A Turbulence Model for the 1-D Dispersive Wave

Chongsheng Cao
Center gor Nonlinear Studies
Los Alamos National Laboratory
Los Alamos, NM 87545, USA
E-mail: ccao@cnls.lanl.gov

Shiyi Chen
Department of Mechanical Engineering
The Johns Hopkins University
200 Latrobe Hall
3400 North Charles Street
Baltimore, MD 21218, USA
E-mail: syc@me.jhu.edu

Edriss S. Titi
Department of Mathematics and
Department of Mechanical and Aerospace Engineering
University of California
Irvine, CA 92697-3875, USA
E-mail: etiti@math.uci.edu

abstract

In this paper we consider a class of one dimensional nonlinear dispersive wave equations. A closure model is developed and the Kolmogorov–type wave spectra are established. Numerical simulation show that the model successfully predicts the inertial range exponents.

1 Introduction

Weak turbulence theories play an important role in the prediction of the spectra for the dispersive waves. The principle of the weak turbulence is that the nonlinear interaction is smaller than the linear effects which disperse the energy. Based on this principle and the fact that the energy transfer takes place through resonant wave interactions, the closures of the statistical kinetic equations have been developed previously (cf. [**1**], [**2**], [**3**], [**4**], [**5**]). Numerical simulations show that the weak turbulence theories are excellent in approximating the statistical stationary solutions of the dispersive wave equations.

Here, following the work of Majda, McLaughlin and Tabak in [**2**], we consider the following dispersive wave equations:

$$i\frac{\partial u}{\partial t} = |\partial_x|^\alpha u + |\partial_x|^{-\beta/4}\left(\left||\partial_x|^{-\beta/4}u\right|^2 |\partial_x|^{-\beta/4}u\right), \tag{1}$$

where α and β are parameters and u is a complex periodic function with zero mean. In [**2**], the authors used the solutions of linear part of the equation (1) to get the fourth order moment. In this article we use similar method developed in [**2**]. First, we solve a linearized phase equation. In addition we assume that the amplitude of the wave is small and varies slowly in time. The Kolmogorov–type wave spectra are established. However, the spectrum is sharper than the one predicted by the previous weak turbulence theories (cf. [**3**], [**4**]). This phenomenon has also been seen in [**2**]. As the authors in [**2**] have pointed out the reason is that in the weak turbulence theory dissipation has been added within the inertial subrange. In order to test our model, a set of direct numerical simulations (DNS) based on pseudo–spectral method of system (1) have been carried out. The obtained spectrum exponents agree well with our model.

2 A model for the kinetic equation

Let

$$\phi(k,t) = \int u(x,t)e^{-2\pi ikx}, \qquad k \in \mathbb{Z}, \tag{2}$$

be the Fourier transform of $u(x,t)$. For $\lambda \in \mathbb{R}$, $|\partial_x|^\lambda$ is defined as:

$$\int |\partial_x|^\lambda u(x,t)e^{-2\pi ikx}dx = |k|^\lambda \phi(k,t),$$

and for any constant c, we define

$$|\partial_x|^\lambda c = 0.$$

Applying the Fourier transform to equation (1), we get

$$i\frac{\partial \phi(k,t)}{\partial t} = |k|^{\alpha}\phi(k,t) + \sum_{k_1+k_2-k_3=k} \frac{\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)}{|k_1k_2k_3k|^{\beta/4}}. \tag{3}$$

Note that the equation (3) can be rewritten as

$$\begin{aligned} i\frac{\partial \phi(k,t)}{\partial t} &= |k|^{\alpha}\phi(k,t) + \left[2\sum_j \frac{|\phi(j)|^2}{|j|^{\beta/2}} - \frac{|\phi(k,t)|^2}{|k|^{\beta/2}}\right]\frac{\phi(k,t)}{|k|^{\beta/2}} + \\ &+ \sum_{\substack{k_1+k_2-k_3=k \\ k_1\neq k \\ k_2\neq k}} \frac{\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)}{|k_1k_2k_3k|^{\beta/4}}. \end{aligned} \tag{4}$$

Denote

$$A(k,t) = \frac{2\sum_j \frac{|\phi(j,t)|^2}{|j|^{\beta/2}} - \frac{|\phi(k,t)|^2}{|k|^{\beta/2}}}{|k|^{\beta/2}}.$$

Equation (4) will be

$$\begin{aligned} i\frac{\partial \phi(k,t)}{\partial t} &= |k|^{\alpha}\phi(k,t) + A(k,t)\phi(k,t) + \\ &+ \sum_{\substack{k_1+k_2-k_3=k \\ k_1\neq k \\ k_2\neq k}} \frac{\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)}{|k_1k_2k_3k|^{\beta/4}}. \end{aligned} \tag{5}$$

Let

$$n(k,t) =< \phi(k,t)\bar{\phi}(k,t) >$$

be the energy spectrum, where $<>$ denotes an ensemble average. It is clear that

$$\begin{aligned} \frac{\partial n(k,t)}{\partial t} &= 2\mathrm{Im}\left\langle \sum_{k_1+k_2-k_3=k} \frac{\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)}{|k_1k_2k_3k|^{\beta/4}} \right\rangle = \\ &= 2 \sum_{\substack{k_1+k_2-k_3=k \\ k_1\neq k \\ k_2\neq k}} \frac{\mathrm{Im}\left\langle \phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)\right\rangle}{|k_1k_2k_3k|^{\beta/4}}. \end{aligned} \tag{6}$$

