], [72]). As the Reynolds number increases, one expects that this steady state loses its sta-bility and goes through a bifurcation (however, this scenario is not always true, see, e.g., [21], [62]). Moreover, this system could go through successive bifur-cation, and exhibit chaotic and turbulent behavior (see, e.g., [59], [73]). Such turbulent dynamics is not restricted to transient solutions, but it can sustain itself for infinitely long intervals of time. To study rigorously this long-time turbulent dynamics one has to make sure that the solutions of (1.1)-(1.4) remain regular globally in time. This is known to be true in the two-dimensional case. How-ever, in the case of three-dimensional flows this is still one of the most challenging open scientific problems. As a result, the study of the long-time behavior of the solutions and the global attractor for the system (1.1)-(1.4) has been restricted to the two-dimensional case. Indeed, the first such rigorous study was presented by Foias and Prodi in [29], where the authors have shown that the global dynamics

is "determined" by finitely many degrees of freedom. Since then lots of work concerning this issue has been done, and we refer the reader to, for example, [16], [19], [21], [32], [33], [51] and the references therein, for the best available results in this regard.

In this paper we are interested in developing a numerical criterion for the stabilization of steady state solutions of system (1.1)-(1.4), using linear feedback control operators. The results and techniques presented here are inspired by [20] and [75].

Our main result states that if we can design a linear feedback controller that stabilizes a steady state solution to the closed-loop m−state Galerkin approximation or m−state nonlinear Galerkin approximation, where m is bigger than some M_{GLK} (in the Galerkin case) or some M_{NLG} (in the nonlinear Galerkin case), then there is a nearby steady state solution to the closed-loop Navier-Stokes system with the same linear feedback controller. Our objective echoes the approach in [55]: we are concerned with "an entirely new *stable* system" (the closed loop system). We provide explicit bounds on M_{GLK} and M_{NLG} in terms of the physical parameters of the equations. Moreover, we show that in this case the steady state of the nonlinear Galerkin converges to the exact solutions much faster than the one of the standard Galerkin.

It is worth mentioning that all the conditions of this criterion are checkable by the computed numerical solutions, and no other conditions are imposed on the unknown exact solution. Furthermore, let us stress the fact that in general there are no error estimates for nonlinear elliptic steady state problem. This is due to the fact that such equations might have multiple solutions and even a continuum of solutions ([30], [35]). This is obvious, for instance, in the case of non-constant steady state solutions of equations with translational symmetry. However, we do not encounter this problem here because the feedback control operator isolates the steady state under consideration and makes the use of standard versions of the implicit function theorem at hand. Here we will present only the two-dimensional NSE case. Similar results are also valid in the three dimensional case, but with different dependence on the physical parameters.

This paper is organized as follows: In Section 2 on the facing page we review preliminaries and functional spaces. In Section 3 on page 44 we give a criterion for the Galerkin approximation case. In Section 4 on page 60 we provide a similar criterion for the nonlinear Galerkin approximation. In Section 5 on page 83 we show that we can use postprocessing Galerkin and nonlinear Galerkin methods to improve the accuracy. In Section 6 on page 88 we provide an algorithm for applying our numerical criterion.

This paper is dedicated to Professor Ciprian Foias on the occasion of his retirement from Indiana University, and to Professor Roger Temam on the occasion of his 60th birthday. We would like to thank them both for their continuous support and encouragement through all over our careers. Their work is a constant source of inspiration for us.

2. Preliminaries and Functional Setting

As we mentioned earlier, we consider here only the two dimensional Navier-Stokes case. Therefore we suppose that Ω is a bounded subset of $\mathbb{R}^2$ with smooth boundary. Let $H^q(\Omega)$ denote the usual Sobolev spaces of index q (cf. [2]). We will denote by $|\cdot|$ the L^2- norm, and $(\cdot, \cdot)$ the L^2- inner product. We set

$$\mathcal{V} = \{\varphi \in (C_0^\infty(\Omega))^2 : \nabla \cdot \varphi = 0\}.$$

The spaces H, V will denote the closures of $\mathcal{V}$ in L^2 and H^1 respectively. We introduce the inner product

$$((u, v)) = \sum_{i=1}^{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx, \quad \text{for } u, v \in V,$$

and denote by $\|\cdot\|$ the norm induced by the inner product $((\cdot, \cdot))$. Thanks to the Poincaré inequality, the norm $\|\cdot\|$ is equivalent to the H^1-norm on V. Let $P : L^2 \to H$ be the orthogonal projection, and let $A = -P\Delta$ be the Stokes operator, subject to the homogeneous Dirichlet, no-slip, boundary condition. It is well known that A^{-1} is a self-adjoint positive compact operator from H onto $\mathcal{D}(A) = H^2 \cap V$ (cf. [17], [37], [58], [73]). Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of A, repeated according to their multiplicities, and let $\{\omega_k\}_{k=1}^\infty$ be the corresponding eigenfunctions. We denote by H_m the linear space spanned by $\{\omega_1, \ldots, \omega_m\}$, by $P_m : H \to H_m$ the orthogonal projection, and by $Q_m = I - P_m$. Moreover, we observe that $V' = \mathcal{D}(A^{-1/2})$ is the dual space of V.

It is well known that the Navier-Stokes equations (NSE) (1.1)-(1.4) are equivalent to the functional differential equation (see, e.g., [17], [72])

(2.1) $$\frac{du}{dt} + \nu Au + B(u, u) = g,$$

(2.2) $$u(0) = u_{\text{in}},$$

where $B(u, u) = P((u \cdot \nabla)u)$, the nonlinear (bilinear) term.

The following are well-known facts (cf. [17], [37], [58], [73])

(2.3) $$\|v\| \geq \lambda_1^{1/2} |v|, \quad \forall v \in V',$$

(2.4) $$|Av| \geq \lambda_1^{1/2} \|v\|, \quad \forall v \in \mathcal{D}(A).$$

There is a constant $C_0 > 0$ such that (cf. [64])

(2.5) $$\frac{m}{C_0} \leq \frac{\lambda_m}{\lambda_1} \leq C_0 m.$$

(2.6) $(B(u,v),w) = -(B(u,v),w); \; (B(u,v),v) = 0, \quad \forall u,v,w \in V,$

(2.7) $|(B(u,v),w)| \leq C_1 |u|^{1/2} \|u\|^{1/2} \|v\| \, |w|^{1/2} \|w\|^{1/2}, \quad \forall u,v,w \in V,$

(2.8) $|(B(u,v),w)| \leq C_1 |u|^{1/2} |Au|^{1/2} \|v\| \, |w|,$
$$\forall u \in \mathcal{D}(A), \forall v \in V, \forall w \in H,$$

(2.9) $|(B(u,v),w)| \leq C_1 |u| \, \|v\| \, |w|^{1/2} |Aw|^{1/2},$
$$\forall u \in H, \; \forall v \in V, \; \forall w \in \mathcal{D}(A),$$

(2.10) $|(B(u,v),w)| \leq C_1 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w|,$
$$\forall u \in V, \; \forall v \in \mathcal{D}(A), \forall w \in H,$$

(2.11) $|(B(u,v),w)| \leq C_1 \big(\|u\|^{1/2} |Au|^{1/2} \|v\|^{1/2} |Av|^{1/2}$
$$\times + |u|^{1/2} |Au|^{1/2} |Av| \big) |A^{-1/2} w|,$$
$$\forall u,v \in \mathcal{D}(A), \; \forall w \in \mathcal{D}(A^{-1/2}).$$

Next we recall the following two-dimensional version of the Lieb-Thirring inequality (cf. [60], see also [74]), for the two dimensional case.

Proposition 2.1. (Lieb-Thirring inequality) *Let $\{\varphi_j\}_{j=1}^N \subset V$ be a subset of orthogonal functions in H with $\|\varphi_j(x)\|_{L^2} = v$, for $j = 1, 2, \ldots, N$. Then there is an absolute constant C_2, which is independent of N, such that*

(2.12) $$\int_\Omega \Big[\sum_{j=1}^N |\varphi_j(x)|^2 \Big]^2 \, dx \leq C_2 v^2 \sum_{j=1}^N \|\varphi_j(x)\|^2,$$

where $|\varphi_j(x)|$ denotes the absolute value of $\varphi_j(x)$.

The Galerkin approximation to the Navier-Stokes equations (NSE) (2.1) and (2.2), which is based on the first $m-$eigenmodes of the Stokes operator A, is given by the system of ordinary differential equations

(2.13) $$\frac{du_m}{dt} + vAu_m + P_m B(u_m, u_m) = P_m g,$$

(2.14) $$u_m(0) = P_m u_{\text{in}},$$

where $u_m(t) \in H_m = \text{Span}\{\omega_1, \ldots, \omega_m\}$.

Suppose that the equations (2.13) and (2.14) have a stationary solution u_m^*, which is unstable. Let $L_m : H_m \to H_m$ be a linear feedback stabilizing operator. That is, u_m^* is a stable steady state solution to the closed-loop

(2.15) $$\frac{du_m}{dt} + vAu_m + P_m B(u_m, u_m) + L_m(u_m - u_m^*) = P_m g.$$

The above equation can be viewed as the m−state Galerkin approximation to the following closed-loop Navier-Stokes system

$$(2.16) \qquad \frac{du}{dt} + \nu Au + B(u, u) + L_m(u - u_m^*) = g.$$

As a result, the n−state Galerkin approximation associated with (2.16), for $n \geq m$, is given by

$$(2.17) \qquad \frac{du_n}{dt} + \nu Au_n + P_n B(u_n, u_n) + P_n L_m(u_n - u_m^*) = P_n g.$$

The question is: provided that m is large enough, is L_m stabilizing a steady state solution u_n^* of (2.17) which is nearby u_m^*? In this chapter we will give an affirmative answer to this question, provided L_m belongs to a certain class of linear operators. Moreover, we will provide an explicit estimate on how large m should be in order for the answer to be affirmative.

Motivated by the above discussion we will consider, by analogy with (2.16), the more general closed-loop NSE

$$(2.18) \qquad \frac{du}{dt} + \nu Au + B(u, u) + \gamma Lu = f,$$
$$(2.19) \qquad u(0) = u_{\text{in}},$$

where $\gamma \geq 0$, is a tracking parameter, and L is a linear operator which satisfies the following properties:

(i) There is a fixed $\alpha \in [0, 1)$ such that $L : \mathcal{D}(A^\alpha) \to H$.
(ii) There is a dimensionless constant $C_L > 0$ such that for any $\beta \in [0, 1]$ we have

$$(2.20) \qquad |A^{\beta-\alpha}Lu| \leq \nu \lambda_1^{1-\alpha} C_L^{1-\alpha}|A^\beta u|, \quad \forall u \in \mathcal{D}(A^\beta).$$

In particular, when $\beta = \max\{0, \alpha - \frac{1}{2}\}$ in (2.20) we have

$$(2.21) \qquad |(Lu, u)| \leq \nu \lambda_1^{1-\alpha} C_L^{1-\alpha}|u|^{2-2\alpha}\|u\|^{2\alpha}, \quad \forall u \in V.$$

Notice that for $\gamma = 0$, the equation (2.18) is the NSE (1.1), and that f contains the term $L_m u_m^*$.

The feedback linear operator L is designed to stabilize the steady state of the Galerkin approximation. Therefore, roughly speaking, it stabilizes the lower modes. However, when it is introduced into the full PDE it could destabilize (excite) the higher modes (a phenomenon known as spillover). The above technical conditions on the operator L simply mean that the viscous linear term νA in the closed-loop system is stronger than γL, and hence will "suppress" this potential spillover phenomenon.

3. A stabilization criterion for the standard Galerkin approximation case

The Galerkin approximation equation of the closed-loop NSE based on m-eigenmodes of the Stokes operator is (cf. [17], [37], [58], [73])

$$(3.1) \qquad \frac{du_m}{dt} + \nu Au_m + P_m B(u_m, u_m) + \gamma P_m L u_m = P_m f,$$

$$(3.2) \qquad u_m(0) = P_m u_{\text{in}},$$

where $u_m \in H_m$.

Following the notation in [20] and [75] we denote, for $1 \leq m \leq \infty$, by u_m^∞ the stationary solution of the equations (3.1), and by $u^\infty \equiv u_\infty^\infty$ the steady state of the equations (2.18). Furthermore, we also denote by $\mathcal{A}_m(u_0) : H_m \to H_m$ the linearized operator about $u_0 \in \mathcal{D}(A)$, defined as

$$(3.3) \quad \mathcal{A}_m(u_0)v = \nu Av + P_m(B(u_0, v) + B(v, u_0)) + \gamma P_m L v, \quad \forall v \in H_m.$$

When $m = \infty$, we will denote by $\mathcal{A} = \mathcal{A}_\infty$. We will denote $\text{Sp}(\mathcal{A}_m(u_0))$ the spectrum of $\mathcal{A}_m(u_0)$; $B(E, \rho) = \{z \in \mathbb{C} : \text{dist}(z, E) < \rho\}$, for any subset $E \subset \mathbb{C}$; and by $(E)_\sigma = \{z \in E : |\Re z| \leq \sigma\}$. Notice that our definition of $(E)_\sigma$ is different than the one in [20] and [75].

In the following Lemma we follow closely the papers [20] and [75]. However, we adjust the details to take into consideration the effect of the feedback operator L.

Lemma 3.1. *Let* $1 \leq m \leq \infty$, *let* $u_m^\infty \in V$ *be a steady state solution of the equation* (3.1).

(i) *If*

$$(3.4) \qquad\qquad\qquad |u_m^\infty| \leq K_1,$$

then

$$(3.5) \qquad \|u_m^\infty\| \leq K_2 = \left(\frac{\nu \lambda_1 G K_1}{1 - \alpha} \right)^{1/2} + \gamma^{1/[2(1-\alpha)]} C_L^{1/2} \lambda_1^{1/2} K_1,$$

$$(3.6) \qquad |Au_m^\infty| \leq \lambda_1^{1/2} K_3 = \frac{8\nu \lambda_1 G}{1 - \alpha} + \frac{32^{1/2} C_1^2 K_1 K_2^2}{\nu^2 (1 - \alpha)^2} + 8^{1/2} \gamma^{1/(1-\alpha)} C_L \lambda_1 K_1,$$

where $G = |f|/(\lambda_1 \nu^2)$, *the Grashoff number.*

(ii) *If* $\gamma C_L^{1/2} < 1$, *then*

$$(3.7) \qquad \|u_m^\infty\| \leq \bar{K}_2 = \frac{|f|}{\nu \lambda_1^{1/2}(1 - \gamma C_L^{1/2})},$$

$$(3.8) \quad |Au_m^\infty| \leq \lambda_1^{1/2} \bar{K}_3 = \frac{8\nu \lambda_1 G}{1 - \alpha} + \frac{32^{1/2} C_1^2 \bar{K}_2^3}{\nu^2 \lambda_1^{1/2}(1 - \alpha)^2} + 8^{1/2} \gamma^{1/(1-\alpha)} C_L \lambda_1^{1/2} \bar{K}_2.$$

Notice in particular that $\gamma = 0$, i.e., the open-loop case falls into this situation.

Proof. Let u_m^∞ be a stationary solution of the equation (3.1) with $|u_m^\infty| \le K_1$. Taking the inner product of the equation (3.1) with u_m^∞ in H_m, and using (2.6), we get

$$\nu \|u_m^\infty\|^2 \le |f|\,|u_m^\infty| + \gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |u_m^\infty|^{2-2\alpha} \|u_m^\infty\|^{2\alpha}.$$

By Young's inequality, we obtain

$$\|u_m^\infty\|^2 \le \frac{|f|K_1}{\nu(1-\alpha)} + \gamma^{1/(1-\alpha)} C_L \lambda_1 K_1^2.$$

Thus we have proved the estimate (3.5). Taking the inner product of equation (3.1) with Au_m^∞ in H_m gives us

$$\nu |Au_m^\infty|^2 \le |f|\,|Au_m^\infty| + |(B(u_m^\infty, u_m^\infty), Au_m^\infty)| + \gamma |(Lu_m^\infty, Au_m^\infty)|.$$

By using the Cauchy-Schwarz inequality, and (2.8) and (2.20) with $\beta = \alpha$, we reach

$$\nu |Au_m^\infty|^2 \le |f|\,|Au_m^\infty| + C_1 |u_m^\infty|^{1/2} \|u_m^\infty\|\,|Au_m^\infty|^{3/2}$$
$$+ \gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |u_m^\infty|^{1-\alpha} |Au_m^\infty|^{1+\alpha}.$$

Thanks to Young's inequality, we get

$$|Au_m^\infty| \le \frac{8|f|}{\nu(1-\alpha)} + \frac{32^{1/2} C_1^2 K_1 K_2^2}{\nu^2 (1-\alpha)^2} + 8^{1/2} \gamma^{1/(1-\alpha)} C_L \lambda_1 K_1,$$

which completes the proof part (i) of the lemma. The proof of the part (ii) of the lemma is similar to the one in the case of $\gamma = 0$, i.e., the NSE. We refer to [17], [73] for details. $\square$

Remark. The condition $\gamma C_L^{1/2} < 1$ in part (ii) of the above lemma means that the linear instabilities (about zero) that might be caused by the linear feedback operator, L, are weak and can be suppressed by the viscous term A.

