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A PDE approach to fractional diffusion: A posteriori error analysis

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ABSTRACT

We derive a computable a posteriori error estimator for the α-harmonic extension problem, which localizes the fractional powers of elliptic operators supplemented with Dirichlet boundary conditions. Our a posteriori error estimator relies on the solution of small discrete problems on anisotropic *cylindrical stars*. It exhibits built-in flux equilibration and is equivalent to the energy error up to data oscillation, under suitable assumptions. We design a simple adaptive algorithm and present numerical experiments which reveal a competitive performance.

1. Introduction

The objective of this work is the derivation and analysis of a computable, efficient and, under certain assumptions, reliable a posteriori error estimator for problems involving fractional powers of the Dirichlet Laplace operator $(-\Delta)^s$ with $s \in (0,1)$, which for convenience we will simply call the fractional Laplacian. Let Ω be an open, connected and bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with boundary $\partial\Omega$, $s \in (0,1)$ and let $f : \Omega \to \mathbb{R}$ be given. We shall be concerned with the following problem: find u such that

$$(-\Delta)^s u = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega. \tag{1.1}$$

One of the main difficulties in the study of problem (1.1) is that the fractional Laplacian is a nonlocal operator; see [5,7,24]. To localize it, Caffarelli and Silvestre showed in [7] that any power of the fractional Laplacian in $\mathbb{R}^n$ can be realized

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as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space $\mathbb{R}_+^{n+1}$. For a bounded domain Ω, the result by Caffarelli and Silvestre has been adapted in [4,8,35], thus obtaining an extension problem which is now posed on the semi-infinite cylinder $\mathcal{C} = \Omega \times (0,\infty)$. This extension is the following mixed boundary value problem:

$$\begin{cases} \operatorname{div}\left(y^\alpha \nabla \mathscr{U}\right) = 0, \quad \text{in } \mathcal{C}, \\ \mathscr{U} = 0, \quad \text{on } \partial_L \mathcal{C}, \qquad \dfrac{\partial \mathscr{U}}{\partial \nu^\alpha} = d_s f, \quad \text{on } \Omega \times \{0\}, \end{cases} \tag{1.2}$$

where $\partial_L \mathcal{C} = \partial\Omega \times [0,\infty)$ is the lateral boundary of $\mathcal{C}$, and d_s is a positive normalization constant that depends only on s; see [5,7] for details. The parameter α is defined as

$$\alpha = 1 - 2s \in (-1,1), \tag{1.3}$$

and the so-called conormal exterior derivative of $\mathscr{U}$ at $\Omega \times \{0\}$ is

$$\frac{\partial \mathscr{U}}{\partial \nu^\alpha} = - \lim_{y \to 0^+} y^\alpha \mathscr{U}_y. \tag{1.4}$$

We will call y the *extended variable* and the dimension $n+1$ in $\mathbb{R}_+^{n+1}$ the *extended dimension* of problem (1.2). The limit in (1.4) must be understood in the distributional sense; see [5,7,8] for more details. As noted in [4,7,8,35], the fractional Laplacian and the Dirichlet to Neumann operator of problem (1.2) are related by

$$d_s(-\Delta)^s u = \frac{\partial \mathscr{U}}{\partial \nu^\alpha} \quad \text{in } \Omega.$$

Based on the ideas presented above, the following simple strategy to find the solution of (1.1) has been proposed and analyzed by the last three authors in [29]: given a sufficiently smooth function f we solve (1.2), thus obtaining a function $\mathscr{U} = \mathscr{U}(x',y)$; setting $u : x' \in \Omega \mapsto u(x') = \mathscr{U}(x',0) \in \mathbb{R}$, we obtain the solution of (1.1). The results of [29] provide an a priori error analysis which combines asymptotic properties of Bessel functions with polynomial interpolation theory on weighted Sobolev spaces. The latter is valid for tensor product elements which may be graded in Ω and exhibit a large aspect ratio in y (anisotropy) which is necessary to fit the behavior of $\mathscr{U}(x',y)$ with $x' \in \Omega$ and $y > 0$. The resulting a priori error estimate is quasi-optimal in both order and regularity for the extended problem (1.2). These results are summarized in Section 3.

The main advantage of the algorithm described above, is that we are solving the local problem (1.2) instead of dealing with the nonlocal operator $(-\Delta)^s$ of problem (1.1). However, this comes at the expense of incorporating one more dimension to the problem, thus raising the question of computational efficiency. A quest for the answer has been the main drive in our recent research program and motivates the study of a posteriori error estimators and adaptivity. The latter is also motivated by the fact that the a priori theory developed in [29] requires $f \in \mathbb{H}^{1-s}(\Omega)$ and Ω convex. If one of these conditions is violated the solution $\mathscr{U}(x',y)$ may have singularities in the direction of the x'-variables and thus exhibit fractional regularity. As a consequence, quasi-uniform refinement of Ω would not result in an efficient solution technique and then an adaptive loop driven by an a posteriori error estimator is essential to recover optimal rates of convergence.

In this work we derive a computable, efficient and, under certain assumptions, reliable a posteriori error estimator and design an adaptive procedure to solve problem (1.2). As the results of [29] show, meshes must be highly anisotropic in the extended dimension y if one intends for the method to be optimal. For this reason, it is imperative to design an a posteriori error estimator which is able to deal with such anisotropic behavior. Before proceeding with our analysis, it is instructive to comment about the anisotropic a posteriori error estimators and analysis advocated in the literature.

A posteriori error estimators are computable quantities, i.e., they may depend on the computed solution, mesh and data, but *not* on the exact solution. They provide information about the quality of approximation of the numerical solution. They are problem-dependent and may be used to make a judicious mesh refinement in order to obtain the best possible approximation with the least amount of computational resources. For isotropic discretizations, i.e., meshes where the aspect ratio of all cells is bounded independently of the refinement level, the theory of a posteriori error estimation is well understood. Starting with the pioneering work of Babuška and Rheinboldt [3], a great deal of work has been devoted to its study. We refer to [1,37] for an overview of the state of the art. However, despite of what might be claimed in the literature, the theory of a posteriori error estimation on anisotropic discretizations, i.e., meshes where the cells have disparate sizes in each direction, is still in its infancy.

To the best of our knowledge the first work that attempts to deal with anisotropic a posteriori error estimation is [34]. In this work, a residual a posteriori error estimator is introduced and allegedly analyzed on anisotropic meshes. However, such analysis relies on assumptions on the exact and discrete solutions and on the mesh, which are neither proved nor there is a way to explicitly enforce them in the course of computations; see [34, §6, Remark 3]. Subsequently, in [21] the concept of *matching function* is introduced in order to derive anisotropic a posteriori error indicators. The correct alignment of the grid with the exact solution is crucial to derive an upper bound for the error. Indeed, this upper bound involves the matching function, which depends on the error itself and then it does not provide a *real computable* quantity; see [21, Theorem 2]. For similar works in this direction see [23,22,28]. In [33], the anisotropic interpolation estimates derived in [16] are used to derive a Zienkiewicz–Zhu type of a posteriori error estimator. However, as properly pointed out in [33, Proposition 2.3], the ensuing upper bound for the error depends on the error itself, and thus, it is not *computable*.

In our case, since the coefficient y^α in (1.2) either degenerates ($s < 1/2$) or blows up ($s > 1/2$), the usual residual estimators do not apply: integration by parts fails! Inspired by [2,26], we deal with both the natural anisotropy of the mesh in the extended variable y and the nonuniform coefficient y^α, upon considering local problems on *cylindrical stars*. The solutions of these local problems allow us to define a computable and anisotropic a posteriori error estimator which, under certain assumptions, is equivalent to the error up to data oscillations terms. In order to derive such a result, a computationally implementable geometric condition needs to be imposed on the mesh, which does not depend on the exact solution of problem (1.2). This approach is of value not only for (1.2), but in general for anisotropic problems since rigorous anisotropic a posteriori error estimators are not available in the literature.