This is the kinetic equation of (3). For convenience, we set

$$\sum_{k^*} = \sum_{\substack{l_1+l_2-l_3=k \\ l_1\neq k \\ l_2\neq k}},$$

and denote

$$N(k,t) = 2\sum_{k^*} \frac{\mathrm{Im}\left\langle \phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)\right\rangle}{|k_1k_2k_3k|^{\beta/4}}. \tag{7}$$

From (3), we have the following equation for $\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)$.

$$\begin{aligned}
& i\frac{\partial}{\partial t}[\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)] = \\
& = [|k_1|^\alpha + |k_2|^\alpha - |k_3|^\alpha - |k|^\alpha][\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)] + \\
& \quad + [A(k_1,t) + A(k_2,t) - A(k_3,t) - A(k,t)][\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)] \\
& \quad + \sum_{k_1^*} \frac{\phi(j_1,t)\phi(j_2,t)\bar{\phi}(j_3,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)}{|j_1j_2j_3k_1|^{\beta/4}} + \\
& \quad + \sum_{k_2^*} \frac{\phi(j_1,t)\phi(j_2,t)\bar{\phi}(j_3,t)\phi(k_1,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)}{|j_1j_2j_3k_2|^{\beta/4}} - \\
& \quad - \sum_{k_3^*} \frac{\bar{\phi}(j_1,t)\bar{\phi}(j_2,t)\phi(j_3,t)\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k,t)}{|j_1j_2j_3k_3|^{\beta/4}} - \\
& \quad - \sum_{k^*} \frac{\bar{\phi}(j_1,t)\bar{\phi}(j_2,t)\phi(j_3,t)\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)}{|j_1j_2j_3k|^{\beta/4}}
\end{aligned} \tag{8}$$

There are two secular terms in equation (8) as follows:

$$[|k_1|^\alpha + |k_2|^\alpha - |k_3|^\alpha - |k|^\alpha][\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)] \tag{9}$$

$$[A(k_1,t) + A(k_2,t) - A(k_3,t) - A(k,t)][\phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)]. \tag{10}$$

The magnitude of the first term is much larger than that of the second term. Therefore, in order to reach equabilium the first one should balance itself. Thus

$$|k_1|^\alpha + |k_2|^\alpha - |k_3|^\alpha - |k|^\alpha = 0. \tag{11}$$

This is four wave resonant condition. As a result, when $|k_1|^\alpha + |k_2|^\alpha - |k_3|^\alpha - |k|^\alpha \neq 0$

$$\mathrm{Im}\left\langle \phi(k_1,t)\phi(k_2,t)\bar{\phi}(k_3,t)\bar{\phi}(k,t)\right\rangle = 0. \tag{12}$$

Denote

$$\phi(k,t) = R(k,t)e^{i\theta(k,t)}.$$

We have

$$n(k,t) = < \phi(k,t)\bar{\phi}(k,t) > = < R^2(k,t) > .$$

It is easy to obtain the following equations:

$$\begin{aligned}
\frac{\partial R(k,t)}{\partial t} &= \sum_{k^*} \frac{R(k_1,t)R(k_2,t)R(k_3,t)}{|k_1k_2k_3k|^{\beta/4}} \times \\
& \times \sin(\theta(k_1,t) + \theta(k_2,t) - \theta(k_3,t) - \theta(k,t)),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{\partial \theta(k,t)}{\partial t} &= -|k|^\alpha - A(k,t) - \sum_{k^*} \frac{R(k_1,t)R(k_2,t)R(k_3,t)}{R(k,t)|k_1k_2k_3k|^{\beta/4}} \times \\
& \times \cos(\theta(k_1,t) + \theta(k_2,t) - \theta(k_3,t) - \theta(k,t)).
\end{aligned} \tag{14}$$

$N(k,t)$ can be rewritten as function of R and θ:

$$N(k,t) = 2\left\langle \sum_{k^*} \frac{R(k_1,t)R(k_2,t)R(k_3,t)R(k,t)}{|k_1k_2k_3k|^{\beta/4}} \times \right. \\ \left. \times \sin(\theta(k_1,t)+\theta(k_2,t)-\theta(k_3,t)-\theta(k,t)) \right\rangle . \tag{15}$$

By integrating equation (14) on $[0,t]$, we can get a formal solution of $\theta(k,t)$:

$$\theta(k,t)-\theta(k,0) = -|k|^\alpha t - \int_0^t A(k,s)\,ds - \int_0^t \sum_{k^*} \frac{[\,R(l_1,s)R(l_2,s)R(l_3,s)\,]^{1/2}}{(R(k,s))^{1/2}|l_1l_2l_3k|^{\beta/4}} \times \\ \times \cos(\theta(l_1,s)+\theta(l_2,s)-\theta(l_3,s)-\theta(k,s))\ ds.$$