Lemma 3.2. *Let $R > 0$, $u_0 \in \mathcal{D}(A)$ with $|Au_0| \le \lambda_1^{1/2} R$. If w is an eigenfunction of $\mathcal{A}_m(u_0)$, then $w \in \mathcal{D}(A)$. Moreover, if $|w| = v$ and $\mathcal{A}_m(u_0)w = \Lambda w$ for some $\Lambda \in \mathbb{C}$, then we have*

(3.9) $$\|w\| \le K_4(\gamma, \Lambda, \nu, R),$$

(3.10) $$|Aw| \le K_5(\gamma, \Lambda, \nu, R),$$

where

$$(3.11) \quad K_4(\gamma, \Lambda, \nu, R) = \left(\frac{2\nu|\Lambda|}{1 - \alpha} + 2\gamma^{1/(1-\alpha)} \nu^2 \lambda_1 C_L \right)^{1/2} + \frac{C_1 R}{1 - \alpha},$$

and

$$(3.12) \quad K_5(\gamma, \Lambda, \nu, R) = \frac{|\Lambda|}{1 - \alpha} + \frac{C_1 R K_4(\gamma, \Lambda, \nu, R)}{\nu(1 - \alpha)}$$

$$+ \frac{C_1 \lambda_1^{1/4} R K_4^{1/2}(\gamma, \Lambda, \nu, R)}{1 - \alpha} + \gamma^{1/(1-\alpha)} \nu C_L \lambda_1.$$

Proof. Let $|w| = \nu$ and

$$(3.13) \qquad\qquad\qquad \mathcal{A}_m(u_0) w = \Lambda w.$$

Taking the inner product of the above equation with w in H_m, and using (2.6) leads to

$$\nu \|w\|^2 \le |\Lambda| \, |w|^2 + |(B(w, u_0), w)| + \gamma |(Lw, w)|.$$

The inequalities (2.7) and (2.21) give us

$$\nu \|w\|^2 \le |\Lambda| \, |w|^2 + C_1 \|u_0\| \, |w| \, \|w\| + \gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |w|^{2-2\alpha} \|w\|^{2\alpha}.$$

By Young's inequality and the Cauchy-Schwarz inequality, we get

$$\|w\|^2 \le \frac{2\nu|\Lambda|}{1 - \alpha} + \frac{C_1^2 R^2}{(1 - \alpha)^2} + 2\gamma^{1/(1-\alpha)} \nu^2 \lambda_1 C_L,$$

which leads to (3.9). Now by taking the L^2−norm in the equation (3.13), we reach

$$\nu |Aw| \le |\Lambda| \, |w| + |B(u_0, w)| + |B(w, u_0)| + \gamma |Lw|$$

By using (2.8), (2.10), and (2.20) with $\beta = \alpha$, we get

$$\nu |Aw| \le |\Lambda| \, |w| + C_1 |u_0|^{1/2} |Au_0|^{1/2} \|w\| + C_1 |w|^{1/2} \|w\|^{1/2} \|u_0\|^{1/2} |Au_0|^{1/2}$$
$$+ \gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |w|^{1-\alpha} |Aw|^{\alpha}.$$

Following Young's inequality, we conclude that

$$|Aw| \le \frac{|\Lambda|}{1 - \alpha} + \frac{C_1 R K_4(\gamma, \Lambda, \nu, R)}{\nu(1 - \alpha)} + \frac{C_1 \lambda_1^{1/4} R K_4^{1/2}(\gamma, \Lambda, \nu, R)}{1 - \alpha}$$
$$+ \gamma^{1/(1-\alpha)} \nu C_L \lambda_1. \qquad \square$$

Lemma 3.3. *Let $R > 0$, $u_0 \in V$ with $\|u_0\| \le R$ and $s \in \mathbb{R}$, and let $M_1(s)$ be the number of the eigenvalues of $\mathcal{A}_m(u_0)$ whose real parts are less than or equal to s, i.e.,*

$$M_1(s) = \# \text{ of } \{\lambda \in \text{Sp}(\mathcal{A}_m(u_0)) : \mathfrak{R}\lambda < s\}.$$

Then we have

(3.14) $M_1(s) \le K_6(\gamma, s, \nu, R),$

where

$$(3.15) \quad K_6(\gamma, s, \nu, R) = \frac{1}{2} \left\{ \left(\frac{4sC_0}{\nu \lambda_1 (1 - \alpha)} - 1 \right) + \left[\left(\frac{4sC_0}{\nu \lambda_1 (1 - \alpha)} - 1 \right)^2 \right. \right.$$

$$\left. \left. + \frac{8C_0 C_2 R^2}{\nu^2 \lambda_1 (1 - \alpha)^2} + 16\gamma^{1/(1-\alpha)} C_0 C_L \right]^{1/2} \right\}.$$

Proof. Let $m < \infty$ be finite, and let $\Lambda_1, \ldots, \Lambda_K$ be the eigenvalues of $\mathcal{A}_m(u_0)$, repeated according to their multiplicities, such that $\mathfrak{R}\Lambda_j \le s$, for $j = 1, 2, \ldots, K$. We will denote by E_K the linear space spanned by the generalized eigenfunctions of $\mathcal{A}_m(u_0)$ corresponding to the eigenvalues $\Lambda_1, \ldots, \Lambda_K$. Let $\psi_1, \ldots, \psi_K$ be an orthogonal basis of E_K with $|\psi_j| = \nu$ for $j = 1, 2, \ldots, K$, and let $\Pi_K : H_m \to E_K$ be the orthogonal projection. Then we get

$$\mathfrak{R} \left(\sum_{j=1}^{K} \Lambda_j \right) = \mathfrak{R} \{ \text{trace}(\Pi_K \mathcal{A}_m(u_0) \Pi_K) \} = \nu^{-2} \mathfrak{R} \left(\sum_{j=1}^{K} (\mathcal{A}_m(u_0) \psi_j, \psi_j) \right).$$

By using (2.6) and (2.21), we reach

$$\mathfrak{R} \left(\sum_{j=1}^{K} \Lambda_j \right)$$

$$= \nu^{-2} \mathfrak{R} \left\{ \sum_{j=1}^{K} [\nu (A\psi_j, \psi_j) + (B(\psi_j, u_0), \psi_j) + \gamma (L\psi_j, \psi_j)] \right\}$$

$$\ge \nu^{-2} \sum_{j=1}^{K} \left[\nu \|\psi_j\|^2 - |(B(\psi_j, u_0), \psi_j)| - \gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |\psi_j|^{2-2\alpha} \|\psi_j\|^{2\alpha} \right].$$

By using the Cauchy-Schwarz and Lieb-Thirring (2.12) inequalities, we obtain

$$\left| \sum_{j=1}^{K} (B(\psi_j, u_0), \psi_j) \right| \le \left(\int_\Omega \left(\sum_{j=1}^{K} \psi_j^2(x) \right)^2 dx \right)^{1/2} \|u_0\|$$

$$\le \left(C_2 \nu^2 \sum_{j=1}^{K} \|\psi_j\|^2 \right)^{1/2} \|u_0\|$$

$$\le \frac{\nu(1 - \alpha)}{2} \left(\sum_{j=1}^{K} \|\psi_j\|^2 \right) + \frac{C_2 \nu \|u_0\|^2}{2(1 - \alpha)}.$$

On the other hand, and by Young's inequality, we have

$$\gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |\psi_j|^{2-2\alpha} \|\psi_j\|^{2\alpha} \leq \nu \alpha \|\psi_j\|^2 + \nu(1-\alpha)\gamma^{1/(1-\alpha)} C_L \lambda_1 |\psi_j|^2,$$

that is

$$\gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha} |\psi_j|^{2-2\alpha} \|\psi_j\|^{2\alpha} \leq \nu \alpha \|\psi_j\|^2 + \nu^3(1-\alpha)\gamma^{1/(1-\alpha)} C_L \lambda_1.$$

Moreover,

$$\sum_{j=1} K\|\psi_j\|^2 = \sum_{j=1} K(A\psi_j, \psi_j) \geq \nu^2 \sum_{j=1} K\lambda_j.$$

Therefore,

$$Ks \geq \Re\left(\sum_{j=1} K\Lambda_j\right) \geq \frac{\nu(1-\alpha)}{2} \sum_{j=1} K\lambda_j - \frac{C_2\|u_0\|^2}{2\nu(1-\alpha)}$$
$$- \sum_{j=1} K\nu(1-\alpha)\gamma^{1/(1-\alpha)} C_L \lambda_1.$$

By using (2.5), we get

$$Ks \geq \frac{\nu \lambda_1(1-\alpha)}{2C_0} \frac{K(K+1)}{2} - \frac{C_2\|u_0\|^2}{2\nu(1-\alpha)} - \nu(1-\alpha)\gamma^{1/(1-\alpha)} C_L \lambda_1 K,$$

which implies (3.14). □

Remark. In a sense, the estimate (3.14) is sharp because the constant $K_6(\gamma, s, \nu, R)$ is growing linearly in s. For example, in the case when $u_0 = 0$ and $\gamma = 0$ we have $\mathcal{A}(u_0) = \nu A$, and in this case the $\Lambda_j = \lambda_j$ behaves like j asymptotically (see (2.5)). Thus, in this special case, $M_1(s) \sim s$.

3.1. Estimates for the resolvents in the Galerkin case. For every $R > 0$ we define with Prodi [67] the parabola-like set $\sigma(R) \subset \mathbb{C}$ by setting

$$(3.16) \quad \sigma(R) = \{z \in \mathbb{C} : \exists x \geq \lambda_1, |z - \nu x| \leq C_1 R x^{1/2} + \gamma \nu \lambda_1^{1-\alpha} C_L^{1-\alpha} x^{\alpha}\},$$

where C_1 and C_L are as in (2.7) and (2.21) respectively. It is worth mentioning that when one deals with three dimensional Navier-Stokes equations, the set $\sigma(R)$ is bigger, but it has the same properties as far as the spectra of the operator $\mathcal{A}_m(u_0)$, for $\|u_0\| \leq R$ and $1 \leq m \leq \infty$, is concerned.

As in [20] and [75], one can show that the spectrum of $\mathcal{A}_m(u_0)$ consists of isolated eigenvalues of finite multiplicities. The following lemmas are similar to those of [20] and [75], but we add the contribution of the linear control operator L. The proofs will be omitted here, and can be found for the case where $L = 0$ in [20].

Lemma 3.4. *Let $R > 0$, $u_0 \in V$ with $\|u_0\| \le R$. Then*
- *the spectra of $\mathcal{A}_m(u_0)$, $m = 1, \ldots, \infty$, are in $\sigma(R)$,*
- *for every $z \notin \sigma(R)$ the following inequalities hold:*

(3.17) $|(\mathcal{A}_m(u_0) - z)^{-1}v| \le \operatorname{dist}(z, \sigma(R))^{-1}|v| = K_H(\gamma, \nu, R, z)|v|,$

(3.18) $\|(\mathcal{A}_m(u_0) - z)^{-1}v\| \le K_V(\gamma, \nu, R, z)|v|, \quad \forall v \in H_m,$

(3.19) $|A(\mathcal{A}_m(u_0) - z)^{-1}v| \le K_A(\gamma, \nu, R, z)|v|, \quad \forall v \in H_m,$

where

(3.20) $K_V(\gamma, \nu, R, z) = \left[\left(2\lambda_1 \gamma^{1/(1-\alpha)} + \dfrac{C_1^2 R^2}{(1-\alpha)^2 \nu^2} \right. \right.$

$$\left. + \frac{2|z|}{(1-\alpha)\nu} \right) K_H^2(\gamma, \nu, R, z) + \frac{2K_H(\gamma, \nu, R, z)}{(1-\alpha)\nu} \Bigg]^{1/2},$$

(3.21) $K_A(\gamma, \nu, R, z) = K_H(\gamma, \nu, R, z) \left[\dfrac{4}{\nu(1-\alpha)} + \left(\dfrac{8|z|}{(1-\alpha)\nu} \right. \right.$

$$\left. + \frac{16 C_1^2 R^2}{\nu^2 (1-\alpha)^2} + 4\gamma^{2/(1-\alpha)} C_L^2 \lambda_1^2 \right)^{1/2} K_V(\gamma, \nu, R, z) \Bigg].$$

Following [75], we will use Carleman's inequality for the Hilbert-Schmidt (H-S) operators to prove this section's main result, Lemma 3.6 on the next page. First let us recall some well known facts about H-S operators (cf. [24])

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space, $F : \mathcal{H} \to \mathcal{H}$ be an H-S operator with non-zero eigenvalues $\mu_1, \mu_2, \ldots$, repeated according to their multiplicities. Then*

(i) *the following infinite product*

$$\varphi_\lambda(F) = \prod_{k \ge 1} \left(1 - \frac{\mu_k}{\lambda} \right) e^{\mu_k/\lambda}$$

converges and defines an analytic function for every $\lambda \ne 0$, $\lambda \notin \operatorname{Sp}(F)$;

(ii) *(Carleman's inequality) the following inequality*

$$\|\varphi_\lambda(F)\lambda I - F)^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \le |\lambda| e^{1/2[1+(\|F\|_2/|\lambda|)^2]}$$

holds for every $\lambda \ne 0$, $\lambda \notin \operatorname{Sp}(F)$, where $\|F\|_2$ denotes the H-S norm;

(iii) *for every $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$, TF and FT are H-S operators. Moreover,*

$$\max\{\|FT\|_2, \|TF\|_2\} \le \|F\|_2 \|T\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}.$$

We recall the following Lemma, which was presented in [75]. We refer to [75] for details of the proof.

Lemma 3.5. *Let $R > 0$, $u_0 \in V$ with $\|u_0\| \leq R$. Then for every $z \notin \sigma(R)$ the operators $(\mathcal{A}_m(u_0) - z)^{-1}$, for $m = 1, \ldots, \infty$, are H-S operators. Moreover,*

$$\tag{3.22} \|(\mathcal{A}_m(u_0) - z)^{-1}\|_2 \leq K_{HS}(\gamma, \nu, R, z),$$

where

$$\tag{3.23} K_{HS}(\gamma, \nu, R, z) = \frac{C_0 \pi}{\sqrt{6}\lambda_1} K_A(\gamma, \nu, R, z),$$

where $K_A(\gamma, \nu, R, z)$ is as in (3.21).

Lemma 3.6. *Let $R > 0$, $u_0 \in \mathcal{D}(A)$, with $|Au_0| \leq \lambda_1^{1/2}R$ and ρ, $s \geq 0$, and let $\lambda \in \mathbb{C}$ and satisfy $|\lambda| \leq s$. Let*

$$\tag{3.24} \lambda_c = -\frac{C_1^2 R^2}{4\nu(1 - \alpha)} - \gamma^{1/(1-\alpha)}\nu(1 - \alpha)C_L\lambda_1.$$

Then for every $m < \infty$, if

$$\tag{3.25} \text{dist}(\lambda, \text{Sp}(\mathcal{A}_m(u_0))) \geq \rho,$$

then we have

$$\tag{3.26} |(\mathcal{A}_m(u_0) - \lambda)^{-1}v| \leq K_7(\gamma, \rho, s, \nu, R)|v|, \quad \forall v \in H_m,$$
$$\tag{3.27} \|(\mathcal{A}_m(u_0) - \lambda)^{-1}v\| \leq K_8(\gamma, \rho, s, \nu, R)|v|, \quad \forall v \in H_m,$$
$$\tag{3.28} |A(\mathcal{A}_m(u_0) - \lambda)^{-1}v| \leq K_9(\gamma, \rho, s, \nu, R)|v|, \quad \forall v \in H_m,$$

where

(3.29) $K_7(\gamma, \rho, s, \nu, R)$

$$= \begin{cases} K_H(\gamma, R, \nu, f, z), & \lambda \notin \sigma(R - \lambda_c/2) \\ \exp\left\{\frac{1}{2} + \frac{5}{2}|s - 2\lambda_c|^2 K_{HS}^2(\gamma, \nu, R, |s - 2\lambda_c|)\right\} \\ \quad \times \frac{1}{\lambda_c|\lambda - 2\lambda_c|^2}\left(\frac{2|s - \lambda_c|}{\rho}\right)^{\widetilde{M}}, & \lambda \in \sigma(R - \lambda_c/2), \end{cases}$$

(3.30) $K_8(\gamma, \rho, s, \nu, R)$

$$= \left[\left(2\lambda_1 \gamma^{1/(1-\alpha)} + \frac{C_1^2 R^2}{(1-\alpha)^2 \nu^2} + + \frac{2s}{(1-\alpha)\nu} \right) K_7^2(\gamma, \nu, R, z) \right.$$
$$\left. + \frac{2K_7(\gamma, \nu, R, z)}{(1-\alpha)\nu} \right]^{1/2},$$

(3.31) $K_9(\gamma, \rho, s, \nu, R)$

$$= \left[\left(\frac{8s}{(1-\alpha)\nu} + \frac{16C_1^2 R^2}{\nu^2(1-\alpha)^2} + 4\gamma^{2/(1-\alpha)} C_L^2 \lambda_1^2 \right)^{1/2} K_8(\gamma, \nu, R, z) \right.$$
$$\left. + \frac{4}{\nu(1-\alpha)} \right] K_7(\gamma, \nu, R, z),$$

where

$$\widetilde{M} = K_6 \left(\gamma, 2\lambda_c + \frac{2}{|\lambda_c|}, \nu, R \right),$$

λ_c *is defined earlier in* (3.24), *and* K_{HS} *is as in* (3.23).

   Proof. By applying Lemma 3.4 on page 49, we only consider the case of $\lambda \in \sigma(R - \lambda_c/2)$. One can check that $2\lambda_c \notin \sigma(R)$. Thus, by Lemma 3.5, $F_m = (\mathcal{A}_m(u_0) - 2\lambda_c)^{-1}$ is an H-S operator. On the other hand, we can write

$$(\mathcal{A}_m(u_0) - \lambda)^{-1} = (\lambda - 2\lambda_c)^{-1}(\mathcal{A}_m(u_0) - 2\lambda_c)^{-1}$$
$$\times ((\lambda - 2\lambda_c)^{-1}I - (\mathcal{A}_m(u_0) - 2\lambda_c)^{-1})^{-1}.$$

Notice that $(\lambda - 2\lambda_c)^{-1} \notin \mathrm{Sp}(F_m)$. We apply Lemma 3.4 on page 49 and the Carleman inequality to get

$$\|(\mathcal{A}_m(u_0) - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}_m, \mathcal{H}_m)}$$
$$\leq \frac{1}{\lambda_c|\lambda - 2\lambda_c|^2} |\varphi_{\lambda - 2\lambda_c}(F_m)|^{-1} e^{1/2[1 + \|F_m\|_2^2(|\lambda - 2\lambda_c|^2]}.$$

Now we apply Lemma 3.5 on the facing page and the estimate for $|\varphi_{\lambda - 2\lambda_c}(F_m)|$ in [75] (see [75] for details), and we have

$$|(\mathcal{A}_m(u_0) - \lambda)^{-1}v| \leq K_7(\gamma, \rho, s, \nu, R)|v|,$$

where $K_7(\gamma, \rho, s, \nu, R)$ is as in (3.29).
   Now we show the estimates (3.27) and (3.28). For each $g \in H_m$, let $(\mathcal{A}_m(u_0) - \lambda)^{-1}g = v$, i.e.,

(3.32) $\nu A\nu + P_m(B(u_0, \nu) + B(\nu, u_0) + \gamma P_m L\nu) - \lambda\nu = g.$

Taking the inner product of equation (3.32) with ν in H, we obtain

$$\nu\|\nu\|^2 \le |\lambda|\,|\nu|^2 + |(g, \nu)| + |(B(\nu, u_0), \nu)| + \gamma|(L\nu, \nu)|.$$

By using (2.8) and (2.21), we reach

$$\nu\|\nu\|^2 \le s|\nu|^2| + |g|\,|\nu| + C_1\|u_0\|\,|\nu|\,\|\nu\| + \gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}|\nu|^{2-2\alpha}\|\nu\|^{2\alpha}.$$

Thanks to Young's inequality and the estimate (3.26), we get

$$\frac{\nu(1-\alpha)}{2}\|\nu\|^2 \le s|\nu|^2 + |g|\,|\nu| + \frac{C_1^2\|u_0\|^2|\nu|^2}{2\nu(1-\alpha)} + \gamma^{1/(1-\alpha)}\nu(1-\alpha)C_L\lambda_1|\nu|^2$$

$$\le \left[\left(s + \frac{C_1^2 R^2}{2\nu(1-\alpha)} + \gamma^{1/(1-\alpha)}\nu C_L\lambda_1\right)K_7^2(\gamma, \rho, s, \nu, R)\right.$$

$$\left. + K_7(\gamma, \rho, s, \nu, R)\right]|g|^2,$$

which implies (3.27). Moreover, taking the inner product of (3.32) with $A\nu$ in H gives us

$$\nu|A\nu|^2 \le |\lambda|\,\|\nu\|^2 + |(g, A\nu)| + |(B(\nu, u_0), A\nu)|$$

$$+ |(B(u_0, \nu), A\nu)| + \gamma|(L\nu, A\nu)|.$$

By using (2.9), (2.10), and (2.20) with $\beta = \alpha$, we get

$$\nu|A\nu|^2 \le s\|\nu\|^2 + |g|\,|A\nu| + C_1\|u_0\|^{1/2}|Au_0|^{1/2}|\nu|^{1/2}\|\nu\|^{1/2}|A\nu|$$

$$+ C_1|u_0|^{1/2}|Au_0|^{1/2}\|\nu\|\,|A\nu| + \gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}|\nu|^{1-\alpha}|A\nu|^{1+\alpha}.$$

By Young's inequality again, and the estimates (3.26) and (3.27), we reach

$$\frac{\nu(1-\alpha)}{8}|A\nu|^2 \le s\|\nu\|^2 + \frac{2|g|^2}{\nu(1-\alpha)} + \frac{2C_1^2\|u_0\|\,|Au_0|\,|\nu|\,\|\nu\|}{\nu(1-\alpha)}$$

$$+ \frac{2C_1^2|u_0|\,|Au_0|\,\|\nu\|^2}{\nu(1-\alpha)} + \gamma^{2/(1-\alpha)}\nu\frac{(1-\alpha)}{2}C_L^2\lambda_1^2|\nu|^2$$

$$\le \left[\frac{2}{\nu(1-\alpha)} + \left(s + \frac{4C_1^2 R^2}{2\nu(1-\alpha)} + \gamma^{2/(1-\alpha)}\nu\frac{(1-\alpha)}{2}C_L^2\lambda_1^2\right)\right]$$

$$\times K_8^2(\gamma, \rho, s, \nu, R)\Big]|g|^2,$$

which implies (3.28) and completes the proof. ∎

We now show that the spectra of $\mathcal{A}_m(u_0)$ are, in some sense, close to each other when m is large enough.