The outline of this paper is as follows. Section 2 sets the framework in which we will operate. Notation and terminology are introduced in Section 2.1. We recall the definition of the fractional Laplacian on a bounded domain via spectral theory in Section 2.2 and, in Section 2.3, we introduce function spaces that are suitable to study problems (1.1) and (1.2). In Section 3 we review the a priori error analysis developed in [29]. The need for a new approach in a posteriori error estimation is examined in Section 4, where we show that the standard approaches either do not work or produce suboptimal results. This justifies the introduction of our new error estimator on *cylindrical stars*. Section 5 is the core of this work and is dedicated to the development and analysis of our new error estimator. After some preliminary setup carried out in Sections 5.1–5.2, in Section 5.3 we introduce and analyze an ideal error estimator that, albeit not computable, sets the stage for Section 5.4 where we devise a fully computable error estimator and show its equivalence, under suitable assumptions, to the error up to data oscillation terms. In Section 6 we review the components of a standard adaptive loop and comment on some implementation details pertinent to the problem at hand. Finally, we present numerical experiments that illustrate and extend our theory.

2. Notation and preliminaries

2.1. Notation

Throughout this work Ω is an open, bounded and connected domain of $\mathbb{R}^n$, $n \geq 1$, with polyhedral boundary $\partial\Omega$. We define the semi-infinite cylinder with base Ω and its lateral boundary, respectively, by

$$\mathcal{C} := \Omega \times (0, \infty), \qquad \partial_L \mathcal{C} := \partial\Omega \times [0, \infty).$$

Given $\mathcal{Y} > 0$ we define the truncated cylinder with base Ω by $\mathcal{C}_\mathcal{Y} := \Omega \times (0, \mathcal{Y})$. The lateral boundary $\partial_L \mathcal{C}_\mathcal{Y}$ is defined accordingly.

Throughout our discussion we will be dealing with objects defined in $\mathbb{R}^{n+1}$ and it will be convenient to distinguish the extended dimension. A vector $x \in \mathbb{R}^{n+1}$, will be denoted by

$$x = \left(x^1, \ldots, x^n, x^{n+1}\right) = \left(x', x^{n+1}\right) = \left(x', y\right),$$

with $x^i \in \mathbb{R}$ for $i = 1, \ldots, n+1$, $x' \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to denote that $\mathcal{X}$ is continuously embedded in $\mathcal{Y}$. We denote by $\mathcal{X}'$ the dual of $\mathcal{X}$ and by $\|\cdot\|_\mathcal{X}$ the norm of $\mathcal{X}$. The relation $a \lesssim b$ indicates that $a \leq Cb$, with a constant C that does not depend on a or b nor the discretization parameters. The value of C might change at each occurrence.

2.2. The fractional Laplace operator

Our definition is based on spectral theory. For any $f \in L^2(\Omega)$, the Lax–Milgram Lemma provides the existence and uniqueness of $w \in H_0^1(\Omega)$ that solves

$$-\Delta w = f \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

The operator $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact, symmetric and positive, whence its spectrum $\{\lambda_k^{-1}\}_{k\in\mathbb{N}}$ is discrete, real, positive and accumulates at zero. Moreover, there exists $\{\varphi_k\}_{k\in\mathbb{N}} \subset H_0^1(\Omega)$, which is an orthonormal basis of $L^2(\Omega)$ and satisfies

$$-\Delta \varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega, \qquad \varphi_k = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Fractional powers of the Dirichlet Laplace operator can then be defined for $w \in C_0^\infty(\Omega)$ by

$$(-\Delta)^s w = \sum_{k=1}^{\infty} \lambda_k^s w_k \varphi_k, \tag{2.2}$$

where $w_k = \int_\Omega w \varphi_k$. By density $(-\Delta)^s$ can be extended to the space

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k \varphi_k : \sum_{k=1}^{\infty} \lambda_k^s w_k^2 < \infty \right\} = \begin{cases} H^s(\Omega), & s \in (0, \frac{1}{2}), \\ H_{00}^{1/2}(\Omega), & s = \frac{1}{2}, \\ H_0^s(\Omega), & s \in (\frac{1}{2}, 1). \end{cases} \tag{2.3}$$

The characterization given by the second equality is shown in [25, Chapter 1]. For $s \in (0,1)$ we denote by $\mathbb{H}^{-s}(\Omega)$ the dual of $\mathbb{H}^s(\Omega)$.

2.3. The Caffarelli–Silvestre extension problem

To exploit the Caffarelli–Silvestre result [7], or its variants [4,6,8], we need to deal with a nonuniformly elliptic equation. To this end, we consider weighted Sobolev spaces with the weight $|y|^\alpha$, $\alpha \in (-1,1)$. If $D \subset \mathbb{R}^{n+1}$, we then define $L^2(|y|^\alpha, D)$ to be the space of all measurable functions defined on D such that

$$\|w\|^2_{L^2(|y|^\alpha,D)} = \int_D |y|^\alpha w^2 < \infty.$$

Similarly we define the weighted Sobolev space

$$H^1\big(|y|^\alpha, D\big) = \big\{w \in L^2\big(|y|^\alpha, D\big) : |\nabla w| \in L^2\big(|y|^\alpha, D\big)\big\},$$

where ∇w is the distributional gradient of w. We equip $H^1(|y|^\alpha, D)$ with the norm

$$\|w\|_{H^1(|y|^\alpha,D)} = \big(\|w\|^2_{L^2(|y|^\alpha,D)} + \|\nabla w\|^2_{L^2(|y|^\alpha,D)}\big)^{1/2}. \tag{2.4}$$

Since $\alpha \in (-1,1)$ we have that $|y|^\alpha$ belongs to the so-called Muckenhoupt class $A_2(\mathbb{R}^{n+1})$; see [15,17,27,36]. This, in particular, implies that $H^1(|y|^\alpha, D)$ equipped with the norm (2.4), is a Hilbert space and the set $C^\infty(D) \cap H^1(|y|^\alpha, D)$ is dense in $H^1(|y|^\alpha, D)$ (cf. [36, Proposition 2.1.2, Corollary 2.1.6], [20] and [17, Theorem 1]). We recall now the definition of Muckenhoupt classes; see [27,36].

Definition 2.5 *(Muckenhoupt class A_2).* Let ω be a weight and $N \geq 1$. We say $\omega \in A_2(\mathbb{R}^N)$ if

$$C_{2,\omega} = \sup_B \left(\fint_B \omega\right)\left(\fint_B \omega^{-1}\right) < \infty, \tag{2.6}$$

where the supremum is taken over all balls B in $\mathbb{R}^N$.

If ω belongs to the Muckenhoupt class $A_2(\mathbb{R}^N)$, we say that ω is an A_2-weight, and we call the constant $C_{2,\omega}$ in (2.6) the A_2-constant of ω.

To study problem (1.2) we define the weighted Sobolev space

$$\mathring{H}^1_L\big(y^\alpha, \mathcal{C}\big) = \big\{w \in H^1\big(y^\alpha, \mathcal{C}\big) : w = 0 \text{ on } \partial_L \mathcal{C}\big\}. \tag{2.7}$$

As [29, (2.21)] shows, the following *weighted Poincaré inequality* holds:

$$\int_{\mathcal{C}} y^\alpha w^2 \lesssim \int_{\mathcal{C}} y^\alpha |\nabla w|^2, \quad \forall w \in \mathring{H}^1_L\big(y^\alpha, \mathcal{C}\big). \tag{2.8}$$

Then, the seminorm on $\mathring{H}^1_L(y^\alpha, \mathcal{C})$ is equivalent to the norm (2.4). For $w \in H^1(y^\alpha, \mathcal{C})$, we denote by $\operatorname{tr}_\Omega w$ its trace onto $\Omega \times \{0\}$, and we recall that the trace operator tr_Ω satisfies, (see [29, Proposition 2.5], [8, Proposition 2.1])

$$\operatorname{tr}_\Omega \mathring{H}^1_L\big(y^\alpha, \mathcal{C}\big) = \mathbb{H}^s(\Omega), \qquad \|\operatorname{tr}_\Omega w\|_{\mathbb{H}^s(\Omega)} \leq C_{\operatorname{tr}_\Omega} \|w\|_{\mathring{H}^1_L(y^\alpha,\mathcal{C})}. \tag{2.9}$$

