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A simple construction of a Fortin operator for the two dimensional Taylor–Hood element

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ABSTRACT

A Fortin operator is constructed to verify the discrete inf–sup condition for the lowest order Taylor–Hood element and its variant in two dimensions. The approach is closely related to the recent work by Mardal et al. (2013). That is based on the isomorphism of the tangential edge bubble function space to a subspace of the lowest order edge element space. A more precise characterization of this subspace and a numerical quadrature are introduced to simplify the analysis and remove the mesh restriction. The constructed Fortin operator is stable in both H^1 and L^2 norm for general shape regular triangulations.

1. Introduction

In this paper we construct a Fortin operator to verify the discrete inf–sup condition for the lowest order Taylor–Hood element [1] for solving the following two dimensional Stokes problem:

$$\begin{cases} -\Delta u + \operatorname{grad} p = f & \text{in } \Omega, \\ -\operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, u is the velocity field, p is the pressure, and f is the external force field.

The key to establishing the well posedness of (1) is known as the following inf–sup condition or the so-called div-stability:

$$\inf_{q\in\mathbb{P}} \sup_{v\in\mathbb{V}} \frac{(\operatorname{div} v, q)}{|v|_1 \|q\|} = \alpha > 0, \tag{B}$$

where $\mathbb{P} = L_0^2(\Omega) := \{q \in L^2(\Omega), \int_\Omega q = 0\}$ endowed with L^2-norm $\|\cdot\|$ and L^2-inner product $(\cdot,\cdot)$, and $\mathbb{V} = H_0^1(\Omega;\mathbb{R}^2) := \{u \in L^2(\Omega;\mathbb{R}^2), \nabla u \in L^2(\Omega;\mathbb{R}^4), u|_{\partial\Omega} = 0\}$ with norm $|\cdot|_1 := \|\nabla(\cdot)\|$. Here $X(\Omega;\mathbb{R}^n)$ is used to denote vector function spaces on Ω. Proofs of (B) can be found in many textbooks; see for example [2].

For simplicity of exposition, we assume Ω is a polygon. Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω with size $h = \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$. Based on $\mathcal{T}_h$, finite element spaces $\mathbb{V}_h \subset \mathbb{V}$ and $\mathbb{P}_h \subset \mathbb{P}$ can be constructed. A conforming finite element discretization of (1) is as follows: find $u_h \in \mathbb{V}_h$ and $p_h \in \mathbb{P}_h$ such that

$$\begin{cases} (\nabla u_h, \nabla v_h) + (\operatorname{div} v_h, p_h) = (f, v_h) & \text{for all } v_h \in \mathbb{V}_h, \\ -(\operatorname{div} u_h, q_h) = 0 & \text{for all } q_h \in \mathbb{P}_h, \end{cases} \tag{2}$$

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for given data $f \in L^2(\Omega; \mathbb{R}^2)$. The key to obtaining a stable discretization of Stokes equations is the following discrete inf–sup condition: there exists a constant β independent of the mesh size h such that

$$\inf_{q_h\in\mathbb{P}_h} \sup_{v_h\in\mathbb{V}_h} \frac{(\operatorname{div} v_h, q_h)}{|v_h|_1 \|q_h\|} = \beta > 0. \tag{B_h}$$

The lowest order Taylor–Hood element [1] takes $\mathbb{V}_h$ to be the space of continuous and piecewise quadratic functions and takes $\mathbb{P}_h$ to be the space of continuous and piecewise linear functions. Denoted by $\mathcal{P}^k := \{u \in C(\Omega), u|_T$ is a polynomial of degree $k, \forall T \in \mathcal{T}_h\}$ as the standard kth order Lagrange finite element and $\mathcal{P}_0^k := \mathcal{P}^k \cap H_0^1(\Omega)$. The lowest order Taylor–Hood element can be simply denoted by $\mathcal{P}_0^2 \times \mathcal{P}_0^2 - \mathcal{P}^1$ element. The discrete inf–sup condition for Taylor–Hood $\mathcal{P}_0^2 \times \mathcal{P}_0^2 - \mathcal{P}^1$ element is, however, not easy to verify. See for example Bercovier and Pironneau [3], Verfürth [4], Boland and Nicolaides [5], Stenberg [6], Brezzi and Falk [7], and Boffi [8].