Replacing $A(k,s), R(j,s), \theta(l_1,s)+\theta(l_2,s)-\theta(l_3,s)-\theta(k,s)$ by their linear approximations:

$$A(k) = \frac{2\sum_j \frac{n(j)}{|j|^{\beta/2}} - \frac{n(k)}{|k|^{\beta/2}}}{|k|^{\beta/2}}, \tag{16}$$

$$R(j,s) = (n(j))^{1/2}, \tag{17}$$

$$\theta(l_1,0)+\theta(l_2,0)-\theta(l_3,0)-\theta(k,0) - [|l_1|^\alpha + |l_2|^\alpha - |l_3|^\alpha - |k|^\alpha]\ s- \\ -[A(l_1)+A(l_2)-A(l_3)-A(k)]\ s, \tag{18}$$

in the above equation, respectively, we get

$$\theta(k,t) = \theta(k,0) - |k|^\alpha t - A(k)t - \sum_{k^*} \frac{[\,n(k_1)n(k_2)n(k_3)\,]^{1/2}}{n^{1/2}(k)|k_1k_2k_3k|^{\beta/4}} \times \\ \times \frac{\sin((B(l_1)+B(l_2)-B(l_3)-B(k))t + \Theta_0^k) - \sin\Theta_0^k}{B(l_1)+B(l_2)-B(l_3)-B(k)},$$

where

$$B(k) = |k|^\alpha + A(k), \tag{19}$$

$$\Theta_0^{lk} = \theta(l_1,0)+\theta(l_2,0)-\theta(l_3,0)-\theta(k,0), \tag{20}$$

and $A(k)$ is as in (16). Let $R(k,t) = n^{1/2}(k) + \epsilon(r,t)$, we have

$$\langle R(k_1,t)R(k_2,t)R(k_3,t)R(k,t)\sin(\theta(j_1,t)+\theta(j_2,t)-\theta(j_3,t)-\theta(j_4,t))\rangle \\ = \frac{1}{T}\int_0^T [R(k_1,t)R(k_2,t)R(k_3,t)R(k,t)\times \\ \times \sin(\theta(k_1,t)+\theta(k_2,t)-\theta(k_3,t)-\theta(k,t))]\,dt \\ = [n(k_1)n(k_2)n(k_3)n(k)]^{\frac{1}{2}}\,\frac{1}{T}\int_0^T \sin(\theta(k_1,t)+\theta(k_2,t)-\theta(k_3,t)-\theta(k,t))dt + O(\epsilon).$$

Here, we have replaced the ensemble average by time average. Now, we assume that $\epsilon(k,t)$ is very small.

REMARK: The referee asked to show this assumption, numerically. I have not done it yet.

Therefore, we only need to find

$$\frac{1}{T}\int_0^T \sin(\theta(k_1,t)+\theta(k_2,t)-\theta(k_3,t)-\theta(k,t))dt.$$

When $|j_1|^\alpha + |j_2|^\alpha| - j_3|^\alpha - |j_4|^\alpha = 0$, we get

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sin(\theta(j_1,t)+\theta(j_2,t)-\theta(j_3,t)-\theta(j_4,t))\ dt \\
&= \frac{1}{T}\int_0^T \sin\left(\Theta_0 - \mathcal{A}_j t - \sum_{j_1^*} \frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\mathcal{A}_{lj_1}t-\Theta_0^{lj_1})-\sin\Theta_0^{lj_1}\right]}{(n(j_1))^{1/2}|l_1l_2l_3j_1|^{\beta/4}\mathcal{A}_{lj_1}} - \right. \\
&\qquad -\sum_{j_2^*} \frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\mathcal{A}_{lj_2}t-\Theta_0^{lj_2})-\sin\Theta_0^{lj_2}\right]}{(n(j_2))^{1/2}|l_1l_2l_3j_2|^{\beta/4}\mathcal{A}_{lj_2}} + \\
&\qquad +\sum_{j_3^*} \frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\mathcal{A}_{lj_3}t-\Theta_0^{lj_3})-\sin\Theta_0^{lj_3}\right]}{(n(j_3))^{1/2}|l_1l_2l_3j_3|^{\beta/4}\mathcal{A}_{lj_3}} + \\
&\qquad \left. +\sum_{j_4^*} \frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\mathcal{A}_{lj_4}t-\Theta_0^{lj_4})-\sin\Theta_0^{lj_4}\right]}{(n(j_4))^{1/2}|l_1l_2l_3j_4|^{\beta/4}\mathcal{A}_{lj_4}}\right)\ dt,
\end{aligned}$$

where

$$\Theta_0 = \theta(j_1,0)+\theta(j_2,0)-\theta(j_3,0)-\theta(j_4,0), \tag{21}$$

$$\mathcal{A}_j = A(j_1)+A(j_2)-A(j_3)-A(j_4), \tag{22}$$

$$\mathcal{A}_{lj} = B(l_1)+B(l_2)-B(l_3)-B(j), \tag{23}$$

and $A(k), B(k)$ are as in (16) and (19), respectively. By changing integral variable $\Theta_0 - \mathcal{A}_j t$