Let us now describe the Caffarelli–Silvestre result and its extension to bounded domains; see [7,35]. Given $f \in \mathbb{H}^{-s}(\Omega)$, let $u \in \mathbb{H}^s(\Omega)$ be the solution of $(-\Delta)^s u = f$ in Ω. We define the α-harmonic extension of u to the cylinder $\mathcal{C}$, as the function $\mathscr{U} \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$, solution of problem (1.2), namely

$$(-\Delta)^s u = d_s \frac{\partial \mathscr{U}}{\partial \nu^\alpha} \quad \text{in } \Omega, \quad \text{where } d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}.$$

Finally, we must mention that

$$C_{\operatorname{tr}_\Omega} \leq d_s^{-1/2}. \tag{2.10}$$

Indeed, given $\psi \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$ we define $\Psi \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$ as the solution of

$$-\operatorname{div}\big(y^\alpha \nabla \Psi\big) = 0, \quad \text{in } \mathcal{C}, \qquad \Psi = 0, \quad \text{on } \partial_L \mathcal{C}, \qquad \Psi = \operatorname{tr}_\Omega \psi \quad \text{on } \Omega \times \{0\}.$$

It is standard to show that Ψ is the minimal norm extension of $\operatorname{tr}_\Omega \psi$. Moreover, separation of variables gives $d_s \|\operatorname{tr}_\Omega \psi\|^2_{\mathbb{H}^s(\Omega)} = \|\nabla \Psi\|^2_{L^2(y^\alpha,\mathcal{C})}$ [8, Proposition 2.1]. Therefore

$$\|\operatorname{tr}_\Omega \psi\|^2_{\mathbb{H}^s(\Omega)} = \frac{1}{d_s}\|\Psi\|^2_{\mathring{H}^1_L(y^\alpha,\mathcal{C})} \leq \frac{1}{d_s}\|\psi\|^2_{\mathring{H}^1_L(y^\alpha,\mathcal{C})}.$$

Estimate (2.10) will be useful to obtain an upper bound of the error by the estimator.

3. A priori error estimates

In an effort to make this contribution self-contained here we review the main results of [29], which deal with the a priori error analysis of discretizations of problem (1.1). This will also serve to make clear the limitations of this theory, thereby justifying the quest for an a posteriori error analysis. To do so in this section, and this section only, we will assume the following regularity result, which is valid if, for instance, the domain Ω is convex [18]

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta_{x'} w\|_{L^2(\Omega)}, \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega). \tag{3.1}$$

Since $\mathcal{C}$ is unbounded, problem (1.2) cannot be directly approximated with finite-element-like techniques. However, as [29, Proposition 3.1] shows, the solution $\mathscr{U}$ of problem (1.2) decays exponentially in the extended variable y so that, by truncating the cylinder $\mathcal{C}$ to $\mathcal{C}_{\mathcal{Y}}$ and setting a vanishing Dirichlet condition on the upper boundary $y = \mathcal{Y}$, we only incur in an exponentially small error in terms of $\mathcal{Y}$ [29, Theorem 3.5].

Define

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}}) = \left\{ v \in H^1(y^\alpha, \mathcal{C}_{\mathcal{Y}}) : v = 0 \text{ on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\} \right\}.$$

Then, the aforementioned problem reads: find $v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ such that

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla v \nabla \phi = d_s \langle f, \operatorname{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}, \tag{3.2}$$

for all $v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})$, where $\langle \cdot, \cdot \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}$ denotes the duality pairing between $\mathbb{H}^{-s}(\Omega)$ and $\mathbb{H}^s(\Omega)$, which is well defined as a consequence of (2.9).

If $\mathscr{U}$ and v denote the solution of (1.2) and (3.2), respectively, then [29, Theorem 3.5] provides the following exponential estimate

$$\left\| \nabla(\mathscr{U} - v) \right\|_{L^2(y^\alpha, \mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)},$$

where λ_1 denotes the first eigenvalue of the Dirichlet Laplace operator and $\mathcal{Y}$ is the truncation parameter.

In order to study the finite element discretization of problem (3.2) we must first understand the regularity of the solution $\mathscr{U}$, since an error estimate for v, solution of (3.2), depends on the regularity of $\mathscr{U}$ as well [29, §4.1]. We recall that [29, Theorem 2.7] reveals that the second order regularity of $\mathscr{U}$ is much worse in the extended direction, namely

$$\|\Delta_{x'} \mathscr{U}\|_{L^2(y^\alpha, \mathcal{C})} + \|\partial_y \nabla_{x'} \mathscr{U}\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1-s}(\Omega)}, \tag{3.3}$$

$$\|\mathscr{U}_{yy}\|_{L^2(y^\beta, \mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}, \tag{3.4}$$

where $\beta > 2\alpha + 1$. This suggests that *graded* meshes in the extended variable y play a fundamental role. In fact, estimates (3.3)–(3.4) motivate the construction of a mesh over $\mathcal{C}_{\mathcal{Y}}$ as follows. We first consider a graded partition $\mathcal{I}_{\mathcal{Y}}$ of the interval $[0, \mathcal{Y}]$ with mesh points

$$y_k = \left(\frac{k}{M} \right)^\gamma \mathcal{Y}, \quad k = 0, \dots, M, \tag{3.5}$$

where $\gamma > 3/(1-\alpha) = 3/(2s)$. We also consider $\mathscr{T}_\Omega = \{K\}$ to be a conforming and shape regular mesh of Ω, where $K \subset \mathbb{R}^n$ is an element that is isoparametrically equivalent either to the unit cube $[0,1]^n$ or the unit simplex in $\mathbb{R}^n$. The collection of these triangulations $\mathscr{T}_\Omega$ is denoted by $\mathbb{T}_\Omega$. We construct the mesh $\mathscr{T}_{\mathcal{Y}}$ as the tensor product triangulation of $\mathscr{T}_\Omega$ and $\mathcal{I}_{\mathcal{Y}}$. In order to obtain a global regularity assumption for $\mathscr{T}_{\mathcal{Y}}$, we assume that there is a constant $\sigma_{\mathcal{Y}}$ such that if $T_1 = K_1 \times I_1$ and $T_2 = K_2 \times I_2 \in \mathscr{T}_{\mathcal{Y}}$ have nonempty intersection, then

$$\frac{h_{I_1}}{h_{I_2}} \le \sigma_{\mathcal{Y}}, \tag{3.6}$$

where $h_I = |I|$. It is well known that this weak regularity condition on the mesh allows for anisotropy in the extended variable (cf. [14,29]). The set of all triangulations of $\mathcal{C}_{\mathcal{Y}}$ that are obtained with this procedure and satisfy these conditions is denoted by $\mathbb{T}$. Fig. 1 shows an example of this type of meshes in three dimensions.

For $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$, we define the finite element space

$$\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in C^0(\overline{\mathcal{C}_{\mathcal{Y}}}) : W|_T \in \mathcal{P}_1(K) \otimes \mathbb{P}_1(I)\ \forall T \in \mathscr{T}_{\mathcal{Y}},\ W|_{\Gamma_D} = 0 \right\}, \tag{3.7}$$

where $\Gamma_D = \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\}$ is called the Dirichlet boundary; the space $\mathcal{P}_1(K)$ is $\mathbb{P}_1(K)$ — the space of polynomials of total degree at most 1, when the base K of an element $T = K \times I$ is simplicial. If K is an n-rectangle $\mathcal{P}_1(K)$ stands for $\mathbb{Q}_1(K)$ — the space of polynomials of degree not larger than 1 in each variable. We also define $\mathbb{U}(\mathscr{T}_\Omega) = \operatorname{tr}_\Omega \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, i.e., a $\mathcal{P}_1(K)$ finite element space over the mesh $\mathscr{T}_\Omega$.