In a recent work [9], Mardal, Schöberl, and Winther construct a Fortin operator from $H_0^1(\Omega; \mathbb{R}^2) \to \mathcal{P}_0^2 \times \mathcal{P}_0^2$ in two dimensions. The key idea to construct the operator is to identify an isomorphism between a subspace of the lowest order edge element space and the tangential edge bubble function space. A Petrov–Galerkin method can then be used to define the desirable operator. We shall follow closely the work [9]. Our main contribution is a more precise characterization of the subspace containing grad $\mathcal{P}^1$ and the middle points quadrature rule for the L^2-inner product. This quadrature, used in Falk [10], simplifies the construction and analysis which consequently removes the mesh condition in [9] required for the L^2 stability of the Fortin operator.

Another new result is a Fortin operator for the modified Taylor–Hood element by adding piecewise constant functions into the pressure space [11,12], i.e., $\mathcal{P}_0^2 \times \mathcal{P}_0^2 - (\mathcal{P}^1 + \mathcal{P}^0)$. The inclusion of piecewise constant pressure will lead to the local mass conservation in each triangle and thus preserve better physical properties [13,14].

We shall use the standard notation of Sobolev spaces and use notation $a \lesssim b$ to denote that there exists a positive constant C independent of the mesh size h, such that $a \le Cb$, and $a \eqsim b$ denote $a \lesssim b \lesssim a$. For a d-dimensional domain ω, we use $|\omega|$ to denote the d-dimensional Lebesgue measure of ω.

2. Fortin operator and discrete inf–sup condition

We shall construct a Fortin operator [15] to verify the discrete div-stability.

Definition 2.1. A linear operator $\Pi_h : \mathbb{V} \to \mathbb{V}_h$ is called a *Fortin operator* if: for any $v \in \mathbb{V}$

(1) $(\operatorname{div} \Pi_h v, q_h) = (\operatorname{div} v, q_h)$ for all $q_h \in \mathbb{P}_h$.
(2) $|\Pi_h v|_1 \le C_1 |v|_1$.

We call the inequality in condition (2) the H^1-stability of the operator Π_h. If instead, an inequality with L^2-norm holds, i.e., $\|\Pi_h v\| \le C_1\|v\|$ for all $v \in \mathbb{V}$, we say Π_h is L^2-stable. For $T \in \mathcal{T}_h$, define $\Omega_T = \cup_{T'\in\mathcal{T}_h, T'\cap T\neq\emptyset} T'$. If the operator Π_h satisfies $\|\Pi_h v\|_T \le C\|v\|_{\Omega_T}$ for all $v \in \mathbb{V}$, we say Π_h is locally L^2-stable. Here we use notation $\|\cdot\|_\omega$ to denote the norm restricted to a subdomain $\omega \subset \Omega$. Obviously local L^2-stability implies L^2-stability since the mesh is shape regular.

The discrete inf–sup condition (B_h) can be derived from the continuous counterpart (B) with the help of a Fortin operator. The following result is standard and can be found in, e.g., [2].

Theorem 2.2. *If there exists a Fortin operator Π_h, then the discrete inf–sup condition (B_h) holds.*

For velocity spaces containing linear finite element space, it suffices to construct a Fortin operator locally stable in the L^2 norm. The proof of the following result is adapted from [16].