Lemma 3.7. *Let $R > 0$, u_0, $u_1 \in \mathcal{D}(A)$ with $|Au_0| \leq \lambda_1^{1/2}R$, $|Au_1| \leq \lambda_1^{1/2}R$, and let s, $\rho > 0$. Then there exist a constant*

$$(3.33) \quad \eta(\gamma,s,\rho,\nu,R) = \frac{\nu}{3C_1 K_5^{1/2}(\gamma,s,\nu,R)K_7(\gamma,\rho,s,\nu,R)}$$

$$\times \frac{\lambda_1^{1/4}}{K_4^{1/2}(\gamma,s,\nu,R) + \lambda_1^{1/4}},$$

and a constant

$$(3.34) \quad M_2(\gamma,\rho,s,\nu,R)$$

$$= \max \left\{ \frac{C_L}{C_0} \left(4\gamma K_5(\gamma,s,\nu,R)K_7(\gamma,\rho,s,\nu,R) \right)^{1/(1-\alpha)}, \right.$$

$$\left. \frac{2C_0}{\nu} \left(2C_1 R K_5(\gamma,s,\nu,R)K_7(\gamma,\rho,s,\nu,R) + K_4(\gamma,s,\nu,R) \right) \right\},$$

where $K_4(\gamma,s,\nu,R)$, $K_5(\gamma,s,\nu,R)$, and $K_7(\gamma,\rho,s,\nu,R)$ are as in (3.11), (3.12), and (3.29), respectively, such that if $\|u_0 - u_1\| \leq \eta(\gamma,\rho,s,\nu,R)$, and the condition

$$(3.35) \qquad\qquad (\mathrm{Sp}(\mathcal{A}_m(u_0)))_s = \varnothing$$

holds for some $m \in [M_2(\gamma,\rho,s,\nu,R),\infty)$, then for every $n \in [m,\infty]$ we have

$$(3.36) \qquad\qquad (\mathrm{Sp}(\mathcal{A}_n(u_1)))_s \subset B(\mathrm{Sp}(\mathcal{A}_m(u_0)),\rho).$$

Proof. Let $n \geq m$, and let $\lambda \in (\mathrm{Sp}(\mathcal{A}_n(u_1)))_s$, i.e., there is a $w \in H_n$, $|w| = \nu$ such that $\mathcal{A}_n(u_1)w = \lambda w$ and $|\Re\lambda| \leq s$. Denote by

$$w_m = P_m w \quad\text{and}\quad \widetilde{w}_m = Q_m w,$$

where P_m and Q_m are the orthogonal projections defined in Section 2 on page 41. Suppose that $\lambda \notin B(\mathrm{Sp}(\mathcal{A}_m(u_0)),\rho)$, i.e., $\mathrm{dist}(\lambda, \mathrm{Sp}(\mathcal{A}_m(u_0))) \geq \rho$. Then by Lemma 3.6 on page 50, we get

$$|\mathcal{A}_m(u_0)w_m - \lambda w_m| \geq \frac{|w_m|}{K_7(\gamma,\rho,s,\nu,R)}.$$

Since

$$|w_m| \geq |w| - |\widetilde{w}_m|,$$

then by applying Lemma 3.3 on page 47, we have

$$|\widetilde{w}_m| \leq \frac{\|\widetilde{w}_m\|}{\lambda_{m+1}^{1/2}} \leq \frac{K_4(\gamma, s, \nu, R)}{\lambda_{m+1}^{1/2}},$$

where $K_4(\gamma, s, \nu, R)$ is given by (3.11). From the above we reach

$$|\mathcal{A}_m(u_0)w_m - \lambda w_m| \geq \frac{1}{K_7(\gamma, \rho, s, \nu, R)}\left(\nu - \frac{K_4(\gamma, s, \nu, R)}{\lambda_{m+1}^{1/2}}\right),$$

where $K_7(\gamma, \rho, s, \nu, R)$ is as in (3.29). On the other hand,

$$\begin{aligned}
\mathcal{A}_m(u_0)w_m - \lambda w_m &= \mathcal{A}_m(u_0)w_m - P_m(\mathcal{A}_n(u_1)w) \\
&= P_m[B(u_0 - u_1, w_m) - B(u_1, \widetilde{w}_m) - B(\widetilde{w}_m, u_0) \\
&\qquad\qquad + B(w, u_0 - u_1) - \gamma L\widetilde{w}_m].
\end{aligned}$$

By using the inequalities (2.8)-(2.10) and (2.20) with $\beta = \alpha$, we obtain

$$\begin{aligned}
&|(\mathcal{A}_m(u_0) - \lambda)w_m| \\
&\leq C_1\left(\frac{\|u_0 - u_1\|}{\lambda_1^{1/4}}\|w_m\|^{1/2}|Aw_m|^{1/2} + |u_1|^{1/2}|Au_1|^{1/2}\|\widetilde{w}_m\|\right) \\
&\quad + C_1\left(\|u_0\|^{1/2}|Au_0|^{1/2}|\widetilde{w}_m|^{1/2}\|\widetilde{w}_m\|^{1/2} + \|u_0 - u_1\|\,|w|^{1/2}|Aw|^{1/2}\right) \\
&\quad + \gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}|\widetilde{w}_m|^{1-\alpha}|A\widetilde{w}_m|^{\alpha} \\
&\leq C_1 K_5^{1/2}(\gamma, s, \nu, R)\left(\frac{K_4^{1/2}(\gamma, s, \nu, R)}{\lambda_1^{1/4}} + 1\right)\|u_0 - u_1\| \\
&\quad + \frac{2C_1 R K_5(\gamma, s, \nu, R)}{\lambda_{m+1}^{1/2}} + \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}K_5(\gamma, s, \nu, R)}{\lambda_{m+1}^{1-\alpha}},
\end{aligned}$$

where $K_4(\gamma, s, \nu, R)$ and $K_5(\gamma, s, \nu, R)$ are given by (3.11) and (3.12), respectively.

Now if we let

$$\|u_0 - u_1\| \leq \eta(\gamma, \rho, s, \nu, R) \quad \text{and} \quad m \geq M_2(\gamma, \rho, s, \nu, R),$$

where $\eta(\gamma, \rho, s, \nu, R)$ and $M_2(\gamma, \rho, s, \nu, R)$ are as in (3.33) and (3.34), respectively, then

$$\frac{\nu}{K_7(\gamma,\rho,s,\nu,R)} \geq C_1 K_5^{1/2}(\gamma,s,\nu,R)\left[\frac{K_4^{1/2}(\gamma,s,\nu,R)}{\lambda_1^{1/4}} + 1\right]\|u_0 - u_1\|$$

$$+ \frac{2C_1 R K_5(\gamma,s,\nu,R)}{\lambda_{m+1}^{1/2}} + \frac{K_4(\gamma,s,\nu,R)}{K_7(\gamma,\rho,s,\nu,R)\lambda_{m+1}^{1/2}}$$

$$+ \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}K_5(\gamma,s,\nu,R)}{\lambda_{m+1}^{1-\alpha}}.$$

This leads into a contradiction. Thus, $\lambda \in B(\mathrm{Sp}(\mathcal{A}_m(u_0)),\rho)$, i.e.,

$$(\mathrm{Sp}(\mathcal{A}_n(u_1)))_s \subset B(\mathrm{Sp}(\mathcal{A}_m(u_0)),\rho).$$

□

Since $f \in H$, then there exist a constant $M_3(\gamma,\sigma,\nu,f,K_3)$ such that

$$(3.37) \qquad |Q_m f| < \frac{\lambda_1}{8C_1 K_9(\gamma,\sigma/2,\sigma,\nu,K_3)} \quad \forall m \geq M_3(\gamma,\sigma,\nu,f),$$

where K_3 is as in (3.6) and $K_9(\sigma/2,\sigma,\nu,K_3)$ is as in (3.31).

Indeed, given f, we can compute K_3 and $K_9(\sigma/2,\sigma,\nu,K_3)$, and then estimate $M_3(\gamma,\sigma,\nu,f,K_3)$. However, we do not have a priori knowledge about the magnitude of $M_3(\gamma,\sigma,\nu,f,K_3)$. For the case $f \in V$, we have an explicit a priori estimate

$$M_3(\gamma,\sigma,\nu,f,K_3) = \frac{(8C_1\|f\|K_9(\gamma,\sigma/2,\sigma,\nu,K_3))^2}{C_0\lambda_1^3}.$$

3.2. The main result in the Galerkin case. Next we present our main theorem of this section.

Theorem 3.2. *Let $\sigma > 0$ be given, and let $u_m^\infty \in V$ be a steady state solution of the Galerkin approximation (3.1) of the closed-loop NSE (2.18) with $|u_m^\infty| \leq K_1$. Then there exists a constant*

$$(3.38) \quad M_{\mathrm{GLK}}(\gamma,\sigma,\nu,f,K_1) = \max\{M_4(\gamma,\sigma,\nu,f,K_1), M_5(\gamma,\sigma,\nu,f,K_1)\},$$

where $M_4(\gamma,\sigma,\nu,f,K_1)$, $M_5(\gamma,\sigma,\nu,f,K_1)$ are specified in (3.49) and (3.59), respectively, such that if the condition

$$(3.39) \qquad\qquad (\mathrm{Sp}(\mathcal{A}_m(u_m^\infty)))_\sigma = \varnothing$$

holds for some $m \geq M_{\mathrm{GLK}}(\gamma,\sigma,\nu,f,K_1)$, then we have

(i) *For every $n \in [m, \infty]$ there is a hyperbolic stationary solution u_n^∞ of the Galerkin approximation equation (3.1) of order n, for which*

$$(3.40) \qquad (\mathrm{Sp}(\mathcal{A}(u_n^\infty)))_{\sigma/2} = \varnothing,$$

$$(3.41) \qquad |u_m^\infty - u_n^\infty| \le \frac{K_{10}(\gamma, \sigma, \nu, f, K_1)}{\lambda_{m+1}},$$

$$(3.42) \qquad |A(u_m^\infty - u_n^\infty)| \le K_{11}(\gamma, \sigma, \nu, f, K_1),$$

where

$$(3.43) \qquad K_{10}(\gamma, \sigma, \nu, f, K_1) = 1 + \frac{4C_1 K_3 K_8(\gamma, \sigma/2, \sigma, \nu, K_3)}{\lambda_1^{1/2}}$$
$$+ 2\gamma\nu \left(\frac{\lambda_1 C_L}{\lambda_{m+1}}\right)^{1-\alpha},$$

$$(3.44) \qquad K_{11}(\gamma, \sigma, \nu, f, K_1) = \frac{\lambda_1 K_9(\gamma, \sigma/2, \sigma, \nu, K_3)}{2C_1},$$

where $K_8(\gamma, \sigma/2, \sigma, \nu, K_3))$ and $K_9(\gamma, \sigma/2, \sigma, \nu, K_3))$ are as in (3.30) and (3.31), respectively.

(ii) *There is a stationary solution u^∞ of the closed-loop system (2.18) such that*

$$(3.45) \qquad u_n^\infty \to u^\infty \quad \text{as } n \to \infty.$$

Moreover, u^∞ is hyperbolic and

$$(3.46) \qquad (\mathrm{Sp}(\mathcal{A}(u^\infty)))_{\sigma/2} = \varnothing,$$

$$(3.47) \qquad |u^\infty - u_n^\infty| \le \frac{K_{10}(\gamma, \sigma, \nu, f, K_1)}{\lambda_{n+1}}.$$

Proof. First we will show the existence of u_n^∞. Let $v = u_n^\infty - u_m^\infty$. If u_n^∞ is a stationary solution of the Galerkin equations (3.1) of order n, then v should satisfy

$$(3.48) \quad \nu A v + P_n B(u_m^\infty + v, u_m^\infty + v) - P_m B(u_m^\infty, u_m^\infty)$$
$$+ \gamma P_n L(u_m^\infty + v) - \gamma P_m L u_m^\infty = P_n f - P_m f.$$

Thus,

$$\mathcal{A}_n(u_m^\infty) v = \nu A v + P_n[B(u_m^\infty, v) + B(v, u_m^\infty) + \gamma L v]$$
$$= Q_m P_n f - Q_m P_n B(u_m^\infty, u_m^\infty) - P_n B(v, v) - \gamma Q_m P_n L u_m^\infty.$$

By Lemma 3.7 on page 53, we have $(\mathrm{Sp}(\mathcal{A}_n(u_m^\infty)))_{\sigma/2} = \varnothing$, when $m \geq M_2(\sigma, \sigma/2, \nu, K_3)$. Therefore, $\mathcal{A}_n(u_m^\infty)$ is invertible. Denote by

$$\mathcal{F}(\nu) = (\mathcal{A}_n(u_m^\infty))^{-1}(Q_m P_n f - Q_m P_n B(u_m^\infty, u_m^\infty) - P_n B(\nu, \nu) - \gamma Q_m P_n L u_m^\infty).$$

By Lemma 3.6 on page 50, we get

$$|A\mathcal{F}(\nu)| \leq K_9\left(\gamma, \frac{\sigma}{2}, \sigma, K_3, \nu, f\right)$$
$$\times |Q_m P_n f - Q_m P_n B(u_m^\infty, u_m^\infty) - P_n B(\nu, \nu) - \gamma Q_m P_n L u_m^\infty|.$$

By using (2.9), (2.11), and (2.20) with $\beta = \alpha$, we obtain

$$|A\mathcal{F}(\nu)| \leq K_9\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)\left[|Q_m f| + \frac{2C_1 K_3^2}{\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\right.$$
$$\left. + \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}K_3}{\lambda_{m+1}^{1-\alpha}} + \frac{C_1|A\nu|^2}{\lambda_1}\right].$$

Therefore, if $m \geq M_4(\gamma, \sigma, \nu, f)$, where

(3.49) $M_4(\gamma, \sigma, \nu, f, K_1)$

$$= \max\left\{M_2\left(\sigma, \frac{\sigma}{2}, \nu, K_3\right), M_3, \frac{1024 C_1^4 K_3^4 K_9^4(\gamma, \sigma/2, \sigma, \nu, K_3)}{C_0\lambda_1^4},\right.$$

$$\left.\frac{C_L}{C_0}\left(\frac{16 C_1 \gamma\nu K_3 K_9^2(\gamma, \sigma/2, \sigma, \nu, K_3)}{\lambda_1}\right)^{1/(1-\alpha)}\right\},$$

then

$$|Q_m f| + \frac{2C_1 K_3^2}{\lambda_1^{1/2}\lambda_{m+1}^{1/2}} + \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}K_3}{\lambda_{m+1}^{1-\alpha}} < \frac{\lambda_1}{4 C_1 K_9^2(\gamma, \sigma/2, \sigma, \nu, K_3)}.$$

As a result, $\mathcal{F}$ maps

(3.50) $$\bar{B}_r = \{\nu \in H_n : |A\nu| \leq r\}$$

into $\bar{B}_r$, where

(3.51) $$r = 2\left[|Q_m f| + \frac{C_1 K_3^2}{\lambda_1^{1/2}\lambda_{m+1}^{1/2}} + \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}K_3}{\lambda_{m+1}^{1-\alpha}}\right]K_9\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)$$

$$< \frac{\lambda_1}{2 C_1 K_9(\gamma, \sigma/2, \sigma, \nu, K_3)}.$$

Moreover,

$$|A(\mathcal{F}(v_1) - \mathcal{F}(v_2))| \le \frac{2C_1 r K_9(\gamma, \sigma/2, \sigma, \nu, K_3)}{\lambda_1} |A(v_2 - v_1)|,$$

and

$$\frac{2C_1 r K_9(\gamma, \sigma/2, \sigma, \nu, K_3)}{\lambda_1} < 1.$$

Therefore, $\mathcal{F}$ is a contraction map on $\bar{B}_r$, given by (3.50), and as a result it has a unique $v^\infty \in \{v \in H_n : |Av| \le r\}$ such that $u_n^\infty = u_m^\infty + v^\infty$ is a steady state of the Galerkin equations (3.1) of order n, and

$$(3.52) \qquad |A(u_n^\infty - u_m^\infty)| \le r \le \frac{\lambda_1}{2C_1 K_9(\gamma, \sigma/2, \sigma, \nu, K_3)},$$

when $m \ge M_4(\gamma, \sigma, \nu, f, K_1)$, and $M_4(\gamma, \sigma, \nu, f, K_1)$ and $K_9(\gamma, \sigma/2, \sigma, \nu, K_3)$ are as in (3.49) and (3.31), respectively.