to t, we obtain

$$
\begin{aligned}
&\langle \sin(\theta(j_1,t)+\theta(j_2,t)-\theta(j_3,t)-\theta(j_4,t))\rangle \\
&= -\frac{1}{T}\int_0^T \sin\left(t+\sum_{j_1^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_1}}{\mathcal{A}_j}(t-\Theta_0)-\Theta_0^{lj_1})-\sin\Theta_0^{lj_1}\right]}{(n(j_1))^{1/2}|l_1l_2l_3j_1|^{\beta/4}\mathcal{A}_{lj_1}}+\right. \\
&\quad +\sum_{j_2^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_2}}{\mathcal{A}_j}(t-\Theta_0)-\Theta_0^{lj_2})-\sin\Theta_0^{lj_2}\right]}{(n(j_2))^{1/2}|l_1l_2l_3j_2|^{\beta/4}\mathcal{A}_{lj_2}}- \\
&\quad -\sum_{j_3^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_3}}{\mathcal{A}_j}(t-\Theta_0)-\Theta_0^{lj_3})-\sin\Theta_0^{lj_3}\right]}{(n(j_3))^{1/2}|l_1l_2l_3j_3|^{\beta/4}\mathcal{A}_{lj_3}}- \\
&\quad \left.-\sum_{j_4^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_4}}{\mathcal{A}_j}(t-\Theta_0)-\Theta_0^{lj_4})-\sin\Theta_0^{lj_4}\right]}{(n(j_1))^{1/2}|l_1l_2l_3j_4|^{\beta/4}\mathcal{A}_{lj_4}}\right)\frac{dt}{\mathcal{A}_j} \\
&= -\frac{1}{T}\int_0^T \sin\left(t+\sum_{j_1^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_1}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lj_1})-\sin\Theta_0^{lj_1}\right]}{(n(j_1))^{1/2}|l_1l_2l_3j_1|^{\beta/4}\mathcal{A}_{lj_1}}+\right. \\
&\quad +\sum_{j_2^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_2}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lj_2})-\sin\Theta_0^{lj_2}\right]}{(n(j_2))^{1/2}|l_1l_2l_3j_2|^{\beta/4}\mathcal{A}_{lj_2}}- \\
&\quad -\sum_{j_3^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_3}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lj_3})-\sin\Theta_0^{lj_3}\right]}{(n(j_3))^{1/2}|l_1l_2l_3j_3|^{\beta/4}\mathcal{A}_{lj_3}}- \\
&\quad \left.-\sum_{j_4^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\left[\sin(\frac{\mathcal{A}_{lj_4}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lj_4})-\sin\Theta_0^{lj_4}\right]}{(n(j_1))^{1/2}|l_1l_2l_3j_4|^{\beta/4}\mathcal{A}_{lj_4}}\right)\frac{dt}{\mathcal{A}_j}.
\end{aligned}
$$

Using linear approximation $\sin(x+\epsilon)=\sin x+\epsilon\cos x$ to above equation, and notice that

$$\int_0^{2\pi} C\sin t\,dt = 0,$$

we get

$$
\begin{aligned}
&\langle \sin(\theta(j_1,t)+\theta(j_2,t)-\theta(j_3,t)-\theta(j_4,t))\rangle \\
&= -\frac{1}{2m\pi}\sum_{\xi=0}^{m-1}\int_0^{2\pi}\cos t\left(\sum_{j_1^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\sin(\frac{\mathcal{A}_{lj_1}}{\mathcal{A}_j}((2\xi\pi+t-\Theta_0)-\Theta_0^{lj_1})}{(n(j_1))^{1/2}|l_1l_2l_3j_1|^{\beta/4}\mathcal{A}_{lj_1}} + \right.\\
&\quad +\sum_{j_2^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\sin(\frac{\mathcal{A}_{lj_2}}{\mathcal{A}_j}((2\xi\pi+t-\Theta_0)-\Theta_0^{lj_2})}{(n(j_2))^{1/2}|l_1l_2l_3j_2|^{\beta/4}\mathcal{A}_{lj_2}} - \\
&\quad -\sum_{j_3^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\sin(\frac{\mathcal{A}_{lj_3}}{\mathcal{A}_j}((2\xi\pi+t-\Theta_0)-\Theta_0^{lj_3})}{(n(j_3))^{1/2}|l_1l_2l_3j_3|^{\beta/4}\mathcal{A}_{lj_3}} - \\
&\quad \left.-\sum_{j_4^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\sin(\frac{\mathcal{A}_{lj_4}}{\mathcal{A}_j}((2\xi\pi+t-\Theta_0)-\Theta_0^{lj_4})}{(n(j_1))^{1/2}|l_1l_2l_3j_4|^{\beta/4}\mathcal{A}_{lj_4}}\right)\frac{dt}{\mathcal{A}_j}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\int_0^{2\pi}\cos t\sin(a_1t+a_2)\,dt = \frac{a_1}{1-a_1^2}[\cos(2\pi a_1+a_2)-\cos a_2] \qquad |a_1|\neq 1,\\
&\int_0^{2\pi}\cos t\sin(a_1t+a_2)\,dt = \pi\sin a_2 \qquad\qquad |a_1| = 1.
\end{aligned}
$$