Theorem 2.3. *Assume the triangulation $\mathcal{T}_h$ is shape regular and the velocity space $\mathbb{V}_h$ contains piecewise linear and continuous function space. If there exists an operator $\Pi_B : H_0^1(\Omega; \mathbb{R}^2) \to \mathbb{V}_h$ such that $(\operatorname{div} v - \operatorname{div} \Pi_B v, q_h) = 0$ for all $q_h \in \mathbb{P}_h$ and locally stable in L^2 norm, then there exists a Fortin operator $\Pi_h : H_0^1(\Omega; \mathbb{R}^2) \to \mathbb{V}_h$ and stable in both H^1 and L^2 norm.*

Proof. Let $\Pi_1 : H_0^1(\Omega; \mathbb{R}^2) \to \mathcal{P}_0^1 \times \mathcal{P}_0^1$ be the Scott–Zhang quasi-interpolation [17] which satisfies

$$|\Pi_1 u|_{1,T} + h_T^{-1}\|u - \Pi_1 u\|_T \lesssim |u|_{1,\Omega_T}, \qquad \|\Pi_1 u\|_T \lesssim \|u\|_{\Omega_T}, \tag{3}$$

where $h_T = \operatorname{diam}(T)$. We define the Fortin operator as

$$\Pi_h u = \Pi_1 u + \Pi_B(u - \Pi_1 u).$$

Then $(\operatorname{div} u - \operatorname{div} \Pi_h u, q_h) = 0$ for all $q_h \in \mathbb{P}_h$ by definition.

The L^2-stability of Π_h is trivial. Now we prove the H^1-stability. By the inverse inequality, stability of Π_B in L^2-norm, and the property (3) of Π_1

$$|\Pi_B(u - \Pi_1 u)|_1^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\|\Pi_B(u - \Pi_1 u)\|_T^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\|u - \Pi_1 u\|_T^2 \lesssim |u|_1.$$

The desired inequality then follows from the triangle inequality and the H^1 stability of Π_1. $\square$

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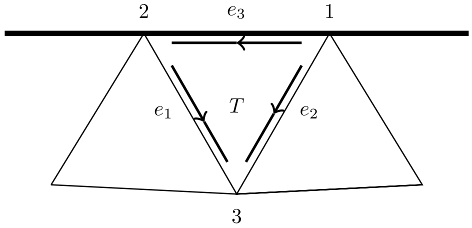

Fig. 1. A typical boundary triangle. The bold line is on the boundary.

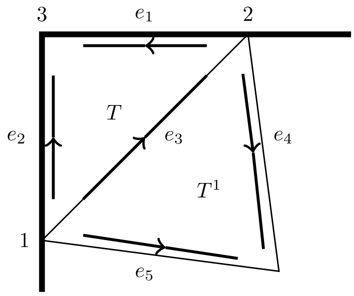

Fig. 2. A typical corner triangle. Bold lines are on the boundary.

3. Modification of edge element spaces

For continuous pressure spaces, by integration by parts, it is equivalent to construct $\Pi_h u \in \mathbb{V}_h$ satisfying $(u - \Pi_h u, \operatorname{grad} q_h) = 0$ for all $q_h \in \mathbb{P}_h$. A key observation in [9] is that grad $\mathcal{P}^1$ is contained in a proper subspace of the lowest order edge element space which is isomorphism to the tangential edge bubble function space. Then a Petrov–Galerkin formulation of the L^2-projection can be used to define $\Pi_h u$. In this section, a constructive characterization of this subspace will be given under a geometric assumption.

For a mesh $\mathcal{T}$ in two dimensions, let $\mathcal{E}$ be the set of all edges, $\mathcal{E}^\partial$ the set of all edges sitting on the boundary, and $\mathcal{E}^0$ the set of all interior edges, i.e. non-boundary edges. We further define $\mathcal{E}_0^0$ as the set of edges with two interior vertices, $\mathcal{E}_\partial^0$ for edges with one interior vertex and one boundary vertex, and $\mathcal{E}_\partial^\partial$ for edges with two boundary vertices. Note that for a boundary edge, two vertices are on the boundary but the converse may not be true. There exists a non-boundary edge with two vertices on the boundary; see edge e_3 in Fig. 2 for such an example. Namely $\mathcal{E}^\partial \subset \mathcal{E}_\partial^\partial$ but $\mathcal{E}_\partial^\partial \cap \mathcal{E}^0$ may not be empty.