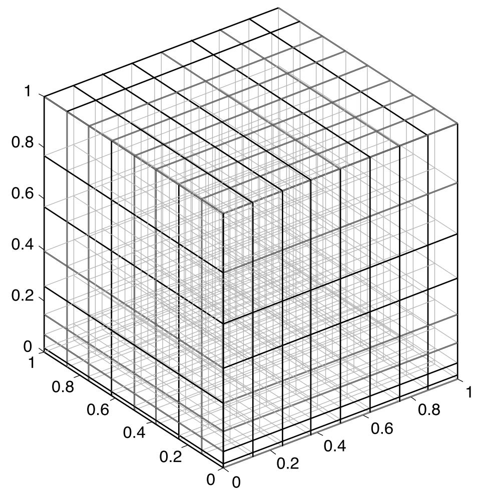


Fig. 1. A three dimensional graded mesh of the cylinder $(0,1)^2 \times (0, \mathcal{Y})$ with 392 degrees of freedom. The mesh is constructed as a tensor product of a quasi-uniform mesh of $(0,1)^2$ with cardinality 49 and the image of the quasi-uniform partition of the interval $(0,1)$ with cardinality 8 under the mapping (3.5).

The Galerkin approximation of (3.2) is given by the unique function $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ such that

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla W = d_s \langle f, \operatorname{tr}_\Omega W\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)}, \quad \forall W \in \mathbb{V}(\mathscr{T}_\mathcal{Y}). \tag{3.8}$$

Existence and uniqueness of $V_{\mathscr{T}_\mathcal{Y}}$ immediately follows from $\mathbb{V}(\mathscr{T}_\mathcal{Y}) \subset \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ and the Lax–Milgram Lemma. It is trivial also to obtain a best approximation result *à la* Cea. This best approximation result reduces the numerical analysis of problem (3.8) to a question in approximation theory which in turn can be answered with the study of piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces; see [29,30]. Exploiting the Cartesian structure of the mesh is possible to handle anisotropy in the extended variable, construct a quasi-interpolant $\Pi_{\mathscr{T}_\mathcal{Y}} : L^1(\mathcal{C}_\mathcal{Y}) \to \mathbb{V}(\mathscr{T}_\mathcal{Y})$, and obtain

$$\|v - \Pi_{\mathscr{T}_\mathcal{Y}} v\|_{L^2(y^\alpha,T)} \lesssim h_K \|\nabla_{x'} v\|_{L^2(y^\alpha,S_T)} + h_I \|\partial_y v\|_{L^2(y^\alpha,S_T)},$$
$$\left\|\partial_{x_j}(v - \Pi_{\mathscr{T}_\mathcal{Y}} v)\right\|_{L^2(y^\alpha,T)} \lesssim h_K \|\nabla_{x'} \partial_{x_j} v\|_{L^2(y^\alpha,S_T)} + h_I \|\partial_y \partial_{x_j} v\|_{L^2(y^\alpha,S_T)},$$

with $j = 1, \dots, n+1$; see [29, Theorems 4.6–4.8] and [30] for details. However, since $\mathscr{U}_{yy} \approx y^{-\alpha-1}$ as $y \approx 0$, we realize that $\mathscr{U} \notin H^2(y^\alpha, \mathcal{C}_\mathcal{Y})$ and the second estimate is not meaningful for $j = n+1$. In view of estimate (3.4) it is necessary to measure the regularity of $\mathscr{U}_{yy}$ with a stronger weight and thus compensate with a graded mesh in the extended dimension. This makes anisotropic estimates essential.

Notice that $\#\mathscr{T}_\mathcal{Y} = M\,\#\mathscr{T}_\Omega$, and that $\#\mathscr{T}_\Omega \approx M^n$ implies $\#\mathscr{T}_\mathcal{Y} \approx M^{n+1}$. Finally, if $\mathscr{T}_\Omega$ is shape regular and quasi-uniform, we have $h_{\mathscr{T}_\Omega} \approx (\#\mathscr{T}_\Omega)^{-1/n}$. All these considerations allow us to obtain the following result; see [29, Theorem 5.4] and [29, Corollary 7.11].

Theorem 3.9 *(A priori error estimate). Let $\mathscr{T}_\mathcal{Y} \in \mathbb{T}$ be a tensor product grid, which is quasi-uniform in Ω and graded in the extended variable so that (3.5) holds. If $\mathbb{V}(\mathscr{T}_\mathcal{Y})$ is defined by (3.7) and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ is the Galerkin approximation defined by (3.8), then we have*

$$\|\mathscr{U} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(y^\alpha,\mathcal{C})} \lesssim \left|\log(\#\mathscr{T}_\mathcal{Y})\right|^s (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

where $\mathcal{Y} \approx \log(\#\mathscr{T}_\mathcal{Y})$. Alternatively, if u denotes the solution of (1.1), then

$$\left\|u - V_{\mathscr{T}_\mathcal{Y}}(\cdot,0)\right\|_{\mathbb{H}^s(\Omega)} \lesssim \left|\log(\#\mathscr{T}_\mathcal{Y})\right|^s (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

Remark 3.10 *(Domain and data regularity).* The results of Theorem 3.9 hold true only if $f \in \mathbb{H}^{1-s}(\Omega)$ and the domain Ω is such that (3.1) holds.

4. A posteriori error estimators: the search for a new approach

The function $\mathscr{U}$, solution of the α-harmonic extension problem (1.2), has a singular behavior on the extended variable y, which is compensated by considering anisotropic meshes in this direction as dictated by (3.5). However, the solution $\mathscr{U}$,

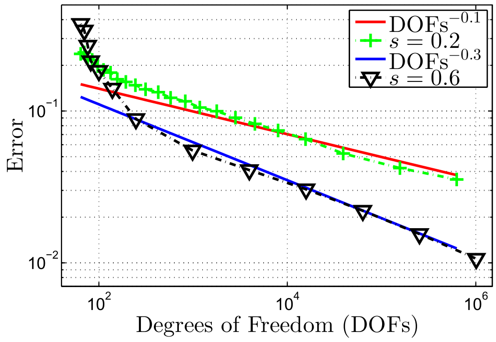


Fig. 2. Computational rate of convergence $\#(\mathscr{T}_{\mathcal{Y}})^{-s/2}$ for an isotropic adaptive algorithm for $n = 1$, $s = 0.2$ and $s = 0.6$.

may also have singularities in the direction of the x'-variables and thus exhibit fractional regularity, which would not allow us to attain the almost optimal rate of convergence given by Theorem 3.9. In fact, as Remark 3.10 indicates, it is necessary to require that $f \in \mathbb{H}^{1-s}(\Omega)$ and that the domain has the property (3.1) in order to have an almost optimal rate of convergence. If any of these two conditions fail singularities may develop in the direction of the x'-variables, whose characterization is as yet an open problem; see [29, §6.3] for an illustration of this situation. The objective of this work is to derive a computable, efficient and, under suitable assumptions, reliable a posteriori error estimator for the finite element approximation of problem (3.2) which can resolve such singularities via an adaptive algorithm.

Let us begin by exploring the standard approaches advocated in the literature. We will see that they fail, thereby justifying the need for a new approach, which we will develop in Section 5.

4.1. Residual estimators

Simply put, the so-called residual error estimators use the strong form of the local residual as an indicator of the error. To obtain the strong form of the equation, integration by parts is necessary. Let us consider an element $T \in \mathscr{T}_{\mathcal{Y}}$ and integrate by parts the term

$$\int_T y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \cdot \nabla W = \int_{\partial T} W y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \cdot \nu - \int_T \operatorname{div}\big(y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}}\big) W,$$

where ν denotes the unit outer normal to T. Since $\alpha \in (-1, 1)$ the boundary integral is meaningless for $y = 0$. As we see, even the very first step (integration by parts) in the derivation of a residual a posteriori error estimator fails! At this point there is nothing left to do but to consider a different type of estimator.