Thus, when $\mathcal{A}_{lp}\neq\mathcal{A}_j$,

$$
\begin{aligned}
&\frac{1}{2m\pi}\sum_{\xi=0}^{m-1}\int_0^{2\pi}\cos t\sin(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lp})\,dt \qquad (24)\\
&= \frac{1}{2m\pi}\sum_{\xi=0}^{m-1}\frac{\mathcal{A}_{lp}\mathcal{A}_j}{\mathcal{A}_j^2-\mathcal{A}_{lp}^2}[\cos(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\pi(\xi+1)-\Theta_0)-\Theta_0^{lp})-\cos(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\pi\xi-\Theta_0)-\Theta_0^{lp})]
\end{aligned}
$$

In case $\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}$ is an integer, it is clear that

$$
\frac{1}{2m\pi}\sum_{\xi=0}^{m-1}\int_0^{2\pi}\cos t\sin(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lp})\,dt = 0.
$$

In case $\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}$ is not an integer, by the ergodicity of $\cos(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\pi(\xi+1)-\Theta_0)-\Theta_0^{lp})$ and $\cos(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\pi\xi-\Theta_0)-\Theta_0^{lp})$, we also have

$$
\lim_{m\to\infty}\frac{1}{2m\pi}\sum_{\xi=0}^{m-1}\int_0^{2\pi}\cos t\sin(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lp})\,dt = 0.
$$

Therefore, the average is not zero only when $\mathcal{A}_{lp}=\mathcal{A}_j$. Moreover,

$$
\frac{1}{2m\pi}\sum_{\xi=0}^{m-1}\int_0^{2\pi}\cos t\sin(\frac{\mathcal{A}_{lp}}{\mathcal{A}_j}(2\xi\pi+t-\Theta_0)-\Theta_0^{lp})\,dt = \frac{1}{2}\sin(\Theta_0+\Theta_0^{lp}).
$$

As a result, we reach

$$
\begin{aligned}
&\langle \sin(\theta(j_1,t)+\theta(j_2,t)-\theta(j_3,t)-\theta(j_4,t))\rangle \\
&= \frac{C}{\mathcal{A}_j}\left[\sum_{j_1^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lj_1}-\mathcal{A}_j)}{(n(j_1))^{1/2}|l_1l_2l_3j_1|^{\beta/4}\mathcal{A}_{lj_1}}+\right.\\
&\quad+\sum_{j_2^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lj_2}-\mathcal{A}_j)}{(n(j_2))^{1/2}|l_1l_2l_3j_2|^{\beta/4}\mathcal{A}_{lj_2}}-\\
&\quad-\sum_{j_3^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lj_3}-\mathcal{A}_j)}{(n(j_3))^{1/2}|l_1l_2l_3j_3|^{\beta/4}\mathcal{A}_{lj_3}}-\\
&\quad\left.-\sum_{j_4^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lj_4}-\mathcal{A}_j)}{(n(j_4))^{1/2}|l_1l_2l_3j_4|^{\beta/4}\mathcal{A}_{lj_4}}\right].
\end{aligned}
$$

Therefore, we can close $N(k,t)$ with a closure $N_c(k,t)$ defined as follow:

$$
\begin{aligned}
N_c(k,t) = C\sum_{k^*}&\frac{(n(k_1)n(k_2)n(k_3)n(k))^{\frac{1}{2}}\delta(|k_1|^\alpha+|k_2|^\alpha|-k_3|^\alpha-|k|^\alpha)}{|k_1k_2k_3k|^{\beta/4}\mathcal{A}}\times\\
\times&\left[\sum_{k_1^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lk_1}-\mathcal{A})}{(n(k_1))^{1/2}|l_1l_2l_3k_1|^{\beta/4}\mathcal{A}_{lk_1}}+\right.\\
&+\sum_{k_2^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lk_2}-\mathcal{A})}{(n(k_2))^{1/2}|l_1l_2l_3k_2|^{\beta/4}\mathcal{A}_{lk_2}}-\\
&-\sum_{k_3^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lk_3}-\mathcal{A})}{(n(k_3))^{1/2}|l_1l_2l_3k_3|^{\beta/4}\mathcal{A}_{lk_3}}-\\
&\left.-\sum_{k^*}\frac{[\,n(l_1)n(l_2)n(l_3)\,]^{1/2}\delta(\mathcal{A}_{lk_4}-\mathcal{A})}{(n(k))^{1/2}|l_1l_2l_3k|^{\beta/4}\mathcal{A}_{lk}}\right],
\end{aligned}
\tag{25}
$$

where $\mathcal{A}=\mathcal{A}(k_1)+\mathcal{A}(k_2)-\mathcal{A}(k_3)-\mathcal{A}(k)$, and $\mathcal{A}_{lk_j}$ is as in (23). Therefore, we get the following closure model for the kinetic equation:

$$
\frac{\partial n(k,t)}{\partial t} = N_c(k,t). \tag{26}
$$

Let us assume that the kinetic energy $n(k)$ has a power law solution $n(k)=|k|^\gamma$, and let