Each edge in $e \in \mathcal{E}$ is assigned a direction. To fix the presentation, we shall use the direction pointing from the vertex with a smaller index to a bigger one. The corresponding unit directional vector will be denoted by t_e.

For a vertex i of the triangulation $\mathcal{T}$, we denote by λ_i the standard hat function, i.e., the basis of linear element at vertex i. For an edge e with vertices i, j, the edge bubble function is $b_e = 4\lambda_i\lambda_j$ which is the nodal basis at middle point m_e of e. Namely $b_{e_i}(m_{e_j}) = \delta_{ij}$ for any $e_i, e_j \in \mathcal{E}$. Define $\mathbb{V}_0^{B,t} = \operatorname{span}\{\psi_e = b_e t_e,\ e \in \mathcal{E}^0\} \subset \mathbb{V}_h$ as the tangential edge bubble function space. Note that to impose the boundary condition, only interior edges are used in $\mathbb{V}_0^{B,t}$.

The lowest order edge element space is defined as $\mathbb{ND} = \operatorname{span}\{\phi_e, e \in \mathcal{E}\}$ and $\mathbb{ND}_0 = \operatorname{span}\{\phi_e, e \in \mathcal{E}^0\}$ with $\phi_e = \lambda_i \nabla\lambda_j - \lambda_j\nabla\lambda_i$, where i, j are indices of two vertices of e and $i < j$. It is easy to verify that $\phi_{e_i}(m_{e_j}) \cdot t_{e_j} = \delta_{i,j}/|e_i|$ for any $e_i, e_j \in \mathcal{E}$.

The mapped space grad $\mathcal{P}^1$ is a proper subspace of $\mathbb{ND}$. Note that grad $\mathcal{P}_0^1 \subset \mathbb{ND}_0$ and there is a natural isomorphism of $\mathbb{ND}_0$ and $\mathbb{V}_0^{B,t}$. Only grad λ_i for boundary nodes i is not contained in $\mathbb{ND}_0$. We shall modify the basis of edges adjacency to boundary edges to find a subspace which contains grad $\mathcal{P}^1$ and isomorphism to $\mathbb{V}_0^{B,t}$. In this section, we assume the triangulation satisfies the following geometric property:

(G) The set $\mathcal{E}_\partial^\partial \cap \mathcal{E}^0$ is empty.

Remark 3.1. This condition will be removed in the next section.

We loop over all boundary triangles, which are defined as triangles containing at least one boundary edge, and modify the basis element-wise. Due to the assumption (G), for each boundary triangle, there exists one and only one boundary edge. Without loss of generality, we assume a typical boundary triangle T consists of two boundary vertices 1, 2 and one interior vertex 3. To fix the presentation, we further assume the global index $n_i,\ i = 1, 2, 3$, of these three vertices satisfies $n_1 < n_2 < n_3$. Different ordering only results in a proper sign change in our construction. Inside this triangle, the three edges are labeled such that the ith edge is opposite to the ith vertex; see Fig. 1. We modify the basis associated to the edge e_2 as $\tilde{\phi}_{e_2}|_T = -\operatorname{grad}(\lambda_1|_T) = \phi_{e_2}|_T + \phi_{e_3}|_T$ and $\tilde{\phi}_{e_1}|_T = -\operatorname{grad}(\lambda_2|_T) = \phi_{e_1}|_T - \phi_{e_3}|_T$. Then locally grad $\mathcal{P}^1(T) \subset \operatorname{span}\{\tilde{\phi}_{e_1}, \tilde{\phi}_{e_2}\}$.