4.2. Local problems on stars over isotropic refinements

Inspired by [2,26] we can construct, over shape regular meshes, a computable error estimator based on the solution of small discrete problems on stars. Its construction and analysis is similar to the developments of Section 5 so we shall not dwell on this any further. Since we consider shape regular meshes, such estimator is equivalent to the error up to data oscillation without any additional conditions on the mesh, but under some suitable assumptions; see Section 5 for details. Then, we have designed an adaptive algorithm driven by such a posteriori error estimator on shape regular meshes [10,26], and here we illustrate its performance with a simple but revealing numerical example. We let $\Omega = (0, 1)$ and $s \in (0, 1)$. The right hand side is $f(x') = \pi^{2s} \sin(\pi x')$, so that $u(x') = \sin(\pi x')$, and the solution $\mathscr{U}$ to (1.2) is

$$\mathscr{U}(x', y) = \frac{2^{1-s}\pi^s}{\Gamma(s)} \sin\big(\pi x'\big) K_s(\pi y),$$

where K_s denotes the modified Bessel function of the second kind; see [29, §2.4] for details. We point out that for the α-harmonic extension we are solving a two dimensional problem so the optimal rate of convergence in the $H^1(y^\alpha, \mathcal{C})$-seminorm that we expect is $\mathcal{O}(\#\mathscr{T}_{\mathcal{Y}}^{-0.5})$. Fig. 2 shows the experimental rate of convergence of this algorithm for the cases $s = 0.2$ and $s = 0.6$ which, as we see, is

$$\mathcal{O}\big(\#\mathscr{T}_{\mathcal{Y}}^{-s/2}\big)$$

and coincides with the suboptimal one obtained with quasi-uniform refinement; see [29, §5.1]. These numerical experiments show that adaptive isotropic refinement cannot be optimal, thus justifying the need to introduce *cylindrical stars* together with a new anisotropic error estimator, which will treat the x'-coordinates and the extended direction, y, separately.

5. A posteriori error estimators: cylindrical stars

It has been proven rather challenging to derive and analyze a posteriori error estimators over a fairly general anisotropic mesh. For this reason, we introduce an *implementable* geometric condition which will allow us to consider graded meshes in Ω in order to compensate for possible singularities in the x'-variables, while preserving the anisotropy in the extended direction, necessary to retain optimal orders of approximation. We thus assume the following condition over the family of meshes $\mathbb{T}$: there exists a positive constant $C_{\mathbb{T}}$ such that for every mesh $\mathscr{T}_{\mathcal{Y}} \in \mathbb{T}$

$$h_{\mathcal{Y}} \le C_{\mathbb{T}} h_{z'}, \tag{5.1}$$

for all the interior nodes z' of $\mathscr{T}_\Omega$, where $h_{\mathcal{Y}}$ denotes the *largest* mesh size in the y direction, and $h_{z'} \approx |S_{z'}|^{1/n}$; see Section 5.1 for the precise definition of $h_{z'}$ and $S_{z'}$. We remark that this condition is satisfied in the case of quasi-uniform refinement in the variable x', which is a consequence of the convexity of the function involved in (3.5). In fact, a simple computation shows

$$h_{\mathcal{Y}} = y_M - y_{M-1} = \frac{\mathcal{Y}}{M^\gamma}\big((M)^\gamma - (M-1)^\gamma\big) \le \gamma \frac{\mathcal{Y}}{M}, \tag{5.2}$$

where $\gamma > 3/(1-\alpha) = 3/(2s)$. We must reiterate that this mesh restriction is fully implementable. We refer the reader to Section 6 for more details on this.

Remark 5.3 *(s-Independent mesh grading).* We point out that the term $\gamma = \gamma(s)$ in (5.2) deteriorates as s becomes small because $\gamma > 3/(2s)$. However, a modified mesh grading in the y-direction has been proposed in [12, §7.3], which does not change the ratio of degrees of freedom in Ω and the extended dimension by more than a constant and provides a uniform bound with respect to $s \in (0,1)$, i.e., $h_{\mathcal{Y}} \le C\mathcal{Y}/M$ where C does not depends on s.

5.1. Preliminaries

Let us begin the discussion on a posteriori error estimation with some terminology and notation. Given a node z on the mesh $\mathscr{T}_{\mathcal{Y}}$, we exploit the tensor product structure of $\mathscr{T}_{\mathcal{Y}}$, and we write $z = (z', z'')$ where z' and z'' are nodes on the meshes $\mathscr{T}_\Omega$ and $\mathcal{I}_{\mathcal{Y}}$ respectively.

Given a cell $K \in \mathscr{T}_\Omega$, we denote by $\mathcal{N}(K)$ and $\mathring{\mathcal{N}}(K)$ the set of nodes and interior nodes of K, respectively. We set

$$\mathcal{N}(\mathscr{T}_\Omega) = \bigcup_{K\in\mathscr{T}_\Omega} \mathcal{N}(K), \qquad \mathring{\mathcal{N}}(\mathscr{T}_\Omega) = \bigcup_{K\in\mathscr{T}_\Omega} \mathring{\mathcal{N}}(K).$$

Given $T \in \mathscr{T}_{\mathcal{Y}}$, we define $\mathcal{N}(T)$ and $\mathring{\mathcal{N}}(T)$ accordingly, i.e., as the set of nodes and interior and Neumann nodes of T, respectively. Similarly, we define $\mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$ and $\mathcal{N}(\mathscr{T}_{\mathcal{Y}})$. Any discrete function $W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ is uniquely characterized by its nodal values on the set $\mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$. Moreover, the functions $\phi_z \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, $z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$, such that $\phi_z(\mathrm{w}) = \delta_{z\mathrm{w}}$ for all $\mathrm{w} \in \mathcal{N}(\mathscr{T}_{\mathcal{Y}})$ are the canonical basis of $\mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, i.e.,

$$W = \sum_{z\in\mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})} W(z)\phi_z.$$

The functions $\{\phi_z : z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})\}$ are the so-called *shape functions* of $\mathbb{V}(\mathscr{T}_{\mathcal{Y}})$. Analogously, given a node $z' \in \mathring{\mathcal{N}}(\mathscr{T}_\Omega)$, we also consider the discrete functions $\varphi_{z'} \in \mathbb{U}(\mathscr{T}_\Omega) = \mathrm{tr}_\Omega\, \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ defined by $\varphi_{z'}(\mathrm{w}') = \delta_{z'\mathrm{w}'}$ for all $\mathrm{w}' \in \mathcal{N}(\mathscr{T}_\Omega)$. The set $\{\varphi_{z'} : z' \in \mathring{\mathcal{N}}(\mathscr{T}_\Omega)\}$ is the canonical basis of $\mathbb{U}(\mathscr{T}_\Omega)$.

The shape functions $\{\phi_z : z \in \mathcal{N}(\mathscr{T}_{\mathcal{Y}})\}$ satisfy two properties which will prove useful in the sequel. First, we have the so-called *partition of unity property*, i.e.,

$$\sum_{z\in\mathcal{N}(\mathscr{T}_{\mathcal{Y}})} \phi_z = 1 \quad \text{in } \bar{\mathcal{C}}_{\mathcal{Y}}. \tag{5.4}$$

Second, for any $z \in \mathring{\mathcal{N}}(\mathscr{T}_{\mathcal{Y}})$, the corresponding shape function ϕ_z belongs to $\mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ whence we have the so-called *Galerkin orthogonality*, i.e.,

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla(v - V_{\mathscr{T}_{\mathcal{Y}}})\nabla\phi_z = 0. \tag{5.5}$$

The partition of unity property also holds for the shape functions $\{\varphi_{z'} : z' \in \mathcal{N}(\mathscr{T}_\Omega)\}$:

$$\sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \varphi_{z'} = 1 \quad \text{in } \bar{\Omega}.$$