Next we will show the estimates (3.41). Let $v_m = P_m u_n^\infty - u_m^\infty$ and $\tilde{v}_m = Q_m u_n^\infty$. Subtract the Galerkin equations (3.1) of order m from the one of order n, to get

$$(3.53) \quad \nu A v_m + P_m(B(u_n^\infty, u_n^\infty) - B(u_m^\infty, u_m^\infty) + \gamma P_m L(u_n^\infty - u_m^\infty)) = 0,$$

$$(3.54) \quad \nu A \tilde{v}_m + Q_m P_n B(u_n^\infty, u_n^\infty) + \gamma L u_n^\infty) = Q_m P_n f.$$

As a consequence of the equation (3.53) we obtain

$$\mathcal{A}_m(u_m^\infty) v_m$$
$$= \nu A v + P_m(B(u_m^\infty, v_m) + B(v_m, u_m^\infty) + \gamma L v_m)$$
$$= -P_m[B(v_m + \tilde{v}_m, v_m + \tilde{v}_m) + B(u_m^\infty, \tilde{v}_m) + B(\tilde{v}_m, u_m^\infty) + \gamma L \tilde{v}_m]$$
$$= -P_m[B(v_m, v_m) + B(u_n^\infty, \tilde{v}_m) + B(\tilde{v}_m, u_n^\infty) + \gamma L \tilde{v}_m].$$

By using (2.6) we have for every $w_1, w_2, w_3 \in H_m$,

$$(3.55) \ |((\mathcal{A}_m(u_m^\infty))^{-1} B(w_1, w_2), w_3)| = |(B(w_1, w_2), ((\mathcal{A}_m(u_m^\infty))^{-1})^* w_3)|$$

$$= |(B(w_1, ((\mathcal{A}_m(u_m^\infty))^{-1})^* w_3), w_2)|.$$

By using (2.8) and (3.27), we get

$$|((\mathcal{A}_m(u_m^\infty))^{-1} B(w_1, w_2), w_3)|$$
$$\le C_1 |w_1|^{1/2} |Aw_1|^{1/2} \|((\mathcal{A}_m(u_m^\infty))^{-1})^* w_3\| \, |w_2|$$
$$\le C_1 K_8\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right) |w_1|^{1/2} |Aw_1|^{1/2} |w_2| \, |w_3|,$$

where $K_8(\gamma, \sigma/2, \sigma, \nu, K_3)$ is given by (3.30). Now by using (2.9) and (3.27) in (3.55), we obtain

$$|((\mathcal{A}_m(u_m^\infty))^{-1}B(w_1, w_2), w_3)|$$
$$\leq C_1|w_1|\,\|((\mathcal{A}_m(u_m^\infty))^{-1})^*w_3)\|\,|w_2|^{1/2}|Aw_2|^{1/2}$$
$$\leq C_1K_8\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)|w_1|\,|w_2|^{1/2}|Aw_2|^{1/2}|w_3|.$$

Therefore, we reach

(3.56) $\quad |(\mathcal{A}_m(u_m^\infty))^{-1}B(w_1, w_2)|$
$$\leq C_1K_8\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)|w_1|^{1/2}|Aw_1|^{1/2}|w_2|,$$

(3.57) $\quad |(\mathcal{A}_m(u_m^\infty))^{-1}B(w_1, w_2)|.$
$$\leq C_1K_8\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)|w_1|\,|w_2|^{1/2}|Aw_2|^{1/2}$$

As a result of these estimates, Lemma 3.6 on page 50, and the inequalities (2.7)-(2.11) and (2.20) with $\beta = 0$, we obtain

$$|v_m| \leq |(\mathcal{A}_m(u_m^\infty))^{-1}P_m[B(v_m, v_m) + B(u_n^\infty, \tilde{v}_m) + B(\tilde{v}_m, u_n^\infty) + L\tilde{v}_m]|$$

$$\leq \frac{C_1}{\lambda_1^{1/4}}K_9\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)|v_m|^{3/2}\|v_m\|^{1/2}$$

$$+ 2C_1K_8\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)K_1^{1/2}K_3^{1/2}|\tilde{v}_m|$$

$$+ \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}K_9(\gamma, \sigma/2, \sigma, \nu, K_3)}{\lambda_{m+1}^{1-\alpha}}|\tilde{v}_m|.$$

Notice that

$$|v_m|^{1/2}\|v_m\|^{1/2} \leq \frac{|Av_m|}{\lambda_1^{3/4}},$$

and

$$\frac{C_1}{\lambda_1}K_9\left(\gamma, \frac{\sigma}{2}, \sigma, \nu, K_3\right)|Av_m| \leq \frac{1}{2}.$$

Therefore, by using the estimates (3.52), we get

$$|v_m| \leq \frac{K_{12}(\gamma, \sigma, \nu, f, K_1)}{\lambda_{m+1}},$$

where

$$(3.58) \quad K_{12}(\gamma, \sigma, \nu, f, K_1) = \frac{4C_1 K_3 K_8(\gamma, \sigma/2, \sigma, \nu, K_3)}{\lambda_1^{1/2}} + 2\gamma\nu \left(\frac{\lambda_1 C_L}{\lambda_{m+1}}\right)^{1-\alpha}.$$

Then the estimate (3.41) is achieved.

Let

$$(3.59) \quad M_5(\gamma, \sigma, \nu, f, K_1) = \frac{K_{12}(\gamma, \sigma, \nu, f, K_1)}{C_0 \lambda_1 \eta(\gamma, \sigma, \sigma/2, \nu, K_3)}.$$

We apply Lemma 3.7 on page 53, when $m \geq M_{\text{GLK}}(\gamma, \sigma, \nu, f, K_1)$, to conclude

$$(\text{Sp}(\mathcal{A}(u^\infty)))_{\sigma/2} = \varnothing. \qquad \blacksquare$$

4. A STABILIZATION CRITERION FOR THE NONLINEAR GALERKIN APPROXIMATION

Without loss of generality, we will assume that $\alpha = \frac{1}{2}$ in the condition (2.20) in this section. By now, it is a well established fact that, for the dissipative parabolic evolution equations, the nonlinear Galerkin method, which is based on the theory of Inertial Manifolds [32] and Approximate Inertial Manifolds (AIM) (see, e.g., [27], [33], [53], [63], [76] and references therein), is a more accurate numerical method than the standard Galerkin method (cf. [23], [39], [47], [48]). Motivated by this superiority in accuracy of the nonlinear Galerkin method, we show here that a similar criterion, yet stronger than the one stated in the previous section, is available in the context of nonlinear Galerkin method.

Here, we will restrict ourselves to the nonlinear Galerkin method that is based on one of the simplest AIM, the one introduced in [27]. That is, for the closed-loop NSE (2.18) and (2.19) we have the following nonlinear Galerkin approximation:

$$(4.1) \quad \frac{dp_m}{dt} + \nu A p_m + P_m[B(p_m + q_m, p_m + q_m) - B(q_m, q_m) \\ + \gamma L(p_m + q_m)] = P_m f,$$

$$(4.2) \quad \nu A q_m + Q_m[B(p_m, p_m) + \gamma L p_m] = Q_m f,$$

$$(4.3) \quad p_m(0) = P_m u_{in},$$

where $p_m \in H_m$ and $q_m \in Q_m$.

For $1 \leq m \leq \infty$ we denote by $p_m^\infty + q_m^\infty$ the stationary solutions of the equations (4.1) and (4.2). Specially, $q_\infty^\infty = 0$ and $u^\infty = p_\infty^\infty$ is a steady state of the equation (2.18). We will denote by

$$(4.4) \quad \Phi_{\text{app}}(u_m) = (\nu A)^{-1}[Q_m f - Q_m B(u_m, u_m) - \gamma Q_m L u_m],$$

for any $u_m \in H_m$, the Foias-Manley-Temam AIM [27]. Furthermore, we also denote by $\mathcal{N}_m(u_m) : H_m \to H_m$ the linearized operator of the nonlinear Galerkin closed-loop operator given in (1.1) and (4.2) about $u_m + \Phi_{\mathrm{app}}(u_m)$, defined as

$$(4.5) \quad \mathcal{N}_m(u_m)w = \nu Aw + P_m[B(u_m + \Phi_{\mathrm{app}}(u_m), w)$$

$$+ B(w, u_m + \Phi_{\mathrm{app}}(u_m)) + B(u_m, \xi(w)) + B(\xi(w), u_m) + \gamma L(w + \xi(w))],$$

for all $w \in H_m$, where $\xi(w) \in Q_m$ satisfies

$$(4.6) \qquad \nu A\xi + Q_m[B(u_m, w) + B(w, u_m) + \gamma Lw] = 0.$$

When $m = \infty$, $\mathcal{N}_\infty(u_\infty) = \mathcal{A}(u_\infty)$.

Lemma 4.1. *For any $m \in [1, \infty]$, if there is a steady state $p_m^\infty + q_m^\infty \in V$ of equations (4.1) and (4.2) with*

$$(4.7) \qquad\qquad\qquad |p_m^\infty + q_m^\infty| \le N_1,$$

then $(p_m^\infty + q_m^\infty) \in \mathcal{D}(A)$, and

$$(4.8) \qquad \|p_m^\infty + q_m^\infty\| \le N_2 = (2\nu\lambda_1 GN_1)^{1/2} + 2\gamma C_L^{1/2}\lambda_1^{1/2}N_1,$$

$$(4.9) \qquad |A(p_m^\infty + q_m^\infty)| \le \lambda_1^{1/2}N_3 = 4\nu\lambda_1 G + \frac{12C_1^2}{\nu^2}N_1N_2^2 + 8\gamma C_L^{1/2}\lambda_1^{1/2}N_2,$$

where again $G = |f|/(\nu^2\lambda_1)$ is the Grashoff number.

Proof. Let $p_m^\infty + q_m^\infty$ be a stationary solution of the equations (4.1) and (4.2), with $|p_m^\infty + q_m^\infty| \le N_1$. Taking the inner product in H_m of the equation (4.1) with p_m^∞ and the equation (4.2) with q_m^∞ in Q_m, and using (2.6), we get

$$\nu\|p_m^\infty\|^2 + (B(p_m^\infty, q_m^\infty), p_m^\infty) + \gamma(L(p_m^\infty + q_m^\infty), p_m^\infty) = (f, p_m^\infty),$$

$$\nu\|q_m^\infty\|^2 + (B(p_m^\infty, p_m^\infty), q_m^\infty) + \gamma(L(p_m^\infty, q_m^\infty)) = (f, q_m^\infty).$$

Adding the above equations up and using (2.6) again, we obtain

$$\nu\|p_m^\infty\|^2 + \nu\|q_m^\infty\|^2 = (f, p_m^\infty + q_m^\infty) - \gamma(L(p_m^\infty + q_m^\infty), p_m^\infty + q_m^\infty) + \gamma(Lq_m^\infty, q_m^\infty).$$

By using (2.21), we obtain

$$\nu\|p_m^\infty\|^2 + \nu\|q_m^\infty\|^2 \le |f|\,|p_m^\infty + q_m^\infty| + 2\gamma\nu C_L^{1/2}\lambda_1^{1/2}|p_m^\infty + q_m^\infty|\,\|p_m^\infty + q_m^\infty\|.$$

By the Cauchy-Schwarz inequality, we reach

$$\frac{\nu}{2}(\|p_m^\infty\|^2 + \|q_m^\infty\|^2) \le |f|N_1 + 2\gamma^2\nu C_L\lambda_1 N_1^2,$$

and the estimate (4.8) follows. Taking the L^2–norm of the equations (4.1) and (4.2) leads to

$$\nu|A(p_m^\infty + q_m^\infty)| \leq |f| + |B(p_m^\infty, p_m^\infty)| + |B(p_m^\infty, q_m^\infty)|$$
$$+ |B(q_m^\infty, p_m^\infty)| + \gamma[|Lp_m^\infty| + |Lq_m^\infty|].$$

By using (2.7)-(2.10) and (2.20) with $\beta = \frac{1}{2}$, we conclude

$$\nu|A(p_m^\infty + q_m^\infty)| \leq |f| + C_1|p_m^\infty|^{1/2}\|p_m^\infty\|\,|Ap_m^\infty|^{1/2} + C_1|p_m^\infty|^{1/2}\|q_m^\infty\|\,|Ap_m^\infty|^{1/2}$$
$$+ C_1|q_m^\infty|^{1/2}\|p_m^\infty\|\,|Aq_m^\infty|^{1/2} + \gamma\nu C_L^{1/2}\lambda_1^{1/2}[\|p_m^\infty\| + \|q_m^\infty\|].$$

Thanks to the Cauchy-Schwarz inequality, we get

$$\frac{\nu}{4}|A(p_m^\infty + q_m^\infty)| \leq |f| + \frac{C_1^2}{\nu}[|p_m^\infty|\,\|p_m^\infty\|^2 + p_m^\infty|\,\|q_m^\infty\|^2 + |q_m^\infty|\,\|p_m^\infty\|^2]$$
$$+ \gamma\nu C_L^{1/2}\lambda_1^{1/2}[\|p_m^\infty\| + \|q_m^\infty\|],$$

which implies the estimate (4.9). $\square$

Lemma 4.2. *Let* $u_m \in H_m$ *with* $|Au_m| \leq \lambda_1^{1/2}R$. *Then*

(4.10)
$$|A\Phi_{\mathrm{app}}(u_m)| \leq \lambda_1^{1/2}N_4(\gamma, R, \nu, f),$$

where

(4.11)
$$N_4(\gamma, R, \nu, f) = \frac{|Q_m f|}{\nu\lambda_1^{1/2}} + \frac{2C_1R^2}{\nu\lambda_{m+1}^{1/2}} + \frac{C_L^{1/2}\nu\lambda_1 R}{\lambda_{m+1}^{1/2}}.$$

Suppose that $\omega \in V$ *and* $\xi(\omega)$ *satisfies (4.6). Then*

(4.12)
$$\|\xi(\omega)\| \leq \frac{2C_1R}{\nu}|\omega| + \gamma C_L^{1/2}\lambda_1^{1/2}|\omega|.$$

Proof. Let $v_m = \Phi_{\mathrm{app}}(u_m)$. Then v_m satisfies

$$\nu Av_m + Q_m[B(u_m, u_m) + \gamma Lu_m] = Q_m f.$$

Thus

$$|Av_m| \leq \frac{1}{\nu}\left[|Q_m f| + \frac{|\nabla B(u_m, u_m)|}{\nu\lambda_{m+1}^{1/2}} + \gamma\frac{|A^{1/2}Lu_m|}{\lambda_{m+1}^{1/2}}\right].$$

By using (2.11) and (2.20) with $\beta = 1$, we get

$$|Av_m| \leq \frac{|Q_m f|}{\nu} + \frac{2C_1|Au_m|^2}{\nu\lambda_1^{1/2}\lambda_{m+1}^{1/2}} + \frac{C_L^{1/2}\nu\lambda_1 R}{\lambda_{m+1}^{1/2}}.$$

Then (4.10) follows. Now, taking the inner product in H of equation (4.6) with ω, we get

$$\nu\|\xi(\omega)\|^2 \leq |(B(u_m, \omega), \xi(\omega))| + |(B(u_m, \omega), \xi(\omega))| + \gamma|(L\omega, \xi(\omega))|.$$

By using (2.7)-(2.10) and (2.21), we obtain

$$\nu\|\xi(\omega)\|^2 \leq 2C_1|u_m|^{1/2}|Au_m|^{1/2}\|\xi(\omega)\|\,|\omega| + \gamma\nu C_L^{1/2}\lambda_1^{1/2}\|\xi(\omega)\|\,|\omega|.$$

Therefore, (4.12) holds. $\qquad\square$

Lemma 4.3. *Let $u_m \in H_m$ with $|Au_m| \leq \lambda_1^{1/2}R$. If w is an eigenfunction of $\mathcal{N}_m(u_m)$, then $w \in \mathcal{D}(A)$. Moreover, if $|w| = v$ and $\mathcal{N}_m(u_m)w = \Lambda w$ for some Λ, then we have*

$$(4.13) \qquad\qquad \|w\| \leq N_5(\gamma, \Lambda, R, \nu, f),$$

$$(4.14) \qquad\qquad |Aw| \leq N_6(\gamma, \Lambda, R, \nu, f),$$

where

(4.15) $N_5(\gamma, \Lambda, R, \nu, f)$

$$= (4|\Lambda|\nu)^{1/2} + 2C_1(R + N_4(\gamma, R, \nu, f)) + \frac{C_1^2\lambda_1^{1/2}R^2}{\nu\lambda_{m+1}^{1/2}} + \sqrt{12}\gamma\nu C_L^{1/2}\lambda_1^{1/2},$$

(4.16) $N_6(\gamma, \Lambda, R, \nu, f)$

$$= |\Lambda| + \gamma C_L^{1/2}\lambda_1^{1/2}[N_5^{1/2}(\gamma, s, R, \nu, f) + 2C_1R + \gamma\nu C_L^{1/2}\lambda_1^{1/2}]$$

$$+ \frac{2C_1}{\nu}[R + N_4(\gamma, R, \nu, f)][N_5(\gamma, \Lambda, R, \nu, f) + 2C_1R + \gamma\nu C_L^{1/2}\lambda_1^{1/2}].$$

Proof. Let $v_m = \Phi_{\mathrm{app}}(u_m)$, and let w be an eigenfunction of $\mathcal{N}_m(u_m)$ with corresponding eigenvalue Λ such that $|w| = v$, i.e.,

(4.17) $\nu Aw + P_m[B(u_m + v_m, w) + B(w, u_m + v_m) + B(u_m, \xi(w))$

$$+ B(\xi(w), u_m) + \gamma L(w + \xi(w))] = \Lambda w,$$

(4.18) $\nu A\xi + Q_m(B(u_m, w) + B(w, u_m) + \gamma Lw) = 0.$

Taking the inner product in H of the equations above with w and ξ respectively, gives

$$(4.19) \qquad \nu\|w\|^2 + (B(w, u_m + v_m), w) + (B(u_m, \xi), w)$$
$$+ (B(\xi, u_m), w) + \gamma(L(w + \xi), w) = \Lambda|w|^2,$$

$$(4.20) \qquad \nu\|\xi\|^2 + (B(u_m, w), \xi) + (B(w, u_m), \xi) + \gamma(Lw, \xi) = 0.$$

Adding up these two equations,

$$\nu\|w\|^2 + \nu\|\xi\|^2 \le |\Lambda|\,|w|^2 + |(B(w, u_m + v_m), w)| + |(B(\xi(w), u_m), w)|$$
$$+ |(B(w, u_m), \xi)| + \gamma|(Lw, w)| + \gamma|(Lw, \xi)| + \gamma|(L\xi, w)|.$$

By using (2.7)-(2.10) and (2.21), we obtain

$$\nu\|w\|^2 + \nu\|\xi\|^2 \le |\Lambda|\,|w|^2 + C_1\|u_m + v_m\|\,|w|\,\|w\|$$
$$+ \frac{2C_1}{\lambda_1^{1/2}}|Au_m|\,|w|^{1/2}\|w\|^{1/2}|\xi| + \gamma\nu C_L^{1/2}\lambda_1^{1/2}|w|\,\|w\|$$
$$+ 2\gamma\nu C_L^{1/2}\lambda_1^{1/2}|w|\,\|\xi\|.$$

By Young's inequality and the Cauchy-Schwarz inequality, we reach

$$\frac{\nu}{2}\|w\|^2 \le \nu^2|\Lambda| + \frac{C_1^4\lambda_1^2 R^4}{4\nu\lambda_{m+1}^2} + \nu C_1^2[R + N_4(\gamma, R, \nu, f)]^2 + 3\gamma^2\nu^3 C_L\lambda_1.$$

Therefore, the estimate (3.9) is proved. Moreover, by taking the L^2-norm of the equation (4.17), we get

$$\nu|Aw| \le \nu|\Lambda| + |B(u_m + v_m, w) + B(w, u_m + v_m)$$
$$+ B(u_m, \xi) + B(\xi, u_m) + \gamma L(w + \xi)|.$$

By using (2.7)-(2.10) and (2.21) again to the above estimate and equation (4.20), we have

$$\nu|Aw| \le \nu|\Lambda| + \frac{C_1}{\lambda_1^{1/2}}|A(u_m + v_m)|\,\|w\| + \frac{C_1}{\lambda_1^{1/4}}|w|^{1/2}\|w\|^{1/2}|A(u_m + v_m)|$$
$$+ \frac{C_1}{\lambda_1^{1/2}}|A(u_m + v_m)|\,\|\xi\| + \frac{C_1}{\lambda_1^{1/4}\lambda_{m+1}^{1/4}}\|\xi\|\,|A(u_m + v_m)|$$
$$+ \gamma\nu C_L^{1/2}\lambda_1^{1/2}\|w + \xi\|.$$

From the estimate (4.12), we obtain

$$\|\xi\| \le 2C_1 R + \gamma \nu C_L^{1/2} \lambda_1^{1/2}.$$

As a result of these estimates we get (4.14), which completes the proof of the lemma. □

Lemma 4.4. *Let $R > 0$, $u_m \in \mathcal{D}(A)$ with $|Au_m| \le \lambda_1^{1/2}R$, and $s \in \mathbb{R}$, and let $T_1(s)$ be the number of the eigenvalues of $\mathcal{N}_m(u_m)$ whose real parts are less than or equal to s, i.e.,*

$$T_1(s) = \# \text{ of } \{\lambda \in \text{Sp}(\mathcal{N}_m(u_m)) : \Re\lambda < s\}.$$

Then

(4.21) $$T_1(s) \le N_7(\gamma, s, R, \nu, f),$$

where

$$
(4.22) \quad N_7(\gamma, s, R, \nu, f) = \frac{1}{2} \Bigg\{ \left(\frac{4sC_0}{\nu\lambda_1} + \frac{8C_1^2 R^2}{\nu^2 \lambda_1^{1/2} \Lambda_{m+1}^{1/4}} + 12\gamma^2 C_L C_0 - 1 \right)
$$

$$
+ \Bigg[\left(\frac{4sC_0}{\nu\lambda_1} + \frac{8C_1^2 R^2}{\nu^2 \lambda_1^{1/2} \Lambda_{m+1}^{1/4}} + 12\gamma^2 C_L C_0 - 1 \right)^2
$$

$$
+ \frac{16 C_0 C_2 (R + N_4(\gamma, R, \nu, f))^2}{\nu^2 \lambda_1} \Bigg]^{1/2} \Bigg\}.
$$

Remark. Here again, in a sense, the estimate (4.21) is sharp since the constant $N_7(\gamma, s, R, \nu, f)$ is growing linearly in s. For example, when $u_{\text{in}} = 0$ and $\gamma = 0$ we have $\mathcal{A}(u_{\text{in}}) = \nu A$, and in this case the $\Lambda_j = \lambda_j$ behaves like j asymptotically (see (2.5)). Indeed, $T_1(s) \sim s$ in this special case.