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To unify the notation, we denote by $\tilde{\phi}_e = \phi_e$ for $e \in \mathcal{E}_0^0$. Since only a boundary basis function is appended to an interior one, the modified basis $\tilde{\phi}_e \in \mathbb{ND}$ and more importantly the following property still holds for interior edges:

$$\tilde{\phi}_{e_i}(m_{e_j}) \cdot t_{e_j} = \delta_{i,j}/|e_i| \quad \text{for any } e_i, e_j \in \mathcal{E}^0. \tag{4}$$

We define the space

$$\widetilde{\mathbb{ND}_0} = \text{span}\{\tilde{\phi}_e, e \in \mathcal{E}^0\}.$$

Then we have grad $\mathcal{P}^1 \subset \widetilde{\mathbb{ND}_0}$ and

$$\widetilde{\mathbb{ND}_0} \cong \mathbb{V}_0^{B,t}$$

by mapping $\tilde{\phi}_e \to \psi_e$ for each $e \in \mathcal{E}^0$.

By standard scaling argument, it is easy to show both bases $\{\psi_e\}$ and $\{\tilde{\phi}_e\}$ are L^2-stable. More specifically, define the patch $\Omega_e = \{T \in \mathcal{T}, e \in \partial T\}$. For $u = \sum_{e\in\mathcal{E}} u_e \psi_e$,

$$\|u\|^2 \eqsim \sum_{e\in\mathcal{E}^0} u_e^2 \|\psi_e\|^2 \eqsim \sum_{e\in\mathcal{E}^0} u_e^2 |\Omega_e|.$$

And for $u = \sum_{e\in\mathcal{E}^0} u_e \tilde{\phi}_e$,

$$\|u\|^2 \eqsim \sum_{e\in\mathcal{E}^0} u_e^2 \|\tilde{\phi}_e\|^2 \eqsim \sum_{e\in\mathcal{E}^0} u_e^2.$$

4. A Fortin operator for Taylor–Hood element

In this section we construct a Fortin operator for $\mathcal{P}_0^2 \times \mathcal{P}_0^2 - \mathcal{P}^1$ element in two dimensions. We follow closely to [9] but simplify the analysis by using a numerical quadrature.

Using the isomorphism between $\widetilde{\mathbb{ND}_0}$ and $\mathbb{V}_0^{B,t}$, in [9], the authors define $\Pi_{B,t} u \in \mathbb{V}_0^{B,t}$ such that

$$(\Pi_{B,t} u, v) = (u, v) \quad \text{for all } v \in \widetilde{\mathbb{ND}_0}. \tag{5}$$

This is a Petrov–Galerkin formulation of the standard L^2-projection with the trial space $\mathbb{V}_0^{B,t}$ and the test space $\widetilde{\mathbb{ND}_0}$.

To get the well-posedness and L^2-stability of $\Pi_{B,t}$, one has to compute the non-symmetric matrix $M = ((\tilde{\phi}_{e_i}, \psi_{e_i}))_{N\times N}$ with N being the number of interior edges, and prove the stability of M in certain norms. In [9] the calculation has been done for a two dimensional triangular grid with certain geometric constraints. More refined analysis is needed to remove these constraints.

We propose to using the middle points quadrature for the L^2 inner product. Define

$$(u, v)_{h,T} = \frac{|T|}{3} \sum_{k=1}^{3} u(m_{e_k}) v(m_{e_k}), \tag{6}$$

and

$$(u, v)_h = \sum_{T\in\mathcal{T}_h} (u, v)_{h,T}.$$

It is well known that the middle points quadrature is exact for quadratic functions, i.e., if uv is a quadratic polynomial on T, then $(u, v)_{h,T} = (u, v)_T$. In particular, if $u \in \mathbb{V}_0^{B,t}$ and v is piecewise constant, $(u, v)_h = (u, v)$.

We consider the modified problem: find $\Pi_{B,t} u \in \mathbb{V}_0^{B,t}$ such that

$$(\Pi_{B,t} u, v)_h = (u, v), \quad \text{for all } v \in \widetilde{\mathbb{ND}_0}. \tag{7}$$

Using this discrete L^2 inner product, the corresponding matrix is diagonal and positive.

Lemma 4.1. *Let* $D = ((\tilde{\phi}_{e_j}, \psi_{e_i})_h)_{N\times N}$. *Then* $D = \text{diag}(d_{e_1}, d_{e_2}, \ldots, d_{e_N})$ *with* $d_e = |\Omega_e|/(3|e|)$.