Given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ and the associated shape function $\varphi_{z'}$, we define the *extended* shape function $\tilde{\varphi}_{z'}$ by $\tilde{\varphi}_{z'}(x', y) = \varphi_{z'}(x')\mathbb{1}_{(0,\mathcal{Y})}(y)$. These functions satisfy the following partition of unity property:

$$\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \tilde{\varphi}_{z'} = 1 \quad \text{in } \bar{\mathcal{C}}_{\mathcal{Y}}. \tag{5.6}$$

Given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, we define the *star* around z' as

$$S_{z'} = \bigcup_{K \ni z'} K \subset \Omega,$$

and the *cylindrical star* around z' as

$$\mathcal{C}_{z'} := \bigcup \left\{ T \in \mathscr{T}_{\mathcal{Y}} : T = K \times I,\ K \ni z' \right\} = S_{z'} \times (0, \mathcal{Y}) \subset \mathcal{C}_{\mathcal{Y}}.$$

Given an element $K \in \mathscr{T}_\Omega$ we define its *patch* as $S_K := \bigcup_{z' \in K} S_{z'}$. For $T \in \mathscr{T}_{\mathcal{Y}}$ its patch S_T is defined similarly. Given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ we define its *cylindrical patch* as

$$\mathcal{D}_{z'} := \bigcup \left\{ \mathcal{C}_{w'} : w' \in S_{z'} \right\} \subset \mathcal{C}_{\mathcal{Y}}.$$

For each $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ we set $h_{z'} := \min\{h_K : K \ni z'\}$.

5.2. Local weighted Sobolev spaces

In order to define the local a posteriori error estimators we first need to define some local weighted Sobolev spaces.

Definition 5.7 *(Local spaces).* Given $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ and its associated cylindrical star $\mathcal{C}_{z'}$, we define

$$\mathbb{W}(\mathcal{C}_{z'}) = \left\{ w \in H^1\left(y^\alpha, \mathcal{C}_{z'}\right) : w = 0 \text{ on } \partial\mathcal{C}_{z'} \setminus \Omega \times \{0\} \right\}.$$

The space $\mathbb{W}(\mathcal{C}_{z'})$ defined above is Hilbert due to the fact that the weight $|y|^\alpha$ belongs to the class $A_2(\mathbb{R}^{n+1})$; see Definition 2.5. Moreover, as the following result shows, a weighted Poincaré-type inequality holds and, consequently, the semi-norm $|||w|||_{\mathcal{C}_{z'}} = \|\nabla w\|_{L^2(y^\alpha, \mathcal{C}_{z'})}$ defines a norm on $\mathbb{W}(\mathcal{C}_{z'})$; see also [29, §2.3].

Proposition 5.8 *(Weighted Poincaré inequality). Let $z' \in \mathcal{N}(\mathscr{T}_\Omega)$. If the function $w \in \mathbb{W}(\mathcal{C}_{z'})$, then we have*

$$\|w\|_{L^2(y^\alpha, \mathcal{C}_{z'})} \lesssim \mathcal{Y} |||w|||_{\mathcal{C}_{z'}}. \tag{5.9}$$

Proof. By density [36, Corollary 2.1.6], it suffices to reduce the considerations to a smooth function w. Given $x' \in S_{z'}$, we have that $w(x', \mathcal{Y}) = 0$ so that

$$w\left(x', y\right) = -\int_y^{\mathcal{Y}} \partial_y w\left(x', \xi\right) \mathrm{d}\xi.$$

Multiplying the expression above by $|y|^\alpha$, integrating over $\mathcal{C}_{z'}$, and using the Cauchy Schwarz inequality, we arrive at

$$\begin{aligned}
\int_{\mathcal{C}_{z'}} |y|^\alpha \left| w\left(x', y\right)\right|^2 \mathrm{d}x' \,\mathrm{d}y &\le \int_{\mathcal{C}_{z'}} |y|^\alpha \left(\int_0^{\mathcal{Y}} |\xi|^\alpha \left|\partial_y w\left(x', \xi\right)\right|^2 \mathrm{d}\xi \int_0^{\mathcal{Y}} |\xi|^{-\alpha} \,\mathrm{d}\xi \right) \mathrm{d}x' \,\mathrm{d}y \\
&= \int_0^{\mathcal{Y}} |y|^\alpha \,\mathrm{d}y \int_0^{\mathcal{Y}} |\xi|^{-\alpha} \,\mathrm{d}\xi \int_{\mathcal{C}_{z'}} |\xi|^\alpha \left|\partial_y w\left(x', \xi\right)\right|^2 \mathrm{d}x' \,\mathrm{d}\xi \\
&\le C_{2,|y|^\alpha} \mathcal{Y}^2 \int_{\mathcal{C}_{z'}} |y|^\alpha \left|\partial_y w\left(x', y\right)\right|^2 \mathrm{d}x' \,\mathrm{d}y,
\end{aligned}$$

where in the third inequality we used that $y^\alpha \in A_2(\mathbb{R}^{n+1})$. In conclusion,

$$\|w\|_{L^2(y^\alpha, \mathcal{C}_{z'})} \lesssim \mathcal{Y} \|\partial_y w\|_{L^2(y^\alpha, \mathcal{C}_{z'})},$$

which is (5.9). □

Remark 5.10 *(Anisotropic weighted Poincaré inequality).* Let $z' \in \mathcal{N}(\mathscr{T}_\Omega)$. If $w \in \mathbb{W}(\mathcal{C}_{z'})$, then by extending the one-dimensional argument in the proof of Proposition 5.8 to an n-dimensional setting, we can also derive

$$\|w\|_{L^2(y^\alpha,\mathcal{C}_{z'})} \lesssim h_{z'} \|\nabla_{x'} w\|_{L^2(y^\alpha,\mathcal{C}_{z'})}.$$

5.3. An ideal a posteriori error estimator

Here we define an *ideal* a posteriori error estimator on anisotropic meshes which is *not computable*. However, it provides the intuition required to define a discrete and computable error indicator, as explained in Section 5.4. On the basis of assumption (5.1), we prove that this ideal error estimator is equivalent to the error without any oscillation terms.

Inspired by [2,9,26] we define $\zeta_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$ to be the solution of

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla \zeta_{z'} \nabla \psi = d_s \langle f, \operatorname{tr}_\Omega \psi \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla \psi, \tag{5.11}$$

for all $\psi \in \mathbb{W}(\mathcal{C}_{z'})$. The existence and uniqueness of $\zeta_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$ is guaranteed by the Lax–Milgram Lemma and the weighted Poincaré inequality of Proposition 5.8. The continuity of the right hand side of (5.11), as a linear functional in $\mathbb{W}(\mathcal{C}_{z'})$, follows from (2.9) and the Cauchy–Schwarz inequality. We then define the global error estimator

$$\tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}} = \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \tilde{\mathscr{E}}_{z'}^2 \right)^{1/2}, \tag{5.12}$$

in terms of the local error indicators

$$\tilde{\mathscr{E}}_{z'} = |||\zeta_{z'}|||_{\mathcal{C}_{z'}}. \tag{5.13}$$

The properties of this ideal estimator are as follows.

Proposition 5.14 *(Ideal estimator). Let $v \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve (3.2) and (3.8), respectively. Then, the ideal estimator $\tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}}$, defined in (5.12)–(5.13), satisfies*

$$\left\|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\right\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \tilde{\mathscr{E}}_{\mathscr{T}_\mathcal{Y}}, \tag{5.15}$$

and for all $z' \in \mathcal{N}(\mathscr{T}_\Omega)$

$$\tilde{\mathscr{E}}_{z'} \le \left\|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\right\|_{L^2(y^\alpha,\mathcal{C}_{z'})}. \tag{5.16}$$

Proof. If $e_{\mathscr{T}_\mathcal{Y}} := v - V_{\mathscr{T}_\mathcal{Y}}$ denotes the error, then for any $w \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ we have

$$\begin{aligned}
\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla w &= d_s \langle f, \operatorname{tr}_\Omega w \rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla w \\
&= d_s \big\langle f, \operatorname{tr}_\Omega (w - W) \big\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla (w - W) \\
&= \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} d_s \big\langle f, \operatorname{tr}_\Omega \big[(w - W)\tilde{\varphi}_{z'}\big] \big\rangle_{\mathbb{H}^{-s}(\Omega)\times\mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla \big[(w - W)\tilde{\varphi}_{z'}\big]
\end{aligned}$$

for any $W \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$, where to derive the expression above we have used Galerkin orthogonality (5.5) and the partition of unity property (5.6).