Proof. Let $m < \infty$ be finite, and let $\Lambda_1, \ldots, \Lambda_K$ be the eigenvalues of $\mathcal{N}_m(u_m)$ repeated according to their multiplicities such that $\Re\Lambda_j \le s$ for $j = 1, 2, \ldots, K$. We will denote by E_K the linear space spanned by the generalized eigenfunctions of $\mathcal{N}_m(u_m)$ corresponding the the eigenvalues $\Lambda_1, \ldots, \Lambda_K$. Let $\psi_1, \ldots, \psi_K$ be an orthogonal basis of E_K with $|\psi_j| = \nu$ for $j = 1, 2, \ldots, K$, and let $\Pi_K : H_m \to E_K$ be the orthogonal projection. We set $v_m = \Phi_{\text{app}}(u_m)$ and observe that

$$(4.23) \quad \Re\Big(\sum_{j=1} K\Lambda_j\Big) = \Re\{\text{trace}(\Pi_K \mathcal{N}_m(u_m)\Pi_K)\}$$

$$= \nu^{-2}\Re\Big(\sum_{j=1} K(\mathcal{N}_m(u_m)\psi_j, \psi_j)\Big)$$

$$= \nu^{-2}\Re\Big\{\sum_{j=1} K\big[\nu(A\psi_j, \psi_j) + (B(\psi_j, u_m + v_m), \psi_j)$$

$$+ (B(u_m, \xi(\psi_j)), \psi_j) + (B(\xi(\psi_j), u_m), \psi_j)$$

$$+ \gamma(L(\psi_j + \xi(\psi_j)), \psi_j)\big]\Big\}.$$

Since

$$\nu A\xi(\psi_j) + Q_m[B(u_m, \psi_j) + B(\psi_j, u_m) + \gamma L\psi_j] = 0,$$

by taking the inner product in H of the above equation with $\xi(\psi_j)$, and by using (2.6), we have

$$(B(u_m, \xi(\psi_j)), \psi_j) = \nu(A\xi(\psi_j), \xi(\psi_j)) + (B(\psi_j, u_m), \xi(\psi_j))$$

$$+ \gamma(L\psi_j, \xi(\psi_j)).$$

Substituting $(B(u_m, \xi(\psi_j), \psi_j))$ in (4.23) gives us

$$\Re\Big(\sum_{j=1} K\Lambda_j\Big) = \nu^{-2}\Re\Big\{\sum_{j=1} K\big[\nu(A\psi_j, \psi_j) + \nu(A\xi(\psi_j), \xi(\psi_j)) + \gamma(L\psi_j, \xi(\psi_j))$$

$$+ (B(\xi(\psi_j), u_m), \psi_j) + (B(\psi_j, u_m), \xi(\psi_j))$$

$$+ (B(\psi_j, u_m + v_m), \psi_j) + \gamma(L(\psi_j + \xi(\psi_j)), \psi_j)\big]\Big\}.$$

By (2.21), we obtain

$$\Re\Big(\sum_{j=1} K\Lambda_j\Big) \geq \nu^{-2}\sum_{j=1} K\Big[\nu\|\psi_j\|^2 + \nu\|\xi(\psi_j)\|^2 - |(B(\psi_j, u_m + v_m), \psi_j)|$$

$$- |(B(\xi(\psi_j), u_m), \psi_j)| - |(B(\psi_j, u_m), \xi(\psi_j))|$$

$$- \gamma\nu C_L^{1/2}\lambda_1^{1/2}|\psi_j|(\|\psi_j\| + 2\|\xi(\psi_j)\|)\Big].$$

Now let us estimate each term in the right hand side of the above inequality. First by the Cauchy-Schwarz inequality and Lieb-Thirring inequality (2.12), we reach

$$\sum_{j=1} K|(B(\psi_j, u_m + v_m), \psi_j)| \leq \left(\int_\Omega \left(\sum_{j=1} K\psi_j(x) \cdot \psi_j(x) \right) dx \right)^{1/2} \|u_m + v_m\|$$

$$\leq \left(C_2 v^2 \sum_{j=1} K\|\psi_j\|^2 \right)^{1/2} \|u_m + v_m\|$$

$$\leq \frac{v}{4} \sum_{j=1} K\|\psi_j\|^2 + C_2 \|u_m + v_m\|^2.$$

By using (2.7)-(2.10), we obtain

$$|(B(\xi(\psi_j), u_m), \psi_j)| \leq C_1 R |\xi(\psi_j)| \, \|\xi(\psi_j)\|,$$

and

$$|(B(\psi_j, u_m), \xi(\psi_j))| \leq C_1 R |\xi(\psi_j)| \, \|\xi(\psi_j)\|.$$

Thanks to the Cauchy-Schwarz inequality, we get

$$\gamma v C_L^{1/2} \lambda_1^{1/2} |\psi_j| (\|\psi_j\| + 2\|\xi(\psi_j)\|)$$

$$\leq \frac{v}{4} \|\psi_j\|^2 + \gamma^2 v C_L \lambda_1 |\psi_j|^2 + 2\gamma v C_L^{1/2} \lambda_1^{1/2}] |\psi_j| \, \|\xi(\psi_j)\|.$$

On the other hand,

$$\sum_{j=1} K\|\psi_j\|^2 = \sum_{j=1} K(A\psi_j, \psi_j) \geq \sum_{j=1} Kv^2 \lambda_j,$$

$$Ks \geq \Re\left(\sum_{j=1} K\Lambda_j \right).$$

Therefore, by applying (4.12), we get

$$Ks \geq \Re\left(\sum_{j=1} K\Lambda_j \right) \geq \sum_{j=1} K \left[\frac{v}{2} \lambda_j - 3v\gamma^2 C_L \lambda_1 - \frac{2C_1^2 \lambda_1^{1/2} R^2}{v\lambda_{m+1}^{1/2}} \right] - \frac{C_2 \|u_m + v_m\|^2}{v}.$$

By using (2.5) we reach

$$Ks \geq \frac{v\lambda_1}{2C_0} \frac{K(K+1)}{2} - 3v\gamma^2 C_L \lambda_1 K - \frac{2C_1^2 \lambda_1^{1/2} R^2}{\lambda_{m+1}^{1/2}} K - \frac{C_2}{v} (R + N_4(\gamma, R, v, f)),$$

which implies (4.21).   ∎

4.1. Estimates for the resolvents in the nonlinear Galerkin case. In this section we define as in Prodi [67] the parabola-like set $\tau(R) \subset \mathbb{C}$, by setting

$$(4.24) \quad \tau(R) = \left\{ z \in \mathbb{C} : \exists x \geq \lambda_1, \ |z - vx| \right.$$

$$\leq [C_1(R + \varphi(R)) + \gamma v \lambda_1^{1/2} C_L^{1/2}] x^{1/2}$$

$$\left. + \left[\frac{3C_1 R}{2\lambda_1^{1/4} \lambda_{m+1}^{1/4}} + \frac{\gamma v \lambda_1^{1/2} C_L^{1/2}}{\lambda_{m+1}^{1/2}} \right] x \right\},$$

where C_1 and C_L are as in (2.7) and (2.21) respectively, and

$$\zeta(R) = \frac{|f|}{v\lambda_{m+1}^{1/2}} + \frac{C_1 R^2}{v\lambda_{m+1}^{1/4}} + \frac{\gamma \lambda_1^{1/2} C_L^{1/2}}{\lambda_{m+1}^{1/2}}.$$

Denote by

$$(4.25) \qquad\qquad \bar{M} = \max \left\{ \frac{1}{C_0 \lambda_1^2} \left(\frac{6C_1 R}{v} \right)^4, \frac{16\gamma^2 C_L}{C_0} \right\}.$$

It is clear that when $m \geq \bar{M}$, $\tau(R)$ is a parabola-like set. Now and later we assume that $m \geq \bar{M}$.

Similar to Section 3 on page 44, we can show the following two lemmas.

Lemma 4.5. *Let $R > 0$, $u_m \in V$ with $\|u_m\| \leq R$. Then*

- *the spectra of $\mathcal{N}_m(u_m)$, $m = 1, \ldots, \infty$, are in $\tau(R)$;*
- *for every $z \notin \tau(R)$ the following inequalities hold:*

$$(4.26) \quad |(\mathcal{N}_m(u_m) - z)^{-1}v| \leq \mathrm{dist}(z, \tau(R))^{-1}|v| = N_H(\gamma, R, v, f, z)|v|,$$

$$(4.27) \quad \|(\mathcal{A}_m(u_0) - z)^{-1}v\| \leq N_V(\gamma, v, R, z)|v|, \quad \forall v \in H_m,$$

$$(4.28) \quad |A(\mathcal{A}_m(u_0) - z)^{-1}v| \leq N_A(\gamma, v, R, z)|v|, \quad \forall v \in H_m,$$

where

(4.29) $N_V(\gamma, R, \nu, f, z)$

$$= \left(\frac{2N_H(\gamma, R, \nu, f, z)}{\nu}\right)^{1/2} + \left(\frac{2|z|}{\nu}\right)^{1/2} N_H(\gamma, R, \nu, f, z)$$

$$+ \left[\frac{\sqrt{2}C_1(R + N_4(\gamma, R, \nu, f))}{\nu} + \gamma\sqrt{3C_L\lambda_1} + \frac{2C_1\lambda_1^{1/4}R}{\nu\lambda_{m+1}^{1/4}}\right]$$

$$\times N_H(\gamma, R, \nu, f, z),$$

(4.30) $N_A(\gamma, R, \nu, f, z)$

$$= \frac{1}{\nu} + \left(\frac{2C_1R}{\nu} + \gamma C_L^{1/2}\lambda_1^{1/2}\right)^2 N_H(\gamma, R, \nu, f, z)$$

$$+ \left[\left(\frac{2|z|}{\nu}\right)^{1/2} + \frac{2C_1}{\nu}(R + N_4(\gamma, R, \nu, f)) + \gamma C_L^{1/2}\lambda_1^{1/2}\right]$$

$$\times N_V(\gamma, R, \nu, f, z),$$

Lemma 4.6. *Let* $R > 0$, $u_m \in V$ *with* $\|u_m\| \le R$. *Then for every* $z \notin \tau(R)$ *the operators* $(\mathcal{N}_m(u_m) - z)^{-1}$, *for* $m = 1, \ldots, \infty$, *are H-S operators. Moreover,*

(4.31) $$\|(\mathcal{N}_m(u_m) - z)^{-1}\|_2 \le N_{HS}(\gamma, \nu, R, z),$$

where

(4.32) $$N_{HS}(\gamma, \nu, R, z) = \frac{C_0\pi}{\sqrt{6}\lambda_1}N_A(\gamma, \nu, R, z),$$

where $N_A(\gamma, R, \nu, f, z)$ *is as in* (4.30).

Following Section 3 on page 44, we denote by

$$\lambda_c = -\frac{1}{2\nu}[C_1(R + \zeta(R)) + \gamma\nu\lambda_1^{1/2}C_L^{1/2}]^2,$$

$$\widetilde{M} = N_7\left(\gamma, 2\lambda_c + \frac{2}{|\lambda_c|}, R, \nu, f\right),$$

where N_7 is as (4.22).

Lemma 4.7. *Let* $R > 0$, $u_m \in V$ *with* $\|u_m\| \le R$, *and* $\rho, s > 0$, *and let* $\Lambda \in \mathbb{C}$ *satisfy* $|\Re\Lambda| \le s$. *Then for every* $m < \infty$, *if*

(4.33) $$\mathrm{dist}(\Lambda, \mathrm{Sp}(\mathcal{N}_m(u_m))) \ge \rho,$$

then we have

$$(4.34) \qquad |(\mathcal{N}_m(u_m) - \Lambda)^{-1}v| \le N_8(\gamma, \rho, s, R, \nu, f)|v|, \qquad \forall v \in H_m,$$

$$(4.35) \qquad \|(\mathcal{N}_m(u_m) - \Lambda)^{-1}v\| \le N_9(\gamma, \rho, s, R, \nu, f)|v|, \qquad \forall v \in H_m,$$

$$(4.36) \qquad |A(\mathcal{N}_m(u_0) - \Lambda)^{-1}v| \le N_{10}(\gamma, \rho, s, R, \nu, f)|v|, \qquad \forall v \in H_m,$$

where

$$(4.37) \quad N_8(\gamma, \rho, s, R, \nu, f)$$

$$= \begin{cases} N_H(\gamma, R, \nu, f, z), & \Lambda \notin \sigma(R - \lambda_c/2), \\[2mm] \exp\left\{\dfrac{1}{2} + \dfrac{5}{2}|s - 2\lambda_c|^2 N_{HS}^2(\gamma, \nu, R, |s - 2\lambda_c|)\right\} \\[2mm] \quad \times \dfrac{1}{\lambda_c|\lambda - 2\lambda_c|^2}\left(\dfrac{2|s - \lambda_c|}{\rho}\right)^{\widetilde{M}}, & \Lambda \in \sigma(R - \lambda_c/2), \end{cases}$$

$$(4.38) \quad N_9(\gamma, \rho, s, R, \nu, f)$$

$$= \left(\frac{2N_8(\gamma, \rho, s, R, \nu, f)}{\nu}\right)^{1/2} + \left(\frac{2s}{\nu}\right)^{1/2} N_8(\gamma, \rho, s, R, \nu, f)$$

$$+ \left[\frac{\sqrt{2}C_1(R + N_4(\gamma, R, \nu, f))}{\nu} + \gamma\sqrt{3C_L\lambda_1} + \frac{2C_1\lambda_1^{1/4}R}{\nu\lambda_{m+1}^{1/4}}\right]$$

$$\times N_8(\gamma, \rho, s, R, \nu, f),$$

$$(4.39) \quad N_{10}(\gamma, \rho, s, R, \nu, f)$$

$$= \frac{1}{\nu} + \left(\frac{2C_1R}{\nu} + \gamma C_L^{1/2}\lambda_1^{1/2}\right)^2 N_8(\gamma, \rho, s, R, \nu, f)$$

$$+ \left[\left(\frac{2s}{\nu}\right)^{1/2} + \frac{2C_1}{\nu}(R + N_4(\gamma, R, \nu, f)) + \gamma C_L^{1/2}\lambda_1^{1/2}\right]$$

$$\times N_9(\gamma, \rho, s, R, \nu, f),$$

and $N_{HS}(\gamma, \nu, R, |s - 2\lambda_c|)$ *is as in* (4.31).

 Proof. By using the same approach as for the estimate (3.26), we can get the estimate (4.34). We only need to show the estimates (4.35) and (4.36).

Let $v_m = \Phi_{\text{app}}(u_m)$ and for each $g \in H_m$, let $(\mathcal{N}_m(u_m) - \lambda)^{-1}g = w$, i.e.,

$$(4.40) \quad \nu A w + P_m[B(u_m + v_m, w) + B(w, u_m + v_m) + B(u_m, \xi(w))$$

$$+ B(\xi(w), u_m) + \gamma L(w + \xi(w))] - \lambda w = g,$$

and

$$(4.41) \qquad \nu A \xi(w) + Q_m(B(u_m, w) + B(w, u_m) + \gamma L w) = 0.$$

Taking the inner product, in H, of the equation (4.40) with w and the equation (4.41) with $\xi(w)$, we obtain

$$\nu \|w\|^2 + (B(w, u_m + v_m), w) + (B(u_m, \xi(w)), w)$$

$$+ (B(\xi(w), u_m), w) + \gamma(L(w + \xi(w)), w) - \lambda |w|^2 = (g, w),$$

and

$$\nu \|\xi(w)\|^2 + (B(u_m, w), \xi(w)) + (B(w, u_m), \xi(w)) + \gamma(L w, \xi(w)) = 0.$$

The summation of these two equations leads to

$$\nu \|w\|^2 + \nu \|\xi\|^2 \leq |(g, w)| + |\lambda| \, |w|^2 + |(B(w, u_m + v_m), w)|$$

$$+ |(B(\xi(w), u_m), w)| + |(B(w, u_m), \xi(w))|$$

$$+ \gamma |(L(w + \xi(w)), w)| + \gamma |(L w, \xi(w))|.$$

By using (2.7)-(2.10) and (2.21), we reach

$$\nu \|w\|^2 + \nu \|\xi\|^2 \leq |g| \, |w| + s|w|^2 + \gamma \nu C_L^{1/2} \lambda_1^{1/2} |w| (\|w\| + \|\xi\|)$$

$$+ C_1 |u_m + v_m|^{1/2} |A(u_m + v_m)|^{1/2} |w| \, \|w\|$$

$$+ \frac{2 C_1 \lambda_1^{1/4} R}{\lambda_{m+1}^{1/4}} |w| \, \|\xi\|.$$

By the Cauchy-Schwarz inequality, we conclude

$$\frac{\nu}{2} \|w\|^2 \leq |g| \, |w| + s|w|^2 + \frac{3}{2} \gamma^2 \nu C_L \lambda_1 |w|^2$$

$$+ \frac{C_1^2 |u_m + v_m| \, |A(u_m + v_m)| \, |w|^2}{\nu} + \frac{2 C_1^2 \lambda_1^{1/2} R^2}{\lambda_{m+1}^{1/2}} |w|^2.$$

By applying the estimate (4.34), we get (4.35).