Proof. D is diagonal since $\tilde{\phi}_{e_i}(m_{e_j}) \cdot t_{e_j} = \delta_{i,j}/|e_i|$ and $\psi_{e_i}(m_j) = \delta_{i,j} t_{e_i}$. The formula of d_{e_i} is obtained by the direct computation. □

Let $u_e = (u, \tilde{\phi}_e), e \in \mathcal{E}^0$. Then $\Pi_{B,t} u = \sum_{e\in\mathcal{E}^0} d_e^{-1} u_e \psi_e$. We then verify $\Pi_{B,t}$ is a Fortin operator and locally stable in L^2-norm.

Theorem 4.2. *Assume the shape-regular triangulation* $\mathcal{T}_h$ *satisfies the assumption* (G). *Let* $\Pi_{B,t} u$ *be defined by* (7). *Then*

$$(\text{div}\, u - \text{div}\,(\Pi_{B,t} u), q_h) = 0 \quad \textit{for all } q_h \in \mathbb{P}_h, \tag{8}$$

and

$$\|\Pi_{B,t} u\|_T \lesssim \|u\|_{\Omega_T}, \quad \textit{for all } u \in \mathbb{V}. \tag{9}$$

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Proof. Notice that for $q_h \in \mathbb{P}_h$, $\operatorname{grad} q_h \in \widetilde{\mathbb{ND}}_0$ and piecewise constant and $\Pi_{B,t} u \in \mathbb{V}_0^{B,t}$ is piecewise quadratic. Thus the quadrature is exact and by the definition of $\Pi_{B,t}$ (7)

$$(\Pi_{B,t} u, \operatorname{grad} q_h) = (\Pi_{B,t} u, \operatorname{grad} q_h)_h = (u, \operatorname{grad} q_h).$$

Integration by parts yields (8). The local L^2 stability follows from

$$\|\Pi_{B,t} u\|_T^2 \lesssim \sum_{e \in \partial T} d_e^{-2} u_e^2 |\Omega_e| \lesssim \sum_{e \in \partial T} (u, \tilde{\phi}_e)^2 \lesssim \|u\|_{\Omega_T}^2 \sum_{e \in \partial T} \|\tilde{\phi}_e\|^2 \lesssim \|u\|_{\Omega_T}^2. \quad \square$$

Remark 4.3. Compare with the original proof in [9], the above analysis is simplified and requires less constraints on the mesh. For example, mesh conditions for the H^1 stability of the L^2 projection are not needed. The stability in both L^2-norm and H^1-norm is crucial for the uniformly stable preconditioner constructed in [9] for a singular perturbed Stokes problem.

5. Extension to general shape regular meshes

In this section, we construct a Fortin operator for general shape regular triangulations without the geometric assumption (G).

We shall apply different modifications for the corner elements, which are triangles with three vertices on the boundary; see Fig. 2 for a typical corner triangle. Without loss of generality, we assume the edge $e_3 \in \mathcal{E}^0$. We could still modify the basis function associated to e_3 by attaching boundary edge basis, i.e., use $\operatorname{grad} \lambda_1$ and $\operatorname{grad} \lambda_2$. But now since there exists only one interior edge of this triangle, the tangential edge bubble space is not big enough. For example, $\operatorname{grad} \lambda_3$ cannot be spanned by basis functions associated to interior edges. Following [9], we enrich the trial space to $\mathbb{V}_0^{B,t} + \{\psi_e^\perp\}$ where $f^\perp$ is the rotation of the vector f by 90° counterclockwise. For the test function space, we chose $\tilde{\phi}_{e_3} = \phi_{e_3} - \phi_{e_1}/2 - \phi_{e_2}/2 = \operatorname{grad}(\lambda_2 - \lambda_1)/2$ and introduce one more function $\phi_\partial = -\operatorname{grad} \lambda_3 = -\phi_{e_1} - \phi_{e_3}$.