$$F(k_1,k_2,k_3,k) = C\frac{(n(k_1)n(k_2)n(k_3)n(k))^{\frac{1}{2}}\delta(|k_1|^\alpha + |k_2|^\alpha| - k_3|^\alpha - |k|^\alpha)}{|k_1k_2k_3k|^{\beta/4}\mathcal{A}} \times$$
$$\times\left[\sum_{k_1^*}\frac{[n(l_1)n(l_2)n(l_3)]^{1/2}\delta(\mathcal{A}_{lk_1}-\mathcal{A})}{(n(k_1))^{1/2}|l_1l_2l_3k_1|^{\beta/4}\mathcal{A}_{lk_1}}+\right.$$
$$+\sum_{k_2^*}\frac{[n(l_1)n(l_2)n(l_3)]^{1/2}\delta(\mathcal{A}_{lk_2}-\mathcal{A})}{(n(k_2))^{1/2}|l_1l_2l_3k_2|^{\beta/4}\mathcal{A}_{lk_2}}- \tag{27}$$
$$-\sum_{k_3^*}\frac{[n(l_1)n(l_2)n(l_3)]^{1/2}\delta(\mathcal{A}_{lk_3}-\mathcal{A})}{(n(k_3))^{1/2}|l_1l_2l_3k_3|^{\beta/4}\mathcal{A}_{lk_3}}-$$
$$\left.-\sum_{k^*}\frac{[n(l_1)n(l_2)n(l_3)]^{1/2}\delta(\mathcal{A}_{lk_4}-\mathcal{A})}{(n(k))^{1/2}|l_1l_2l_3k|^{\beta/4}\mathcal{A}_{lk}}\right].$$

Then

$$N_c(k,t) = C\sum_{k^*}F(k_1,k_2,k_3,k).$$

Using *conformal transformation* developed by Zakharov ([**3**], [**4**]) or the self–similarity method derived by Majda, McLaughlin and Tabak in [**2**], by simple calculation, we obtain the following equation (here, we refer [**2**] for details)

$$\frac{\partial n(k,t)}{\partial t} = C\sum F(k_1,k_2,k_3,k)\left[1+\left|\frac{k_3}{k}\right|^y - \left|\frac{k_1}{k}\right|^y - \left|\frac{k_2}{k}\right|^y\right], \tag{28}$$

where

$$y = 3\gamma + \frac{3}{2}\beta + \alpha - 4. \tag{29}$$

For the steady state $\frac{\partial n(k,t)}{\partial t} = 0$ which requires with $y = 0$ or $y = \alpha$. Therefore,

$$\gamma = \begin{cases} \frac{4}{3} - \frac{\beta}{2} - \frac{\alpha}{3} & \text{for} \quad y = \alpha \\ \frac{4}{3} - \frac{\beta}{2} & \text{for} \quad y = 0. \end{cases} \tag{30}$$

3 Numerical Simulation

In previous section we have established a closure model by solving the linearized phase equation. In this section we carry out direct numerical simulations for Eq. (1) using the pseudo-spectral method. We focus on several cases of the parameters (α, β). The computational grid is 8192. First, we change the variable

$$\psi(k,t) = e^{i|k|^\alpha t}\phi(k, t + t_n).$$

Then, ψ satisfies

$$\frac{\partial\psi(k,t)}{\partial t} = -i\frac{e^{i|k|^\alpha t}}{|k|^{\beta/4}}\mathcal{F}\left[\left|\mathcal{F}^{-1}\left(\frac{\psi(k,t)e^{-i|k|^\alpha t}}{|k|^{\beta/4}}\right)\right|^2 \mathcal{F}^{-1}\left(\frac{\psi(k,t)e^{-i|k|^\alpha t}}{|k|^{\beta/4}}\right)\right], \tag{31}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively. We solve the above equation with a forth–order Runge–Kutta scheme, and after each step Δt, we add force and dissipation by multiplying the $\phi(k, t_{n+1})$ by

$$e^{\left[\sum_j f_j \delta_k^j - \nu^-|k|^{-d} - \nu^+|k|^d\right]\Delta t}.$$

Here, following the work of **[2]**, in our numerical simulation, we choice

$$f_j = \begin{cases} \frac{\epsilon^2}{|k|^{\beta+\alpha\gamma-1}} & 12 \le |j| \le 18 \\ 0 & \text{other} \end{cases},$$

$$\nu^- = \frac{\epsilon^2}{K_-^{\beta+\alpha\gamma-d}},$$

$$\nu^+ = \frac{\epsilon^2}{K_+^{\beta+\alpha\gamma+d}},$$

Here $\epsilon = 0.175$, $d = 12$, $K_- = 5$, and $K_+ = 2250$, and γ is as in (30). The ensemble average was replaced by a time average between $t = 20000$ and $t = 30000$.

FIG. 1–FIG. 5 display the energy spectrum, $n(k)$, versus the wave number k for pairs $(\alpha, \beta) = (0.3, 0.5), (0.25, 0.5), (0.25, 0.25), (0.5, 0.25), (0.5, 1.0)$, respectively. A power law spectrum with $|k|^{-\gamma}$ can be identified in the inertial subrange $50 < |k| < 800$. A solid line has been included in each figure to show the theoretical prediction given by Eq. (30). All numerical simulations show that the range that the spectrum exponents agree with our model is larger when β is smaller.