Moreover, taking the inner product, in H, of the equation (4.40) with Aw gives

$$\nu |Aw|^2 \le s\|w\|^2 + |(g, Aw)|$$
$$+ |(B(u_m + v_m, w), Aw) + (B(w, u_m + v_m), Aw)$$
$$+ (B(u_m, \xi), Aw) + (B(\xi, u_m), Aw) + \gamma(Lw, Aw)|.$$

By using (2.7)-(2.10) and (2.20) with $\beta = \frac{1}{2}$, we get

$$\nu |Aw|^2 \le s\|w\|^2 + |g|\,|Aw| + C_1 |u_m + v_m|^{1/2} |A(u_m + v_m)|^{1/2} \|w\|\,|Aw|$$
$$+ C_1 \|u_m + v_m\|^{1/2} |A(u_m + v_m)|^{1/2} |w|^{1/2} \|w\|^{1/2} |Aw|$$
$$+ \frac{C_1 |Au_m|\,\|\xi\|}{\lambda_1^{1/2}} |Aw| + \frac{C_1 |Au_m|\,\|\xi\|}{\lambda_1^{1/4} \lambda_{m+1}^{1/4}} |Aw|$$
$$+ \gamma \nu C_L^{1/2} \lambda_1^{1/2} (\|w\| + \|\xi\|)|Aw|.$$

By using estimates (4.12) and (4.34), we obtain

$$\|\xi(w)\| \le \left(\frac{2C_1 R}{\nu} + \gamma C_L^{1/2} \lambda_1^{1/2} \right) N_8(\gamma, \rho, s, R, \nu, f)|g|,$$

By the Cauchy-Schwarz inequality, and using estimates (4.34) and (4.35) and the above inequality, we get (4.36). $\qquad\square$

Lemma 4.8. *Let $R > 0$ and u_0, $u_1 \in \mathcal{D}(A)$ with $|Au_0| \le \lambda_1^{1/2}R$, $|Au_1| \le \lambda_1^{1/2}R$, and let s, $\rho > 0$. Then there are two constants*

(4.42) $\qquad \eta(\gamma, \rho, s, R, \nu, f) = \dfrac{\nu}{2C_1 N_8(\gamma, \rho, s, R, \nu, f) N_{11}(\gamma, \rho, s, R, \nu, f)},$

and

(4.43) $\qquad T_2(\gamma, \rho, s, R, \nu, f) = \dfrac{4N_8^2(\gamma, \rho, s, R, \nu, f) N_{12}^2(\gamma, \rho, s, R, \nu, f)}{C_0 \nu^2},$

where

(4.44) $\quad N_{11}(\gamma, \rho, s, R, \nu, f) = \dfrac{N_5^{1/2}(\gamma, s, R, \nu, f) N_6^{1/2}(\gamma, s, R, \nu, f)}{\lambda_1^{1/4}}$

$$+ \nu^{1/2} N_6^{1/2}(\gamma, s, R, \nu, f),$$

(4.45) $\quad N_{12}(\gamma, \rho, s, R, \nu, f)$

$$= C_1 \lambda_1^{1/2} \{R + N_4(\gamma, R, \nu, f)[N_5(\gamma, s, R, \nu, f) + \nu^{1/2} N_6^{1/2}(\gamma, s, R, \nu, f)]$$

$$+ 4C_1 R N_0(\gamma, s, R, \nu, f) + 2C_1 N_6(\gamma, s, R, \nu, f)\}$$

$$+ \gamma \nu C_L^{1/2} \lambda_1^{1/2} [N_6(\gamma, s, R, \nu, f) + 2N_0(\gamma, s, R, \nu, f)],$$

*and $N_4(\gamma, R, \nu, f)$, $N_5(\gamma, s, R, \nu, f)$, $N_6(\gamma, s, R, \nu, f)$, and $N_8(\gamma, \rho, s, R, \nu, f)$
are as in (4.11), (4.15), (4.16), and (4.37) respectively, and $N_0(\gamma, R, \nu, f)$ is as
specified in (4.49), such that if*

$$\|P_m(u_0 - u_1)\| \le \eta(\gamma, \rho, s, R, \nu, f)$$

and

(4.46) $$(\mathrm{Sp}(\mathcal{N}_m(P_m u_0)))_s = \varnothing$$

holds for some $m \in [T_2(\gamma, \rho, s, R, \nu, f), \infty)$, then for every $n \in [m, \infty]$ we have

(4.47) $$(\mathrm{Sp}(\mathcal{N}_n(P_n u_1)))_s \subset B(\mathrm{Sp}(\mathcal{N}_m(P_m u_0)), \rho).$$

Proof. Let $u_m = P_m u_0$, $v_m = \Phi_{\mathrm{app}}(u_m)$; and $p_m = P_m u_1$, $q_m = Q_m u_1$;
and $\tilde{p}_n = P_n u_1$, $\tilde{q}_n = \Phi_{\mathrm{app}}(u_n)$. Suppose that $\lambda \in (\mathrm{Sp}(\mathcal{N}_n(\tilde{p}_n)))_s$, i.e., there is
a $w \in H_n$, $|w| = v$ such that $\mathcal{N}_n(\tilde{p}_n)w = \lambda w$ and $|\Re\lambda| \le s$. Denote by

$$w_m = P_m w \quad \text{and} \quad w_m^{\perp} = Q_m w.$$

If $\lambda \notin B(\mathrm{Sp}(\mathcal{N}_m(u_m)), \rho)$, i.e., $\mathrm{dist}(\lambda, \mathrm{Sp}(\mathcal{N}_m(u_m))) \ge \rho$, by Lemma 4.7 on
page 69, and using the same argument as in Lemma 3.7 on page 53, we get

(4.48) $\quad |\mathcal{N}_m(u_m)w_m - \lambda w_m| \ge \dfrac{|w_m|}{N_8(\gamma, \rho, s, \nu, R)}$

$$\ge \frac{1}{N_8(\gamma, \rho, s, \nu, R)} \left(v - \frac{N_5(\gamma, s, \nu, R)}{\lambda_{m+1}^{1/2}} \right)$$

$$=: \mathcal{LH},$$

where $N_5(\gamma, s, R, \nu, f)$ is as in (4.15), and $N_8(\gamma, \rho, s, R, \nu, f)$ is as in (4.37).
On the other hand,

$$\mathcal{N}_m(u_m)w_m - \lambda w_m$$
$$= \mathcal{N}_m(u_m)w_m - P_m(\mathcal{N}_n(\tilde{p}_n)w)$$
$$= P_m[B(u_m + v_m - \tilde{p}_n - \tilde{q}_n, w_m) + B(\tilde{p}_n + \tilde{q}_n, -w_m^\perp)$$
$$\qquad + B(-w_m^\perp, u_m + v_m) + B(w, u_m + v_m - \tilde{p}_n - \tilde{q}_n) + B(u_m, \xi(w_m))$$
$$\qquad + B(\xi(w_m), u_m) - B(\tilde{p}_n, \zeta(w)) - B(\zeta(w), \tilde{p}_n) - \gamma L w_m^\perp]$$
$$= P_m[B(u_m - p_m, w_m) + B(w, u_m - p_m) + B(\xi(w_m), u_m) + B(u_m, \xi(w_m))$$
$$\qquad + B(v_m - q_m, w_m) + B(w, v_m - q_m) + B(\zeta(w), \tilde{p}_n) + B(\tilde{p}_n, \zeta(w))$$
$$\qquad + B(-w_m^\perp, u_m + v_m) + B(\tilde{p}_n + \tilde{q}_n, -w_m^\perp) - \gamma L w_m^\perp],$$

where $\xi(w_m)$ and $\zeta(w)$ satisfy following equations

$$\nu A\xi + Q_m[B(u_m, w_m) + B(w_m, u_m) + \gamma L w_m] = 0,$$
$$\nu A\zeta + Q_n[B(\tilde{p}_n, w) + B(w, \tilde{p}_n) + \gamma L w] = 0.$$

By using (2.7), (2.8), and (2.20) with $\beta = \frac{1}{2}$, it is clear that

$$|A\xi| \le N_0(\gamma, s, R, \nu, f),$$
$$|A\zeta| \le N_0(\gamma, s, R, \nu, f),$$

where

$$(4.49) \quad N_0(\gamma, s, R, \nu, f) = \frac{C_1 R}{\nu}(N_5(\gamma, s, R, \nu, f) + \nu^{1/2}\lambda_1^{1/2}N_5^{1/2}(\gamma, s, R, \nu, f))$$
$$+ \gamma C_L^{1/2}\lambda_1^{1/2}N_5(\gamma, s, R, \nu, f).$$

By using the inequalities (2.8)-(2.10) and (2.20) with $\beta = \frac{1}{2}$, we obtain

$$|(\mathcal{N}_m(u_m) - \lambda)w_m|$$
$$\le C_1\|u_m - p_m\| \left(\frac{\|w_m\|^{1/2}|Aw_m|^{1/2}}{\lambda_1^{1/4}} + |w|^{1/2}|Aw|^{1/2}\right)$$
$$+ C_1\left[\frac{|Av_m| + |Aq_m|}{\lambda_{m+1}^{1/2}}(\|w_m\| + |w|^{1/2}|Aw|^{1/2})\right.$$
$$\left. + \frac{2(|A\xi||Au_m| + |A\zeta||A\tilde{p}_n|)}{\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\right]$$
$$+ \gamma\nu C_L^{1/2}\lambda_1^{1/2}[\|w_m^\perp\| + \|\xi\| + \|\zeta\|]$$
$$+ C_1\left[\frac{|A(\tilde{p}_n + \tilde{q}_n)|}{\lambda_1^{1/2}}\|w_m^\perp\| + |w_m^\perp|^{1/2}|Aw_m^\perp|^{1/2}\|\tilde{p}_n + \tilde{q}_n\|\right] \quad \le$$

$$\leq C_1 \left[\frac{N_5^{1/2}(\gamma, s, R, \nu, f) N_6^{1/2}(\gamma, s, R, \nu, f)}{\lambda_1^{1/4}} + \nu^{1/2} N_6^{1/2}(\gamma, s, R, \nu, f) \right] \| u_m - p_m \|$$

$$+ \frac{1}{\lambda_{m+1}^{1/2}} [C_1 \lambda_1^{1/2} (R + N_4(\gamma, R, \nu, f))(N_5(\gamma, s, R, \nu, f) + \nu^{1/2} N_6^{1/2}(\gamma, s, R, \nu, f))$$

$$+ 4C_1 R N_0(\gamma, s, R, \nu, f) + 2C_1 N_6(\gamma, s, R, \nu, f) + \gamma \nu C_L^{1/2} \lambda_1^{1/2})$$

$$\times (N_6(\gamma, s, R, \nu, f) + 2N_0(\gamma, s, R, \nu, f))] =: \mathcal{RH},$$

where $N_4(\gamma, R, \nu, f)$, $N_5(\gamma, s, R, \nu, f)$, and $N_6(\gamma, s, R, \nu, f)$ are as in (4.11), (4.15), and (4.16) respectively, and $N_0(\gamma, s, R, \nu, f)$ is as in (4.49). Therefore, we have

$$\mathcal{LH} \leq \mathcal{RH}.$$

But if we let

$$\| u_m - p_m \| \leq \eta(\gamma, \rho, s, R, \nu, f) \quad \text{and} \quad m \geq T_2(\gamma, \rho, s, R, \nu, f),$$

where $\eta(\gamma, \rho, s, R, \nu, f)$ and $T_2(\gamma, s, \rho, R, \nu, f)$ are as in (4.42) and (4.43), then

$$\mathcal{LH} = \frac{1}{N_8(\gamma, \rho, s, R, \nu, f)} \left[\nu - \frac{N_5(\gamma, s, \nu, R)}{\lambda_{m+1}^{1/2}} \right] \geq \mathcal{RH}$$

$$= C_1 \left[\frac{N_5^{1/2}(\gamma, s, R, \nu, f) N_6^{1/2}(\gamma, s, R, \nu, f)}{\lambda_1^{1/4}} + \nu^{1/2} N_6^{1/2}(\gamma, s, R, \nu, f) \right] \| u_m - p_m \|$$

$$+ \frac{1}{\lambda_{m+1}^{1/2}} [C_1 \lambda_1^{1/2} (R + N_4(\gamma, R, \nu, f))(N_5(\gamma, s, R, \nu, f) + \nu^{1/2} N_6^{1/2}(\gamma, s, R, \nu, f))$$

$$+ 4C_1 R N_0(\gamma, s, R, \nu, f) + 2C_1 N_6(\gamma, s, R, \nu, f) + \gamma \nu C_L^{1/2} \lambda_1^{1/2})$$

$$\times (N_6(\gamma, s, R, \nu, f) + 2N_0(\gamma, s, R, \nu, f))].$$

This leads into contradiction. Thus $\lambda \in B(\text{Sp}(\mathcal{N}_m(u_0)), \rho)$, i.e.,

$$(\text{Sp}(\mathcal{N}_n(u_1)))_s \subset B(\text{Sp}(\mathcal{N}_m(u_0)), \rho),$$

which completes the proof of the lemma. $\qquad\qquad\square$

4.2. The main result in the nonlinear Galerkin case. Next we present our main theorem of this section.

Theorem 4.1. *Let $\sigma > 0$ be given, and let $p_m^\infty + q_m^\infty$ be a steady state solution of the Galerkin approximation equation (4.1) of the closed-loop system (2.18) such that $|p_m^\infty + q_m^\infty| \leq N_1$. Then there exists a constant*

$$(4.50) \quad M_{\mathrm{NLG}}(\gamma, \sigma, N_1, \nu, f) = \max\{T_3(\gamma, \sigma, N_1, \nu, f), T_4(\gamma, \sigma, N_1, \nu, f)\},$$

where $T_3(\gamma, \sigma, N_1, \nu, f)$, $T_4(\gamma, \sigma, N_1, \nu, f)$ are specified as in (4.70) and (4.74), such that if

$$(4.51) \quad (\mathrm{Sp}(\mathcal{N}_m(p_m^\infty)))_\sigma = \varnothing$$

holds for some $m \geq M_{\mathrm{NLG}}(\gamma, \sigma, N_1, \nu, f)$, then we have

  (i) *For every $n \geq m$, there is a stationary solution $p_n^\infty + q_n^\infty$ of the nonlinear Galerkin approximation equations (4.1) and (4.2) of order n, for which*

$$(4.52) \quad (\mathrm{Sp}(\mathcal{N}_n(p_n^\infty)))_{\sigma/2} = \varnothing,$$

$$(4.53) \quad |p_n^\infty + q_n^\infty - p_m^\infty - q_m^\infty| \leq \frac{N_{13}(\gamma, \sigma, \nu, f)}{\lambda_{m+1}^{3/2}},$$

  where

$$(4.54) \quad N_{13}(\gamma, \sigma, N_1, \nu, f)$$

$$= \frac{4}{3}\Bigg[\frac{\lambda_1 N_4(\gamma, N_3, \nu, f)}{\nu}$$

$$\times \; (2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2}) \left(\gamma\nu C_L^{1/2} + \frac{1}{\lambda_1^{1/2}} + \frac{2C_1 N_4^2(\gamma, N_3, \nu, f)}{\lambda_{m+1}^{1/2}} \right)$$

$$+ \; 2C_1\lambda_1 N_4^2(\gamma, N_3, \nu, f)$$

$$+ \; \frac{\lambda_1((1 + \lambda_1^{1/2})\lambda_{m+1}^{1/2} + 2C_1\lambda_1^{1/2}N_4^2(\gamma, N_3, \nu, f)N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f))}{2C_1\lambda_{m+1}^{1/2}\nu N_{10}^2(\gamma, \sigma/2, \sigma, N_3, \nu, f)}$$

$$\times \; (4C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})$$

$$+ \; \frac{\lambda_1^{3/2}}{32C_1^2\nu N_{10}^2(\gamma, \sigma/2, \sigma, N_3, \nu, f)}\left(\lambda_1^{1/2} + 2N_{10}\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right) \right)$$

$$\times \left(2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2} + 1 + \frac{2C_1\lambda_1^{1/2}N_4^2(\gamma, N_3, \nu, f)}{\lambda_{m+1}^{1/2}} \right) \Bigg]$$

$$\times N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right),$$

where $N_9(\gamma, \sigma/2, \sigma, N_3, \nu, f)$ and $N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)$ are as (4.39) and (4.39) respectively.

(ii) *There is a stationary solution u^∞ of the closed-loop system (2.18) such that*

$$(4.55) \qquad p_n^\infty + q_n^\infty \to u^\infty \quad \text{as } n \to \infty.$$

Moreover, u^∞ is hyperbolic and

$$(4.56) \qquad (\mathrm{Sp}(\mathcal{N}_\infty(u^\infty)))_{\sigma/2} = \varnothing,$$

$$(4.57) \qquad |u^\infty - p_n^\infty - q_n^\infty| \le \frac{N_{13}(\gamma, \sigma, \nu, f)}{\lambda_{n+1}^{3/2}}.$$

Proof. We will use the contraction mapping theorem to show the existence of $p_n^\infty + q_n^\infty$.

First let us establish the map. Denote by

$$w = p_n^\infty + q_n^\infty - p_m^\infty - q_m^\infty, \quad w_n = P_n w, \quad w_n^\perp = Q_n w, \quad \hat{q}_m^\infty = P_n q_m^\infty,$$

$$\tilde{q}_m^\infty = Q_n q_m^\infty, \quad u_n = p_m^\infty + \hat{q}_m^\infty, \quad v_n = \Phi_{\mathrm{app}}(u_n).$$

If $p_n^\infty + q_n^\infty$ is a stationary solution of the nonlinear Galerkin approximation equations (4.1) and (4.2) based on n-eigenmodes, then w should satisfy

$$\nu A(u_n + w_n) + P_n[B(p_m^\infty + q_m^\infty + w, p_m^\infty + q_m^\infty + w) - B(\tilde{q}_m^\infty + w_n^\perp, \tilde{q}_m^\infty + w_n^\perp)$$
$$+ \gamma P_n L(u_n + v_n + w_n + w_n^\perp)] = P_n f,$$

$$\nu A(\tilde{q}_m^\infty + w_n^\perp) + Q_n[B(u_n + w_n, u_n + w_n) + \gamma L(u_n + w_n)] = Q_n f,$$

that is,

$$(4.58) \quad \mathcal{N}_n(u_n)w_n = \nu A w_n + P_n[B(u_n + v_n, w_n) + B(w_n, u_n + v_n)$$
$$+ B(u_n, \xi(w_n)) + B(\xi(w_n), u_n) + \gamma L(w_n + \xi(w_n))] = \Psi(w_n),$$

$$(4.59) \qquad \nu A\xi + Q_n[B(u_n, w_n) + B(w_n, u_n) + \gamma L w_n] = 0,$$

$$(4.60) \qquad \nu A(v_n - \tilde{q}_m^\infty) + Q_n[B(\hat{q}_m^\infty, u_n) + B(p_m^\infty, \hat{q}_m^\infty) + \gamma L \hat{q}_m^\infty] = 0,$$

$$(4.61) \qquad \nu A w_n^\perp = \nu A(v_n - \tilde{q}_m^\infty) + \nu A\xi(w_n) - Q_n B(w_n, w_n),$$

where

(4.62) $\Psi(w_n)$

$$= P_n[B(q_m^\infty, q_m^\infty) - B(\tilde{q}_m^\infty, \tilde{q}_m^\infty) + B(\hat{q}_m^\infty, w_n^\perp)$$

$$+ \; B(w_n^\perp, \hat{q}_m^\infty) + B(u_n, \xi(w_n) + v_n - \tilde{q}_m^\infty - w_n^\perp)$$

$$+ \; B(\xi(w_n) + v_n - \tilde{q}_m^\infty - w_n^\perp, u_n) + B(w_n, v_n - \tilde{q}_m^\infty - w_n^\perp)$$

$$+ \; B(v_n - \tilde{q}_m^\infty - w_n^\perp, w_n) - B(w_n, w_n) + \gamma L(\xi(w_n) - w_n^\perp)].$$

From (4.61), we get

(4.63) $$w_n^\perp = v_n - \tilde{q}_m^\infty + \xi(w_n) - (\nu A)^{-1} Q_n B(w_n, w_n).$$

In other words, if we can show the existence of w_n, then we get $w_n^\perp$ by (4.63). By Lemma 4.8 on page 72, we have $(\mathrm{Sp}(\mathcal{N}_n(u_n)))_{\sigma/2} = \varnothing$, when $m \geq T_2(\sigma, \sigma/2, \nu, N_3)$. So $\mathcal{N}_n(u_n)$ is invertible. Denote by

(4.64) $$\mathcal{F}(w_n) = (\mathcal{N}_n(u_n))^{-1}\Psi(w_n),$$

where $\Psi(w_n)$ is as in (4.62).