Notice that for each $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ the function $(w - W)\tilde{\varphi}_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$. Indeed, if z' is an interior node,

$$(w - W)\tilde{\varphi}_{z'}|_{\partial\mathcal{C}_{z'} \setminus \Omega\times\{0\}} = 0 \tag{5.17}$$

because of the vanishing property of the shape function $\varphi_{z'}$ on $\partial S_{z'}$ together with the fact that $w = W = 0$ on $\Omega \times \{\mathcal{Y}\}$; otherwise, if z' is a Dirichlet node then $w = W = 0$ on $\partial\mathcal{C}_{z'}$, and we get (5.17).

Set $W = \Pi_{\mathscr{T}_\mathcal{Y}} w$, where $\Pi_{\mathscr{T}_\mathcal{Y}}$ denotes the quasi-interpolation operator introduced in [29, §4]; see also [30]. This yields a bound on $|||(w - W)\tilde{\varphi}_{z'}|||_{\mathcal{C}_{z'}}$ as follows:

$$|||(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\tilde{\varphi}_{z'}|||_{\mathcal{C}_{z'}}^2 \lesssim \int_{\mathcal{C}_{z'}} y^\alpha \left|\nabla(w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\right|^2 \tilde{\varphi}_{z'}^2 + \int_{\mathcal{C}_{z'}} y^\alpha |w - \Pi_{\mathscr{T}_\mathcal{Y}} w|^2 |\nabla_{x'} \tilde{\varphi}_{z'}|^2 \lesssim |||w|||_{\mathcal{D}_{z'}}^2.$$

To bound the first term above we use the local stability of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ [29, Theorems 4.7 and 4.8] together with the fact that $0 \le \tilde{\varphi}_{z'} \le 1$ for all $x \in \mathcal{C}_{\mathcal{Y}}$. For the second term we resort to the local approximation properties of $\Pi_{\mathscr{T}_{\mathcal{Y}}}$ [29, Theorems 4.7 and 4.8]

$$\int_{\mathcal{C}_{z'}} y^\alpha |w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w|^2 |\nabla_{x'} \tilde{\varphi}_{z'}|^2 \lesssim \frac{1}{h_{z'}^2}\left(h_{z'}^2 \|\nabla_{x'} w\|^2_{L^2(y^\alpha, \mathcal{D}_{z'})} + h_{\mathcal{Y}}^2 \|\partial_y w\|^2_{L^2(y^\alpha, \mathcal{D}_{z'})}\right) \lesssim |||w|||^2_{\mathcal{D}_{z'}}, \tag{5.18}$$

where we used that $|\nabla_{x'} \tilde{\varphi}_{z'}| = |\nabla_{x'} \varphi_{z'}| \lesssim h_{z'}^{-1}$ together with (5.1).

Set $\psi_{z'} = (w - \Pi_{\mathscr{T}_{\mathcal{Y}}} w)\tilde{\varphi}_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$ as test function in (5.11) to obtain

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla e_{\mathscr{T}_{\mathcal{Y}}} \nabla w = \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla \zeta_{z'} \nabla \psi_{z'} \lesssim \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} |||\zeta_{z'}|||_{\mathcal{C}_{z'}} |||w|||_{\mathcal{D}_{z'}}$$

$$\lesssim \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} |||\zeta_{z'}|||^2_{\mathcal{C}_{z'}} \right)^{1/2} \|\nabla w\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} = \tilde{\mathscr{E}}_{\mathscr{T}_{\mathcal{Y}}} \|\nabla w\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})},$$

where we used that $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||w|||_{\mathcal{D}_{z'}}$ and the finite overlapping property of the stars $S_{z'}$. Since w is arbitrary, we set $w = e_{\mathscr{T}_{\mathcal{Y}}}$ and obtain (5.15).

Finally, inequality (5.16) is immediate:

$$\tilde{\mathscr{E}}_{z'}^2 = |||\zeta_{z'}|||^2_{\mathcal{C}_{z'}} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla \zeta_{z'} \nabla \zeta_{z'} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_{\mathcal{Y}}} \nabla \zeta_{z'} \le \|\nabla e_{\mathscr{T}_{\mathcal{Y}}}\|_{L^2(y^\alpha, \mathcal{C}_{z'})} |||\zeta_{z'}|||_{\mathcal{C}_{z'}}.$$

This concludes the proof. □

Remark 5.19 *(Anisotropic meshes).* Examining the proof of Proposition 5.14, we realize that a critical part of (5.18) consists of the application of inequality (5.1), which we recall reads: $h_{\mathcal{Y}} \le C_{\mathbb{T}} h_{z'}$ for all $z' \in \mathcal{N}(\mathscr{T}_\Omega)$. Therefore, Proposition 5.14 shows how the resolution of local problems on cylindrical stars allows for anisotropic meshes on the extended variable y and graded meshes in Ω. The latter enables us to compensate possible singularities in the x'-variables.

Remark 5.20 *(Relaxing the mesh condition).* Owing to the anisotropy of the mesh, condition (5.1) can be violated only near the top of the cylinder. Near the bottom of the cylinder, the size of the elements will be much smaller than $h_{z'}$. If one could prove that the error $v - V_{\mathscr{T}_{\mathcal{Y}}}$ decays exponentially (as the exact solution v does) an examination of the proof of Proposition 5.14 reveals that condition (5.1) can be removed. Proving this decay, however, requires local pointwise error estimates which are not available and are currently under investigation.

5.4. A computable a posteriori error estimator

Although Proposition 5.14 shows that the error estimator $\tilde{\mathscr{E}}_{\mathscr{T}_{\mathcal{Y}}}$ is ideal, it has an insurmountable drawback: for each node $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, it requires knowledge of the exact solution $\zeta_{z'}$ to the local problem (5.11) which lies in the infinite dimensional space $\mathbb{W}(\mathcal{C}_{z'})$. This makes this estimator not computable. However, it provides intuition and establishes the basis to define a discrete and computable error estimator. To achieve this, let us now define local discrete spaces and local computable error indicators, on the basis of which we will construct our global error estimator.

Definition 5.21 *(Discrete local spaces).* For $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, define the discrete space

$$\mathcal{W}(\mathcal{C}_{z'}) = \left\{ W \in C^0(\bar{\mathcal{C}}_{\mathcal{Y}}) : W|_T \in \mathcal{P}_2(K) \otimes \mathbb{P}_2(I)\ \forall T = K \times I \in \mathcal{C}_{z'},\ W|_{\partial \mathcal{C}_{z'} \setminus \Omega \times \{0\}} = 0 \right\},$$

where, if K is a quadrilateral, $\mathcal{P}_2(K)$ stands for $\mathbb{Q}_2(K)$ — the space of polynomials of degree not larger than 2 in each variable. If K is a simplex, $\mathcal{P}_2(K)$ corresponds to $\mathbb{P}_2(K) \oplus \mathbb{B}(K)$ where where $\mathbb{P}_2(K)$ stands for the space of polynomials of total degree at most 2, and $\mathbb{B}(K)$ is the space spanned by a local cubic bubble function.