4 Discussion

A closure model has been developed. This model successfully predicted the power law exponents in all of the numerical simulations reported here. As we pointed out before, dispersion is a main factor in weak turbulence. We have found that there are two dispersive terms (cf. (9)–(10)) in the system (1). One term is the linear part of the equation which transforms energy through four wave resonant, consistent with previous results. However, we have also found that the nonlinear term (as shown in (10)), which balances the energy, has an essential effect in weak turbulence. It seems that this nonlinear dispersion has been overlooked in previous studies. We argue in this paper that the latter should be an essential term in weak turbulence.

Acknowledgements

This research is supported by the Department of Energy, under contract W–7405–ENG–36.

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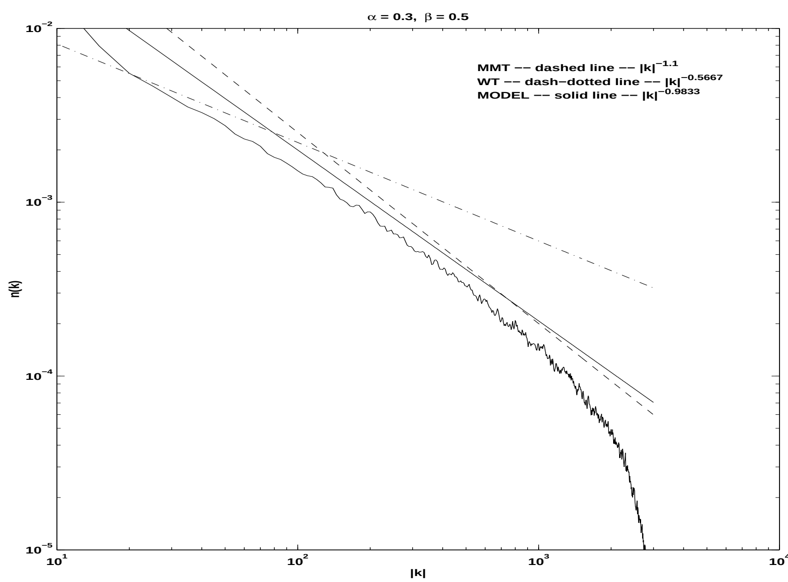

FIG. 1. The energy spectrum, $n(k)$, versus the wave number k for $\alpha = \mathbf{0.3}, \beta = \mathbf{0.5}$. In the inertial range $50 < k < 800$, a power spectrum with $k^{-0.9833}$ can be identified. The solid line is from our theory, the dash-dotted line is from WT (weak turbulent theory) and the dashed line is from MMT (Majda, McLaughlin and Tabak).

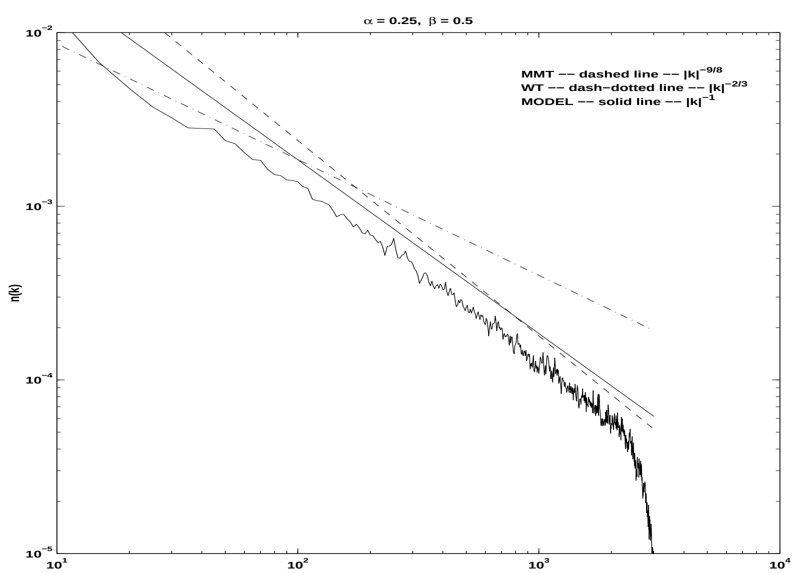

FIG. 2. The energy spectrum, $n(k)$, versus the wave number k for $\alpha = \mathbf{0.25}, \beta = \mathbf{0.5}$. In the inertial range $50 < k < 800$, a power spectrum with $k^{-9/8}$ can be identified.