Next we will show that $\mathcal{F}$ maps

(4.65) $$\mathcal{B}_n(r) = \{w_n \in H_n : |Aw_n| \leq r\}$$

into $\mathcal{B}_n(r)$, where

(4.66) $$r = \frac{\lambda_1}{4C_1 N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)}.$$

By the estimate (3.28) of the Lemma 4.7 on page 69, we get

$$|A\mathcal{F}(w_n)| \leq N_{10}\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)|\Psi(w_n)|.$$

Now we will get an estimate for $|\Psi(w_n)|$. First, by using (2.7)-(2.11) and (2.20) with $\beta = \frac{1}{2}$, we get the following estimates

(4.67) $$|A\xi(w_n)| \leq \frac{4C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2}}{\nu\lambda_{m+1}^{1/2}}|Aw_n|$$

(4.68) $$|A(v_n - \tilde{q}_m^\infty)| \leq \frac{\lambda_1^{1/2} N_4(\gamma, N_3, \nu, f)(2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})}{\nu\lambda_{m+1}^{1/2}}$$

(4.69) $$|Aw_n^\perp| \leq \frac{1}{\nu\lambda_{m+1}^{1/2}}\left(\lambda_1^{1/2} N_4(\gamma, N_3, \nu, f)(2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})\right.$$

$$\left. + \; (4C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})|Aw_1| + \frac{2C_1|Aw_1|^2}{\lambda_1^{1/2}}\right).$$

Moreover, we use (2.7)-(2.11), (2.20) with $\beta = \frac{1}{2}$, and the above estimates to obtain

$|\Psi(w_n)|$

$$\leq \frac{2C_1 N_3 |A(\xi(w_n) + v_n - \tilde{q}_m^\infty - w_2)|}{\lambda_{m+1}^{1/2}} + \frac{2C_1 \lambda_1 N_4^2(\gamma, v, f)}{\lambda_{m+1}}$$

$$+ \frac{2C_1 \lambda_1^{1/2} N_4(\gamma, N_3, v, f)|Aw_n^\perp|}{\lambda_{m+1}} + \frac{2C_1 |Aw_n|}{\lambda_1^{1/2}\lambda_{m+1}^{1/2}}|A(v_n - \tilde{q}_m^\infty - w_n^\perp)|$$

$$+ \frac{C_1 |Aw_n|^2}{\lambda_1} + \gamma v C_L^{1/2} \lambda_1^{1/2} \|\xi(w_n) - w_n^\perp\|$$

$$\leq \frac{4C_1^2}{v\lambda_1 \lambda_{m+1}}|Aw_n|^3 + \frac{2C_1 \lambda_1^{1/2} N_4(\gamma, N_3, v, f)}{v\lambda_{m+1}^{3/2}}(4C_1 N_3 + \gamma v C_L^{1/2}\lambda_1^{1/2})|Aw_n|$$

$$+ \frac{C_1 |Aw_n|^2}{\lambda_1}\left[1 + \frac{1}{\lambda_{m+1}}\left(\frac{12C_1 \lambda_1^{1/2} N_3}{v} + 2\gamma C_L^{1/2}\lambda_1\right.\right.$$

$$\left.\left. + \frac{4C_1 \lambda_1 N_4(\gamma, N_3, v, f)}{v\lambda_{m+1}^{1/2}}\right)\right]$$

$$+ \frac{\lambda_1 N_4(\gamma, N_3, v, f)}{\lambda_{m+1}}(2C_1 N_3 + \gamma v C_L^{1/2}\lambda_1^{1/2})$$

$$\times \left(\frac{2C_1 N_4(\gamma, N_3, v, f)}{v\lambda_{m+1}^{1/2}} + \gamma C_L^{1/2}\right) + \frac{2C_1 \lambda_1 N_4^2(\gamma, N_3, v, f)}{\lambda_{m+1}}.$$

Therefore when $|Aw_n| \leq r$, r is as in (4.66), and

$$(4.70) \qquad m \geq T_3(\gamma, \sigma, N_1, v, f) = \max\left\{T_2\left(\sigma, \frac{\sigma}{2}, v, N_3\right), T_5, T_6\right\},$$

where

$$T_5 = \left[\frac{16C_1 N_{10}^2(\gamma, \sigma/2, \sigma, N_3, v, f)N_{14}(\gamma, \sigma, N_1, v, f)}{3C_0\lambda_1}\right],$$

$$T_6 = \left[\frac{3N_{15}(\gamma, \sigma, N_1, v, f)}{C_0}\right],$$

and

(4.71) $N_{14}(\gamma, \sigma, N_1, \nu, f)$

$$= \frac{\lambda_1}{16C_1\nu N_{10}^3(\gamma, \sigma/2, \sigma, N_3, \nu, f)} + 2C_1 N_4^2(\gamma, N_3, \nu, f)$$

$$+ \frac{12C_1\lambda_1^{1/2}N_3 + 2\gamma\nu C_L^{1/2}\lambda_1 + 4C_1\lambda_1^{1/2}N_4(\gamma, N_3, \nu, f)}{16C_1\nu N_{10}^2(\gamma, \sigma/2, \sigma, N_3, \nu, f)}$$

$$+ N_4(\gamma, N_3, \nu, f)(2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})$$

$$\times \left(\frac{2C_1 N_4(\gamma, N_3, \nu, f)}{\nu\lambda_1^{1/2}} + \gamma C_L^{1/2} \right)$$

$$+ \frac{N_4(\gamma, N_3, \nu, f)(4C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})}{2\nu N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)},$$

(4.72) $N_{15}(\gamma, \sigma, N_1, \nu, f)$

$$= 2\gamma C_1 C_L^{1/2} + \frac{3C_1}{4\nu N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)}$$

$$+ \frac{2C_1^2}{\nu\lambda_1^{1/2}}(3N_3 + N_4(\gamma, N_3, \nu, f))$$

$$+ \frac{2C_1^2 N_4(\gamma, N_3, \nu, f)N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)}{\nu\lambda_1}$$

$$\times(4C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2}),$$

we have $|A\mathcal{F}(w_1)| \leq r$, that is, $\mathcal{F}$ maps $\mathcal{B}_n(r)$ into $\mathcal{B}_n(r)$.

Finally, we will show that the map $\mathcal{F}$ is a contraction map on $\mathcal{B}_n(r)$. Suppose $w_n, \widetilde{w}_n \in \mathcal{B}_n(r)$. Then

$$\mathcal{F}(w_n) - \mathcal{F}(\widetilde{w}_n)$$

$$= (\mathcal{N}_n(u_n))^{-1}\Psi(w_n) - (\mathcal{N}_n(u_n))^{-1}\Psi(\widetilde{w}_n)$$

$$= (\mathcal{N}_n(u_n))^{-1}P_n\big[B(u_n, \xi(w_n) - \xi(\widetilde{w}_1) - w_n^\perp + \widetilde{w}_n^\perp) + B(\hat{q}_m^\infty, w_n^\perp - \widetilde{w}_n^\perp)$$

$$+ B(\xi(w_n) - \xi\widetilde{w}_n - w_n^\perp + \widetilde{w}_n^\perp, u_n)$$

$$+ B(w_n^\perp - \widetilde{w}_n^\perp, \hat{q}_m^\infty) - B(\widetilde{w}_n, w_n^\perp - \widetilde{w}_n^\perp)$$

$$- B(w_n^\perp - \widetilde{w}_n^\perp, \widetilde{w}_n) + B(w_n - \widetilde{w}_n, v_n - \tilde{q}_m^\infty - w_n^\perp)$$

$$+ B(v_n - \tilde{q}_m^\infty - w_n^\perp, w_n - \widetilde{w}_n) - B(w_n - \widetilde{w}_n, w_n)$$

$$- B(\widetilde{w}_n, w_n - \widetilde{w}_n) + \gamma L(\xi(w_n) - \xi(\widetilde{w}_n) - w_n^\perp + \widetilde{w}_n^\perp)\big],$$

$$\nu A(\xi(w_n) - \xi(\widetilde{w}_n))$$

$$+ Q_n[B(u_n, w_n - \widetilde{w}_n) + B(w_n - \widetilde{w}_n, u_n) + \gamma L(w_n - \widetilde{w}_n)] = 0,$$

$$\nu A(w_n^\perp - \widetilde{w}_n^\perp)$$

$$= \nu A(\xi(w_n) - \xi(\widetilde{w}_n)) - Q_n[B(w_n - \widetilde{w}_n, w_n) + B(\widetilde{w}_n, w_n - \widetilde{w}_n)].$$

Then by using (2.7)-(2.11) and (2.20) with $\beta = \frac{1}{2}$, we obtain

$$|A(\xi(w_n) - \xi(\widetilde{w}_n))| \le \frac{4C_1 N_3 + \gamma \nu C_L^{1/2} \lambda_1^{1/2}}{\nu \lambda_{m+1}^{1/2}} |A(w_n - \widetilde{w}_n)|,$$

$$|A(w_n^\perp - \widetilde{w}_n^\perp)| \le \frac{4C_1 \lambda_1^{1/2} N_3 + \gamma \nu C_L^{1/2} \lambda_1 + 4C_1 |Aw_n|}{\nu \lambda_1^{1/2} \lambda_{m+1}^{1/2}} |A(w_n - \widetilde{w}_n)|.$$

As a result of the estimates above, (2.7)-(2.11), and (2.20) with $\beta = \frac{1}{2}$, we have

$$|A(\mathcal{F}(w_n) - \mathcal{F}(\widetilde{w}_n))|$$

$$\le C_1 N_{10}\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right) \left[\frac{2|Aw_n|}{\lambda_1} + \frac{12 C_1^2 |Aw_1|^2}{\nu \lambda_1 \lambda_{m+1}} + \frac{8 C_1 \gamma C_L^{1/2} |Aw_n|}{\lambda_{m+1}} \right.$$

$$+ \frac{8 C_1^2 |Aw_n|}{\nu \lambda_{m+1}} \left(\frac{3 N_3}{\lambda_1^{1/2}} + \frac{N_4(\gamma, \gamma, N_3, \nu, f)}{\lambda_{m+1}^{1/2}} \right)$$

$$\left. + \frac{2 C_1 \lambda_1^{1/2} N_4(\gamma, N_3, \nu, f)}{\nu \lambda_{m+1}^{3/2}} \left(4 C_1 N_3 + \gamma \nu C_L^{1/2} \lambda_1^{1/2} \right) \right].$$

Therefore, by (4.66) and (4.70), we get

$$|A(\mathcal{F}(w_1) - \mathcal{F}(\widetilde{w}_1))| \le \frac{5}{6} |A(w_1 - \widetilde{w}_1)|.$$

Thus, $\mathcal{F}$ is a contraction map on $\mathcal{B}_n(r)$. By the contraction mapping theorem, there are a unique $w_n^* \in \mathcal{B}_n(r)$ and a unique $w_n^{*\perp}$ given by (4.63), such that

$$p_n^\infty + q_n^\infty = p_m^\infty + q_m^\infty + w_n + w_n^{*\perp}$$

is a steady state of the nonlinear Galerkin equations (4.1) and (4.2) of order n, and

$$(4.73) \qquad |A(p_n^\infty + q_n^\infty - p_m^\infty - q_m^\infty)| \le \frac{\lambda_1}{4 C_1 N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)},$$

when $m \geq T_3(\gamma, \sigma, N_1, \nu, f)$ and $T_3(\gamma, \sigma, N_1, \nu, f)$ is as in (4.70).

Now we will show the estimate (4.53). First notice that, by using the same argument as in the proof of (3.56) and (3.57), for all $w_1, w_2 \in H_m$, we obtain

$$|(\mathcal{N}_m(p_m^\infty))^{-1}B(w_1, w_2)| \leq C_1 N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)|w_1|^{1/2}|Aw_1|^{1/2}|w_2|,$$

$$|(\mathcal{N}_m(p_m^\infty))^{-1}B(w_1, w_2)| \leq C_1 N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)|w_1||w_2|^{1/2}|Aw_2|^{1/2}.$$

Then we apply the above estimates and the estimate (4.36) of the Lemma (4.7) to (4.64), and obtain

$$|w_n^*| = |\mathcal{F}(w_n^*)|$$

$$\leq C_1 N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)\left[2N_3|\xi(w_n^*) + v_n - \tilde{q}_m^\infty - w_n^{*\perp}|\right.$$

$$+ \frac{2\lambda_1 N_4^2(\gamma, N_3, \nu, f)}{\lambda_{m+1}^{3/2}} + \frac{2\lambda_1^{1/2} N_4^2(\gamma, N_3, \nu, f)|w_2|}{\lambda_{m+1}^{1/2}}$$

$$\left.+ \frac{2r(|v_n - \tilde{q}_m^\infty - w_n^{*\perp}|)}{\lambda_1^{1/2}}\right]$$

$$+ \frac{C_1 N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)|w_n^*|^{3/2}\|w_n^*\|^{1/2}}{\lambda_1^{1/4}}$$

$$+ \gamma\nu C_L^{1/2}\lambda_1^{1/2}N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)|\xi(w_1) - w_2|.$$

Since $|w_n^*|^{1/2}\|w_n^*\|^{1/2} \leq |Aw_n^*|/\lambda_1^{3/4}$ and $(C_1 r N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f))/\lambda_1 = \frac{1}{4}$, we get

$$|w_n^*| \leq \frac{4}{3}N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)\left[2C_1 N_3|(\nu A)^{-1}Q_n B(w_n^*, w_n^*)|\right.$$

$$+ \frac{2C_1\lambda_1^{1/2}N_4^2(\gamma, N_3, \nu, f)|w_n^{*\perp}|}{\lambda_{m+1}^{1/2}}$$

$$+ \frac{2rC_1(|\xi(w_n^*)| + |(\nu A)^{-1}Q_n B(w_n^*, w_n^*)|)}{\lambda_1^{1/2}}$$

$$+ \frac{2C_1\lambda_1 N_4^2(\gamma, N_3, \nu, f)}{\lambda_{m+1}^{3/2}}$$

$$\left.+ \gamma\nu C_L^{1/2}\lambda_1^{1/2}\left(|v_n - \tilde{q}_m^\infty| + |(\nu A)^{-1}Q_n B(w_n^*, w_n^*)|\right)\right].$$

Therefore, by using the estimates (4.67), (4.68), (4.69), and (4.73), we obtain

$$|w_n^*| \leq \frac{N_{16}(\gamma, \sigma, N_1, \nu, f)}{\lambda_{m+1}^{3/2}},$$

where

$$N_{16}(\gamma, \sigma, N_1, \nu, f) = \frac{4}{3} N_9\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)$$

$$\times \left[\frac{\lambda_1 N_4(\gamma, N_3, \nu, f)(2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})}{\nu} \left(\gamma\nu C_L^{1/2} + \frac{2C_1 N_4^2(\gamma, N_3, \nu, f)}{\lambda_{m+1}^{1/2}}\right) \right.$$

$$+ 2C_1\lambda_1 N_4^2(\gamma, N_3, \nu, f)$$

$$+ \frac{\lambda_1^{3/2}(\lambda_{m+1}^{1/2} + 2C_1 N_4^2(\gamma, N_3, \nu, f))N_{10}(\gamma, \sigma/2, \sigma, N_3, \nu, f)}{2C_1\lambda_{m+1}^{1/2}\nu N_{10}^2(\gamma, \sigma/2, \sigma, N_3, \nu, f)}$$

$$\times (4C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2})$$

$$+ \frac{\lambda_1^{3/2}}{32C_1^2\nu N_{10}^2(\gamma, \sigma/2, \sigma, N_3, \nu, f)}\left(\lambda_1^{1/2} + 2N_{10}\left(\gamma, \frac{\sigma}{2}, \sigma, N_3, \nu, f\right)\right.$$

$$\left.\left.\times \left(2C_1 N_3 + \gamma\nu C_L^{1/2}\lambda_1^{1/2} + \frac{2C_1\lambda_1^{1/2}N_4^2(\gamma, N_3, \nu, f)}{\lambda_{m+1}^{1/2}}\right)\right)\right].$$

The estimate (4.53) follows after this estimate and (4.69).
    Let

(4.74) $$T_4(\gamma, \sigma, N_1, \nu, f) = \left[\frac{N_{16}(\gamma, \sigma, N_1, \nu, f)}{C_0\lambda_1\eta(\gamma, \sigma, \sigma/2, \nu, N_3)}\right],$$

where $\eta(\gamma, \sigma, \sigma/2, \nu, N_3)$ is as in (4.42). When $m \geq M_{\text{NLG}}(\gamma, \sigma, N_1, \nu, f)$, it is clear that $\|w_n^*\| \leq \eta(\gamma, \sigma, \sigma/2, \nu, N_3)$, so the second part of the theorem has been proved.   □

5. Postprocessing the stable Galerkin steady states

In previous sections we have provided an explicit numerical criterion for the stabilization of steady state solutions of the Navier-Stokes equations with a certain class of linear feedback control operators. In this section we use an efficient numerical

method, the postprocessing Galerkin method, which was introduced in [38], [39], [40], [65], to improve the error estimates (3.47) and (4.57).

Here we consider only the Galerkin case. One can obtain similar results for the nonlinear Galerkin case. Suppose u_m^∞, $m \geq M_{\mathrm{GLK}}$ is the stable steady state solution of the Galerkin equations. u^∞ is the steady state solution of the closed-loop Navier-Stokes equations which we obtained by Theorem 3.2 on page 55. By using the postprocessing technique, we are able to get the following result.

Theorem 5.1. *We denote by* $\Phi_s : H_m \to H_m^\perp$ *the so called steady Approximate Inertial Manifold* ([76], *see also* [30], [35]), *which is defined implicitly by*

$$(5.1) \quad \Phi_s(u_m) = (\nu A)^{-1} Q_m \{ f - B(u_m + \Phi_s(u_m), u_m + \Phi_s(u_m)) \\ - \gamma L(u_m + \Phi_s(u_m)) \}$$

for $u_m \in H_m$, *provided* m *is large enough* (*see* [76]). *We define* v_m^∞ *implicitly by*

$$v_m^\infty = -(\mathcal{A}_m(u_m^\infty))^{-1} P_m [B(u_m^\infty, \Phi_s(u_m^\infty + v_m^\infty)) + B(\Phi_s(u_m^\infty + v_m^\infty), u_m^\infty)) \\ + \gamma L \Phi_s(u_m^\infty + v_m^\infty)],$$

and denote by

$$u^* = u_m^\infty + v_m^\infty + \Phi_s(u_m^\infty + v_m^\infty).$$

Then under the same assumptions as in Theorem 3.2 on page 55, we have

$$(5.2) \qquad |u^\infty - u^*| \leq \frac{K_{PP}(\gamma, \sigma, \nu, f, K_1)}{\lambda_{m+1}^2} \left[1 + \frac{1}{2} \log \left(\frac{\lambda_{m+1}}{\lambda_1} \right) \right]^{1/2},$$

where

$$(5.3) \qquad K_{PP}(\gamma, \sigma, \nu, f, K_1) = K_P(\gamma, \sigma, \nu, f, K_1) \left(1 + \frac{K^*}{\lambda_{m+1}^{1/2}} \right),$$

where K^* *and* $K_P(\gamma, \sigma, \nu, f, K_1)$ *are specified as in* (5.11) *and* (5.13).