We define $\bar{\Pi}_{B,t} u \in \mathbb{V}_0^{B,t} + \{\psi_{e_3}^\perp\}$ such that

$$(\bar{\Pi}_{B,t} u, v)_h = (u, v) \quad \text{for all } v \in \widetilde{\mathbb{ND}}_0 + \{\phi_\partial\}. \tag{10}$$

Suppose $\bar{\Pi}_{B,t} u = \sum_{e \in \mathcal{E}^0} c_e \psi_e + c_0 \psi_{e_3}^\perp$. We now compute the coefficients c_e and c_0 by choosing different test functions in (10). Let $u_e = (u, \tilde{\phi}_e)$, $e \in \mathcal{E}^0$ and $u_0 = (u, \phi_\partial)$. For $e \notin \Omega_{e_3}$, we chose $v \in \phi_e$ in (10) and compute the coefficient c_e as before $c_e = d_e^{-1} u_e$. Denote by $a_{\partial n} := (\psi_{e_3}^\perp, \phi_\partial)_h$, $a_{\partial e} := (\psi_e, \phi_\partial)_h$ and $a_{en} := (\psi_{e_3}^\perp, \phi_e)_h$. Direct calculation shows that $a_{\partial n} = |e_3|/6$ and $a_{en} = -|e_3|/12$. Since ϕ_∂ involves only the basis of boundary edges, the entry $a_{\partial e} = 0$ for $e \in \mathcal{E}^0$. Choosing $v = \phi_\partial$ in (10), we then get $c_0 = u_0 / a_{\partial n}$. Choosing $v = \phi_{e_i}$, we get $c_{e_i} = d_{e_i}^{-1}(u_{e_i} - a_{e_i n} c_0) = d_{e_i}^{-1}(u_{e_i} + u_0/2)$ for $i = 4, 5$. The local L^2-stability of $\bar{\Pi}_{B,t}$ then follows easily as before.

We are in a position to summarize our main result in the following theorem.

Theorem 5.1. *For $u \in H_0^1(\Omega; \mathbb{R}^2)$, let $\Pi_h u = u_1 + u_2$ where $u_1 = \Pi_1 u$, $u_2 = \Pi_{B,t}(u - u_1)$ with Π_1 being the Scott–Zhang quasi-interpolation mapped to $\mathcal{P}_0^1 \times \mathcal{P}_0^1$ and $\Pi_{B,t}$ defined by (7)/ (10) for triangulations without/with corner triangles. Then Π_h is a Fortin operator for the Taylor–Hood element $\mathcal{P}_0^2 \times \mathcal{P}_0^2 - \mathcal{P}^1$ and stable in both H^1 and L^2 norm.*

6. Modified Taylor–Hood element

In this section, we shall prove the discrete inf–sup stability of the modified Taylor–Hood element $\mathcal{P}_0^2 \times \mathcal{P}_0^2 - (\mathcal{P}^1 + \mathcal{P}^0)$ in two dimensions. Namely piecewise constant function space is included in the pressure space which will lead to the local mass conservation in each triangle and thus preserve better physical properties [18,11–14]. A proof of the discrete inf–sup condition for the enhanced spaces in general setting (general order and two and three dimensions) can be found in [18]. Here we present a simple proof for the lowest order case. Other proofs can be found in [19,13].

Let us define $\mathbb{V}_0^{B,n} = \operatorname{span}\{\psi_e^\perp = b_e n_e, e \in \mathcal{E}^0\}$. The lowest order Raviart–Thomas space is defined as $RT_0 = \operatorname{span}\{\varphi_e, e \in \mathcal{E}^0\}$ with $\varphi_e = (\lambda_i \nabla^\perp \lambda_j - \lambda_j \nabla^\perp \lambda_i)$ where $\nabla^\perp f = (\nabla f)^\perp$. It is easy to verify that $\varphi_{e_i}(m_{e_j}) \cdot n_{e_j} = \delta_{i,j}/|e_i|$ and both $\{\psi_e^\perp\}$ and $\{\varphi_e\}$ are L^2 stable bases.