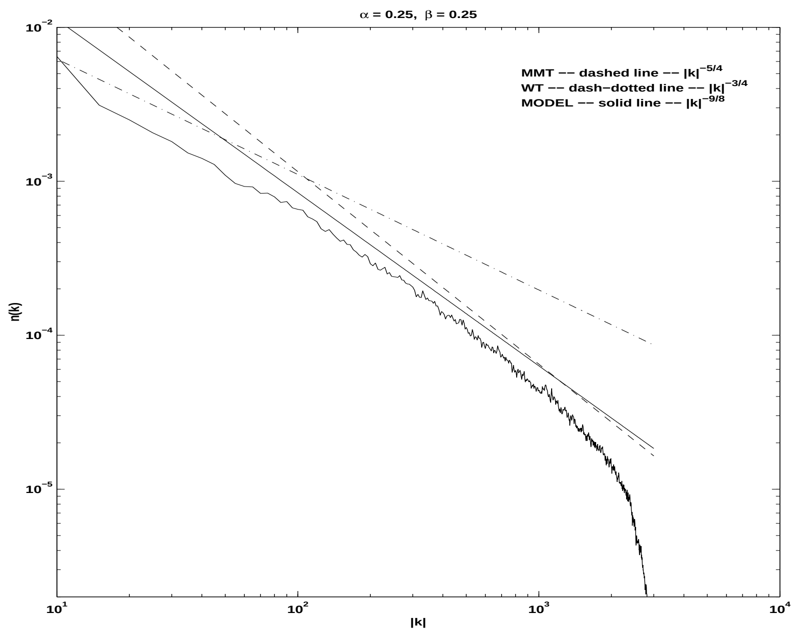

FIG. 3. The energy spectrum, $n(k)$, versus the wave number k for $\alpha = \mathbf{0.25}, \beta = \mathbf{0.25}$. In the inertial range $50 < k < 800$, a power spectrum with $k^{-9/8}$ can be identified.

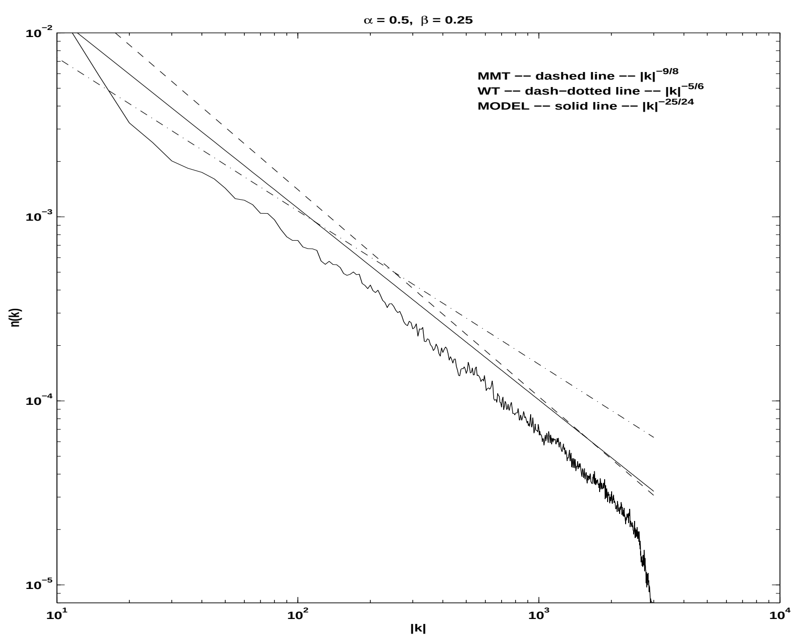

FIG. 4. The energy spectrum, $n(k)$, versus the wave number k for $\alpha = \mathbf{0.5}, \beta = \mathbf{0.25}$. In the inertial range $50 < k < 800$, a power spectrum with $k^{-25/24}$ can be identified.

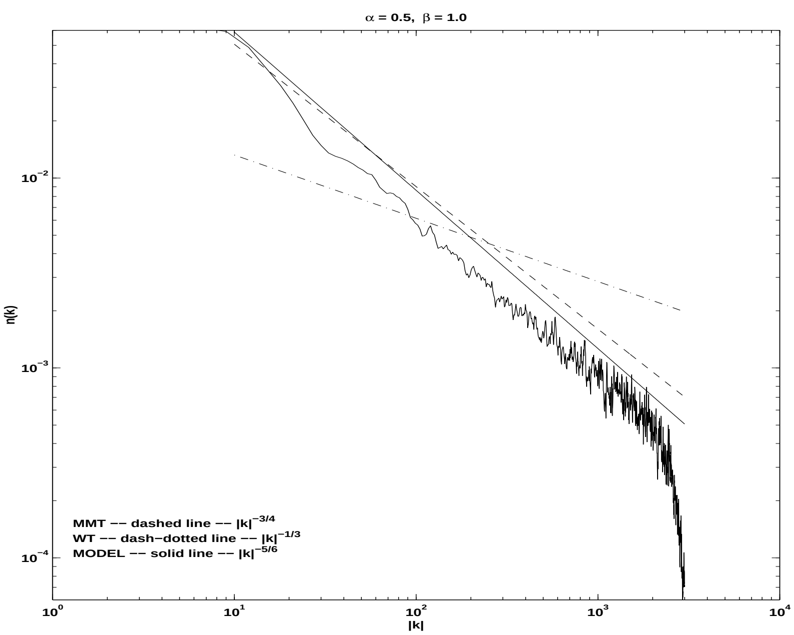

FIG. 5. The energy spectrum, $n(k)$, versus the wave number k for $\alpha = \mathbf{0.5}, \beta = \mathbf{1.0}$. In the inertial range $50 < k < 800$, a power spectrum with $k^{-5/6}$ can be identified.