Proof. We rewrite $u^\infty = p_m^\infty + q_m^\infty$ with $p_m^\infty \in H_m$ and $q_m^\infty \in Q_m$. They satisfy the following equations

$$(5.4) \qquad \nu A p_m^\infty + P_m [B(p_m^\infty + q_m^\infty, p_m^\infty + q_m^\infty) + \gamma L(p_m^\infty + q_m^\infty)] = P_m f,$$

$$(5.5) \qquad \nu A q_m^\infty + Q_m [B(p_m^\infty + q_m^\infty, p_m^\infty + q_m^\infty) + \gamma L(p_m^\infty + q_m^\infty) = Q_m f,$$

$$(5.6) \qquad \nu A u_m^\infty + P_m [B(u_m^\infty, u_m^\infty) + \gamma L u_m^\infty] = P_m f.$$

Thus, we have

$$p_m^\infty - u_m^\infty = -(\mathcal{A}_m(u_m^\infty))^{-1} P_m [B(p_m^\infty - u_m^\infty, p_m^\infty - u_m^\infty) + B(p_m^\infty, q_m^\infty)$$
$$+ \; B(q_m^\infty, p_m^\infty) + B(q_m^\infty, q_m^\infty) + \gamma L q_m^\infty].$$

Therefore, we get

$$p_m^\infty - u_m^\infty - v_m^\infty$$
$$= -(\mathcal{A}_m(u_m^\infty))^{-1} P_m [B(p_m^\infty - u_m^\infty, p_m^\infty - u_m^\infty) + B(p_m^\infty, q_m^\infty)$$
$$+ \; B(q_m^\infty, p_m^\infty) + B(q_m^\infty, q_m^\infty) + \gamma L q_m^\infty - B(u_m^\infty, \Phi_s(u_m^\infty + v_m^\infty))$$
$$- \; B(\Phi_s(u_m^\infty + v_m^\infty), u_m^\infty) - \gamma L \Phi_s(u_m^\infty + v_m^\infty)].$$

Notice that, by using a similar technique as in the proof of (3.56) and (3.57), for all w_1, $w_2 \in H$, and $w_3 \in H_m$, we obtain

$$|((\mathcal{A}_m(u_m^\infty))^{-1} B(w_1, w_2), w_3)| \leq C_1 |w_1| \, |w_2| \, |A^{1/2}(\mathcal{A}_m(u_m^\infty))^{-1} w_3|_{L^\infty}.$$

Now let us recall the following inequality, which holds in the two dimensional case, from [11]

$$(5.7) \qquad \|u\|_{L^\infty} \leq C_2 \|u\| \left[1 + \log \left(\frac{|Au|}{\lambda_1^{1/2} \|u\|} \right) \right]^{1/2}.$$

Therefore,

$$|A^{1/2}(\mathcal{A}_m(u_m^\infty))^{-1} w_3|_{L^\infty}$$
$$\leq C_2 |A(\mathcal{A}_m(u_m^\infty))^{-1} w_3| \left[1 + \log \left(\frac{|A^{3/2}(\mathcal{A}_m(u_m^\infty))^{-1} w_3|}{\lambda_1^{1/2} |A(\mathcal{A}_m(u_m^\infty))^{-1} w_3|} \right) \right]^{1/2}.$$

By using (3.28) with $\lambda = 0$, $\rho = \sigma$, $s = \sigma$, $R = K_3$, and the fact $w_3 \in H_m$, we conclude

$$|A^{1/2}(\mathcal{A}_m(u_m^\infty))^{-1} w_3|_{L^\infty} \leq C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) |w_3| \left[1 + \log \left(\frac{\lambda_{m+1}^{1/2}}{\lambda_1^{1/2}} \right) \right]^{1/2}.$$

Moreover,

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(w_1, w_2)|$$
$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2} \log \left(\frac{\lambda_{m+1}}{\lambda_1} \right) \right]^{1/2} |w_1| \, |w_2|.$$

As a result of the inequality above, we have

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(p_m^\infty - u_m^\infty, p_m^\infty - u_m^\infty)|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} |p_m^\infty - u_m^\infty|^2,$$

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(p_m^\infty - u_m^\infty, q_m^\infty)|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} |p_m^\infty - u_m^\infty| \, |q_m^\infty|,$$

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(q_m^\infty, p_m^\infty - u_m^\infty)|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} |p_m^\infty - u_m^\infty| \, |q_m^\infty|,$$

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(u_m^\infty, q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty))|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} |u_m^\infty| \, |q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)|,$$

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty), u_m^\infty)|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} |u_m^\infty| \, |q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)|,$$

$$|(\mathcal{A}_m(u_m^\infty))^{-1} B(q_m^\infty, q_m^\infty)|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} |q_m^\infty|^2,$$

$$|(\mathcal{A}_m(u_m^\infty))^{-1} \gamma L(q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty))|$$

$$\leq \frac{\gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha}}{\lambda_{m+1}^{1-\alpha}} K_9(\gamma, \sigma, \sigma, \nu, K_3) |q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)|.$$

Therefore,

$$|p_m^\infty - u_m^\infty - v_m^\infty|$$

$$\leq C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3) \left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2}$$

$$\times \; [|p_m^\infty - u_m^\infty|^2 + 2|p_m^\infty - u_m^\infty| \, |q_m^\infty| + 2|u_m^\infty| \, |q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)| + |q_m^\infty|^2]$$

$$+ \frac{\gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha}}{\lambda_{m+1}^{1-\alpha}} K_9(\gamma, \sigma, \sigma, \nu, K_3) |q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)|.$$

Now we assume that m is large enough such that

(5.8)
$$|A(p_m^\infty + q_m^\infty)| \le \frac{3}{2}\lambda_1^{1/2}K_3,$$

$$|A(u_m^\infty + v_m^\infty + \Phi_s(u_m^\infty + v_m^\infty))| \le \frac{3}{2}\lambda_1^{1/2}K_3,$$

(5.9) $$\nu - \frac{3C_2K_2}{2\lambda_{m+1}}\left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} - \frac{3C_1K_3}{2\lambda_{m+1}} - \frac{\nu\gamma C_L^{1-\alpha}\lambda_1^{1-\alpha}}{\lambda_{m+1}^{2-2\alpha}} \ge \frac{\nu}{2}.$$

Notice that all the above assumptions are true if m is at least of order G, the Grashoff number. In fact, to implement the results of our previous sections we require m to be much larger, namely, that $m \ge M_{\mathrm{GLK}}$.

Next we get an estimate for $|q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)|$. Denote by $w_m = p_m^\infty - u_m^\infty - v_m^\infty$ and by $w_m^\perp = q_m^\infty - \Phi_s(u_m^\infty + v_m^\infty)$. Following the equations (5.1) and (5.5), we obtain

$$\nu Aw_m^\perp + Q_m[B(p_m^\infty + q_m^\infty, w_m + w_m^\perp)$$

$$+ B(w_m + w_m^\perp, u_m^\infty + v_m^\infty + \Phi_s(u_m^\infty + v_m^\infty)) + \gamma L(w_m + w_m^\perp) = 0.$$

Following the inequalities (2.7)-(2.11) and (5.7), and (2.20) we get

$$\left(\nu - \frac{3C_2K_2}{2\lambda_{m+1}}\left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2} - \frac{3C_1K_3}{2\lambda_{m+1}} - \frac{\nu\gamma C_L^{1-\alpha}\lambda_1^{1-\alpha}}{\lambda_{m+1}^{2-2\alpha}}\right)\|w_m^\perp\|$$

$$\le \frac{C_1C_2\|p_m^\infty\|\,|w_m|}{\lambda_{m+1}^{1/2}}\left[1 + \frac{1}{2}\log\left(\frac{\lambda_{m+1}}{\lambda_1}\right)\right]^{1/2}$$

$$+ C_1|w_m|^{1/2}\|w_m\|^{1/2}|q_m^\infty|^{1/2}\|q_m^\infty\|^{1/2} + C_1|u_m^\infty|^{1/2}|Au_m^\infty|^{1/2}|w_m|$$

$$+ C_1|w_m|^{1/2}\|w_m\|^{1/2}|\Phi_s(u_m^\infty + v_m^\infty)|^{1/2}\|\Phi_s(u_m^\infty + v_m^\infty)\|^{1/2}$$

$$+ \frac{\gamma\nu C_L^{1-\alpha}\lambda_1^{1-\alpha}}{\lambda_{m+1}^{1/2-\alpha}}|w|.$$

Therefore, by using the assumptions (5.8) and (5.9) we conclude

(5.10)
$$|w_m^\perp| \le \frac{K^*}{\lambda_{m+1}^{1/2}}|w_m|,$$

where

$$(5.11) \quad K^* = \frac{2}{\nu} \left[\frac{3C_1 C_2 \lambda_1^{1/2} K_3}{2\lambda_{m+1}^{1/2}} \left[1 + \frac{1}{2} \log\left(\frac{\lambda_{m+1}}{\lambda_1} \right) \right]^{1/2} \right.$$

$$\left. + \frac{3C_1 \lambda_1^{1/2} K_3}{\lambda_{m+1}^{1/2}} + C_1 K_3 + \frac{\gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha}}{\lambda_{m+1}^{1/2-\alpha}} \right].$$

Thus, if m is large enough such that

$$(5.12) \qquad K^* K_9(\gamma, \sigma, \sigma, \nu, K_3)$$

$$\times \left[\frac{2C_1 C_2 K_1}{\lambda_{m+1}} \left[1 + \frac{1}{2} \log\left(\frac{\lambda_{m+1}}{\lambda_1} \right) \right]^{1/2} + \frac{\gamma \nu C_L^{1-\alpha} \lambda_1^{1-\alpha}}{\lambda_{m+1}^{3/2-\alpha}} \right] \le \frac{1}{2},$$

we get

$$|p_m^\infty - u_m^\infty - v_m^\infty| \le \frac{K_P(\gamma, \sigma, \nu, f, K_1)}{\lambda_{m+1}^2} \left[1 + \frac{1}{2} \log\left(\frac{\lambda_{m+1}}{\lambda_1} \right) \right]^{1/2},$$

where

$$(5.13) \quad K_P(\gamma, \sigma, \nu, f, K_1) = 2C_1 C_2 K_9(\gamma, \sigma, \sigma, \nu, K_3)$$

$$\times \left(K_{10}(\gamma, \sigma, \sigma, \nu, K_3) + \frac{3}{2} K_3 \right)^2.$$

Notice that, when $m \ge M_{\text{GLK}}$, (5.12) is true. Moreover, we have

$$|q_m^\infty - \Phi_s(u_m^\infty)| \le \frac{K^* K_P(\gamma, \sigma, \nu, f, K_1)}{\lambda_{m+1}^{5/2}} \left[1 + \frac{1}{2} \log\left(\frac{\lambda_{m+1}}{\lambda_1} \right) \right]^{1/2}.$$

Then the estimate (5.2) follows.   □

6. CONCLUSION

In this paper we have presented an explicit numerical criterion for the stabilization of steady state solutions of the Navier-Stokes equations with linear feedback control. We will summarize below our results and the steps of how to implement this criterion.

Suppose u^* is an unstable steady state solution of the NSE. It is not practical, computationally, to stabilize the infinite dimensional system. However, by using numerical methods we can find a stationary solution of the Galerkin approximation equations, or the nonlinear Galerkin approximation equations, which is an approximation of u^*. Suppose that we can design a linear feedback control L, whose Galerkin (or nonlinear Galerkin) truncation stabilizes this approximate solution. The question is whether this same linear feedback control, L, stabilizes

u^*. In this paper we have shown that the answer to this question is affirmative, provided we use enough eigenmodes, and the linear feedback operator L belongs to a certain class of linear operators which are dominated by the viscous term νA. This restriction on the feedback controllers is essential in order for the viscous term to suppress the potential spillover that L might cause, by destabilizing the higher eigenmodes. This approach is practical because we are now working with a finite dimensional system.

A general abstract algorithm. Our general algorithm is as follows:

(i)   Find a steady state u_m^* of the m−state Galerkin approximation equations, or a steady state $p_m^* + q_m^*$ of the m−state nonlinear Galerkin approximation equations, by numerical simulation for some m, such that u_m^*, or $p_m^* + q_m^*$, is close to u^*, respectively. More precisely, we assume that, in the Galerkin case,

$$|u^* - u_m^*| \le \varepsilon \ll 1,$$

or in the nonlinear Galerkin case,

$$|u^* - (p_m^* + q_m^*)| \le \varepsilon \ll 1.$$

(ii)   Design a linear feedback controller L, which might act on finite or infinitely many modes, and which belongs to the class of linear operators described in Section 2, such that u_m^* is a stable stationary solution of the closed-loop system

$$\frac{du_m}{dt} + \nu A u_m + P_m B(u_m, u_m) + P_m L(u_m - u_m^*) = P_m g$$

in the Galerkin case, or $p_m^* + q_m^*$ is a stable stationary solution of the following closed-loop system

$$\frac{dp_m}{dt} + \nu A p_m + P_m B(p_m + q_m, p_m + q) - P_m B(q_m, q_m)$$
$$+ P_m L(p_m + q_m - p_m^* - q_m^*) = P_m g,$$
$$\nu A q_m + Q_m B(p_m, p_m) + Q_m L(p_m - p_m^*) = Q_m g$$

in the nonlinear Galerkin case.

(iii)   Let $\sigma > 0$ be the lower bound of the real part of eigenvalues of the linearized operator, $\mathcal{A}_m(u_m^*)$ in the Galerkin case, or $\mathcal{N}_m(p_m^*)$ in the nonlinear Galerkin case ($\mathcal{A}_m(u_m^*)$ and $\mathcal{N}_m(p_m^*)$ are defined in Sections 3 on page 44 and 4 on page 60, respectively).

(iv) Let M_{GLK} and M_{NLG} be as in Section 3 and Section 4, respectively. If $m \geq M_{\mathrm{GLK}}$ in the Galerkin case, or $m \geq M_{\mathrm{NLG}}$ in the nonlinear Galerkin case, then the feedback controller L will stabilize a steady state solution u^{**} of the closed-loop Navier-Stokes equations

$$\frac{du}{dt} + \nu Au + B(u, u) + L(u - u_m^*) = g.$$

Moreover, one has the following error estimate for the Galerkin case

$$|u^{**} - u_m^*| \leq \frac{C}{\lambda_{m+1}} \quad \text{and} \quad |u^{**} - u^*| \leq \frac{C}{\lambda_{m+1}} + \varepsilon,$$

or in the nonlinear Galerkin case,

$$|u^{**} - p_m^* - q_m^*| \leq \frac{C}{\lambda_{m+1}^{3/2}} \quad \text{and} \quad |u^{**} - u^*| \leq \frac{C}{\lambda_{m+1}^{3/2}} + \varepsilon.$$

Furthermore, one can use the postprocessing procedure [38, 39, 65] to improve the accuracy of both (Galerkin as well as nonlinear Galerkin) methods as it has been discussed in Section 5 on page 83. As a result the above error estimates, one may conclude that u^{**} is very close to the original solution u^*, provided m is large enough, and that $p_m^* + q_m^*$ converges to u^{**} faster than u^* does.

(v) Otherwise, if either step 3 or step 4 does not hold we either design another linear feedback control that allows for a larger σ, or increase m and repeat the above procedure.

In this way one is guaranteed that the linear feedback control L for the finite system ODEs will stabilize a steady state solution u^{**} of the closed-loop NSE, and u^{**} is nearby the unstable steady state u^* of the NSE.

A specific implementation of the algorithm. Next, we present a special case of the above general algorithm, which will guarantee an *a priori* upper estimate for ε, given in step 1 of the above general algorithm, of order $O(\lambda_{m+1}^{-1})$ in the Galerkin case, and of order $O(\lambda_{m+1}^{-3/2})$ in the nonlinear Galerkin case.

Let u^* be, as before, an unstable steady state of the NSE (open-loop)

$$\frac{du}{dt} + \nu Au + B(u, u) = g,$$

which we are interested in stabilizing. Set $u^* = \xi_m^* + \eta_m^*$, where $\xi_m^* = P_m u^*$ and $\eta_m^* = (I - P_m)u^*$. It is clear that $u_m = \xi_m^*$ is a steady state of the following Galerkin system

$$\frac{du_m}{dt} + \nu Au_m + P_m B(u_m, u_m) + \gamma P_m Lu_m$$

$$= P_m[g + B(\xi_m^*, \xi_m^*) - B(u^*, u^*) + \gamma L\xi_m^*].$$

The equation above is the Galerkin procedure at level m of the closed-loop system:

$$\frac{du}{dt} + \nu Au + B(u,u) + \gamma Lu = f,$$

where $f = g + B(\xi_m^*, \xi_m^*) - B(u^*, u^*) + \gamma L\xi_m^*$.

Notice that the above equation is within our frame of work. In order for us to derive explicit values to the constants M_{GLK} and M_{NLG} we would need to provide an explicit bound for $\|f\|$ (see the end of Section **3.1 on page 48**). Indeed, it is sufficient to estimate the right hand side of the following

$$\|f\| \le \|g\| + \|B(\xi_m^*, \xi_m^*)\| + \|B(u^*, u^*)\| + \|\gamma L\xi_m^*\|.$$

First, notice that by (2.11)

$$\|B(u^*, u^*)\| \le 2C_1 |u^*|^{1/2} |Au^*|^{3/2},$$

and

$$\|B(\xi_m^*, \xi_m^*)\| \le 2C_1 |\xi_m^*|^{1/2} |A\xi_m^*|^{3/2} \le 2C_1 |u^*|^{1/2} |Au^*|^{3/2}.$$

Based on part (ii) of Lemma 3.1 on page 44 we have

$$|u^*| \le \frac{|g|}{\lambda_1 \nu} \quad \text{and} \quad |Au^*| \le K(|g|) = \frac{2|g|}{\nu} + \frac{C_1^2 |g|^3}{\lambda_1^2 \nu^5}.$$

Thus, we obtain

$$\|B(u^*, u^*)\| + \|B(\xi_m^*, \xi_m^*)\| \le \tilde{K}(|g|),$$

with $\tilde{K}(|g|) = [4C_1 K^{3/2}(|g|)|g|]/(\lambda_1 \nu)$.

As in Section 2 on page 41, we assume that (2.20) holds for $\alpha = 0$ and $\beta = \frac{1}{2}$. Therefore, we have

$$\gamma \|L\xi_m^*\| \le \nu \lambda_1 C_L |A^{1/2}\xi_m^*| \le \nu \lambda_1 C_L |A^{1/2} u^*| \le \lambda_1^{1/2} C_L |g|.$$

From the above we conclude that

$$\|f\| \le \bar{K}(|g|, \|g\|, \gamma, C_L) = \lambda_1^{1/2} C_L |g| + \tilde{K}(|g|) + \|g\|.$$

One can restrict further the class of linear feedback operator L by imposing an upper bound on C_L, and hence give an explicit a priori determined upper bound to $\|f\|$.

Now, we return to apply the general algorithm under these specific choices for u_m^* and f. Indeed, in step 1 of the general algorithm we have $u_m^* = \xi_m^*$ and therefore

$$|u^* - u_m^*| = |\eta_m^*| \le \frac{1}{\lambda_{m+1}}|A\eta_m^*| \le \frac{1}{\lambda_{m+1}}|Au^*| \le \frac{K(|g|)}{\lambda_{m+1}}.$$

Then we take $\varepsilon = K(|g|)/\lambda_{m+1}$, which is very small provided m is large enough. Following the same steps as in the general algorithm for this choice of f and u_m^*, we obtain in step 4 that

$$|u^* - u^{**}| \le \frac{C}{\lambda_{m+1}} + \frac{K(|g|)}{\lambda_{m+1}}.$$

Hence, the stable steady state solution u^{**} of the closed-loop is not far away from the original unstable steady state of the open-loop u^*.

We can design a similar algorithm for the nonlinear Galerkin case with a new specific f which satisfies the right estimates so that

$$\varepsilon = O(\lambda_{m+1}^{-3/2})$$

in the nonlinear Galerkin case, and hence step 4 in this situation would lead to

$$|u^* - u^{**}| = O(\lambda_{m+1}^{-3/2}).$$

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