We define $\Pi_{B,n} u \in \mathbb{V}_0^{B,n}$ such that

$$\int_e \Pi_{B,n} u \cdot n \, ds = \int_e u \cdot n \, ds \quad \text{for all } e \in \mathcal{E}^0. \tag{11}$$

The orthogonality

$$(\operatorname{div} u - \operatorname{div} \Pi_{B,n} u, q_h) = 0$$

for piecewise constant q_h follows from the application of divergence theorem to (11) on each triangle.

We then study the stability of this projection.

Lemma 6.1. *For* $v \in H_0^1(T;\mathbb{R}^2)$, *we have*

$$h_T^{-1}\|\Pi_{B,n}v\|_T + |\Pi_{B,n}v|_{1,T} \lesssim h_T^{-1}\|v\|_T + |v|_{1,T}. \tag{12}$$

Proof. By the inverse inequality, we only need to estimate $h_T^{-1}\|\Pi_{B,n}v\|_{0,T}$ as follows:

$$h_T^{-2}\|\Pi_{B,n}v\|_T^2 \lesssim \sum_{e_i\in\partial T}\left(\int_e v\cdot n\,\mathrm{d}s\right)^2 \lesssim h_T^{-2}\|v\|_T^2 + |v|_{1,T}^2.$$

In the second inequality, we have used Cauchy–Schwarz inequality and the scaled trace theorem for integral on edges: for any function $g \in H^1(T)$

$$\|g\|_e^2 \lesssim h_T^{-1}\|g\|_T^2 + h_T\|\nabla g\|_T^2. \quad \square \tag{13}$$

Theorem 6.2. *Suppose the triangulation satisfies the assumption* (G). *Let* $\Pi_h u = u_1 + u_2 + u_3$ *where* $u_1 = \Pi_1 u$, $u_2 = \Pi_{B,n}(u-u_1)$, *and* $u_3 = \Pi_{B,t}(u-u_1-u_2)$ *with* Π_1 *the Scott–Zhang quasi-interpolation mapped to* $\mathcal{P}_0^1\times\mathcal{P}_0^1$, $\Pi_{B,t}$ *defined in* (7), *and* $\Pi_{B,n}$ *in* (11). *Then* Π_h *is a Fortin operator for the modified Taylor–Hood element* $\mathcal{P}_0^2\times\mathcal{P}_0^2-(\mathcal{P}^1+\mathcal{P}^0)$.

Proof. By construction $(\operatorname{div} u, q_h) = (\operatorname{div}\Pi_h u, q_h)$ for $q_h \in \mathcal{P}^1$. By divergence theorem, $(\operatorname{div} u_3, q_h) = 0$ for all $q_h \in \mathcal{P}_0$ since on edges u_3 contains only tangential component. Therefore $(\operatorname{div} u, q_h) = (\operatorname{div}\Pi_h u, q_h)$ for $q_h \in \mathcal{P}^0$.

We prove the H^1-stability $|\Pi_h u|_1 \lesssim |u|_1$ by considering the three components one by one. By (3), $|u_1|_1 \lesssim |u|$. By (12) and (3), we have

$$|u_2|_1^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\|u_2\|_T^2 \lesssim \sum_{T\in\mathcal{T}_h}\left(h_T^{-1}\|u-u_1\|_{0,T}^2 + |u-u_1|_{1,T}^2\right) \lesssim |u|_1^2.$$

By the inverse equality and the local L^2-stability of $\Pi_{B,t}$, we have

$$|u_3|_1^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\|u_3\|_T^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\|u-u_1\|_T^2 + h_T^{-2}\|u_2\|_T^2 \lesssim |u|_1^2. \quad \square$$

Remark 6.3. The assumption (G) is necessary for the $\mathcal{P}_0^2\times\mathcal{P}_0^2-(\mathcal{P}^1+\mathcal{P}^0)$ element. Suppose there exists a corner element T (shown in Fig. 2). Choosing $q_h = \chi_T$ in (2), the coefficient of the normal edge bubble basis is zero. From the discussion in the previous section, only one tangential edge bubble basis is not large enough to impose the div-stability.

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