We then define the discrete local problems: for each cylindrical star $\mathcal{C}_{z'}$ we define $\eta_{z'} \in \mathcal{W}(\mathcal{C}_{z'})$ to be the solution of

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla W = d_s \langle f, \operatorname{tr}_\Omega W \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V_{\mathscr{T}_{\mathcal{Y}}} \nabla W, \tag{5.22}$$

for all $W \in \mathcal{W}(\mathcal{C}_{z'})$. We also define the global error estimator

$$\mathscr{E}_{\mathscr{T}_{\mathcal{Y}}} = \left(\sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \mathscr{E}_{z'}^2 \right)^{\frac{1}{2}}, \tag{5.23}$$

in terms of the local error indicators

$$\mathscr{E}_{z'} = |||\eta_{z'}|||_{\mathcal{C}_{z'}}. \tag{5.24}$$

We next explore the connection between the estimator (5.23) and the error. We first prove a local lower bound for the error without any oscillation term and free of any constant.

Theorem 5.25 *(Localized lower bound). Let $v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve (3.2) and (3.8) respectively. Then, for any $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, we have*

$$\mathscr{E}_{z'} \leq \left\| \nabla(v - V_{\mathscr{T}_\mathcal{Y}}) \right\|_{L^2(y^\alpha, \mathcal{C}_{z'})}. \tag{5.26}$$

Proof. The proof repeats the arguments employed to obtain inequality (5.16). Let $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, and let $\eta_{z'}$ and $\mathscr{E}_{z'}$ be as in (5.22) and (5.24). Then,

$$\mathscr{E}^2_{z'} = |||\eta_{z'}|||^2_{\mathcal{C}_{z'}} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla \eta_{z'} = \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \eta_{z'} \leq |||e_{\mathscr{T}_\mathcal{Y}}|||_{\mathcal{C}_{z'}} |||\eta_{z'}|||_{\mathcal{C}_{z'}},$$

which concludes the proof. □

Remark 5.27 *(Strong efficiency).* The oscillation and constant free lower bound (5.26) implies a strong concept of efficiency: the relative size of $\mathscr{E}_{z'}$ dictates mesh refinement regardless of fine structure of the data f, and thus works even in the pre-asymptotic regime.

To obtain an upper bound we must assume the following.

Conjecture 5.28 *(Operator $\mathcal{M}_{z'}$). For every $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ there is a linear operator $\mathcal{M}_{z'} : \mathbb{W}(\mathcal{C}_{z'}) \to \mathcal{W}(\mathcal{C}_{z'})$ such that, for all $w \in \mathbb{W}(\mathcal{C}_{z'})$, satisfies:*

- *For every cell $K \in \mathscr{T}_\Omega$ such that $K \subset S_{z'}$*

$$\int_{K \times \{0\}} \operatorname{tr}_\Omega(w - \mathcal{M}_{z'} w) = 0. \tag{5.29}$$

- *For every cell $T \subset \mathcal{C}_{z'}$ and every $W \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$*

$$\int_T y^\alpha \nabla(w - \mathcal{M}_{z'} w) \nabla W = 0. \tag{5.30}$$

- *Stability*

$$|||\mathcal{M}_{z'} w|||_{\mathcal{C}_{z'}} \lesssim |||w|||_{\mathcal{C}_{z'}}, \tag{5.31}$$

  where the hidden constant is independent of the discretization parameters but may depend on α.

We next introduce the so-called *data oscillation*. For every $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, we define the local data oscillation as

$$\operatorname{osc}_{z'}(f)^2 := d_s h^{2s}_{z'} \| f - f_{z'} \|^2_{L^2(S_{z'})} \tag{5.32}$$

where $f_{z'}|_K \in \mathbb{R}$ is the average of f over K, i.e.,

$$f_{z'}|_K := \fint_K f. \tag{5.33}$$

The global data oscillation is defined as

$$\operatorname{osc}_{\mathscr{T}_\Omega}(f)^2 := \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \operatorname{osc}_{z'}(f)^2. \tag{5.34}$$

We also define the *total error indicator*

$$\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_\mathcal{Y}}, S_{z'}) := \left(\mathscr{E}^2_{z'} + \operatorname{osc}_{z'}(f)^2\right)^{1/2} \quad \forall z' \in \mathcal{N}(\mathscr{T}_\Omega), \tag{5.35}$$

which will be used to mark elements for refinement in the adaptive finite element method proposed in Section 6. Let $\mathscr{K}_{\mathscr{T}_\Omega} = \{S_{z'} : z' \in \mathcal{N}(\mathscr{T}_\Omega)\}$ and, for any $\mathscr{M} \subset \mathscr{K}_{\mathscr{T}_\Omega}$, we set

$$\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_\mathcal{Y}},\mathscr{M}) := \Big(\sum_{S_{z'}\in\mathscr{M}} \tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_\mathcal{Y}}, S_{z'})^2 \Big)^{1/2}. \tag{5.36}$$

With the aid of the operators $\mathcal{M}_{z'}$ and under the assumption that Conjecture 5.28 holds, we can bound the error by the estimator, up to oscillation terms.

Theorem 5.37 *(Global upper bound). Let $v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ solve (3.2) and (3.8), respectively. If $f \in L^2(\Omega)$ and Conjecture 5.28 holds, then the total error estimator $\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_\mathcal{Y}}, \mathscr{K}_{\mathscr{T}_\Omega})$, defined in (5.36) satisfies*

$$\big\| \nabla(v - V_{\mathscr{T}_\mathcal{Y}}) \big\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim \tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_\mathcal{Y}}, \mathscr{K}_{\mathscr{T}_\Omega}). \tag{5.38}$$

Proof. Let $e_{\mathscr{T}_\mathcal{Y}} = v - V_{\mathscr{T}_\mathcal{Y}}$ denote the error and let $\psi_{z'} = (w - \Pi_{\mathscr{T}_\mathcal{Y}} w)\tilde{\varphi}_{z'} \in \mathbb{W}(\mathcal{C}_{z'})$, for any $w \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$, where $\Pi_{\mathscr{T}_\mathcal{Y}}$ is the quasi-interpolation operator introduced in [29, §4] and [30, §5]; recall the estimate $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||w|||_{\mathcal{D}_{z'}}$ obtained as part of the proof of Proposition 5.14. Following the proof of Proposition 5.14, we have

$$\begin{aligned}
\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla w &= \sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \psi_{z'} \\
&= \sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \mathcal{M}_{z'}\psi_{z'} - \sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla (\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}).
\end{aligned}$$

We now examine each term separately. First, for every $z' \in \mathcal{N}(\mathscr{T}_\Omega)$ we have $\mathcal{M}_{z'}\psi_{z'} \in \mathcal{W}(\mathcal{C}_{z'})$, whence the definition of the discrete local problem (5.22) yields

$$\begin{aligned}
\sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla \mathcal{M}_{z'}\psi_{z'} &= \sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla \mathcal{M}_{z'}\psi_{z'} \\
&\le \Big(\sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \mathscr{E}_{z'}^2 \Big)^{1/2} \Big(\sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} |||\psi_{z'}|||^2_{\mathcal{C}_{z'}} \Big)^{1/2} \lesssim \mathscr{E}_{\mathscr{T}_\mathcal{Y}} \|\nabla w\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})},
\end{aligned}$$

where in the last inequality we used the stability assumption (5.31) of the operator $\mathcal{M}_{z'}$, the inequality $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||w|||_{\mathcal{D}_{z'}}$, and the finite overlapping property of the stars $\mathcal{C}_{z'}$.

Second, for any $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, we use the conditions (5.29) and (5.30) imposed on the operator $\mathcal{M}_{z'}$ to derive

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla (\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) = d_s \int_{S_{z'}} (f - f_{z'}) \operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}),$$

where we used that $\nabla V_{\mathscr{T}_\mathcal{Y}}$ is constant on every T. Moreover, since $f_{z'}$ is the $L^2(S_{z'})$ projection onto piecewise constants of f we have that, for any ϱ such that $\varrho|_K \in \mathbb{R}$

$$\begin{aligned}
\int_{S_{z'}} (f - f_{z'}) \operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) &= \int_{S_{z'}} (f - f_{z'}) \big(\operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) - \varrho \big) \\
&\le \|f - f_{z'}\|_{L^2(S_{z'})} \big\| \operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) - \varrho \big\|_{L^2(S_{z'})}.
\end{aligned}$$

After suitably choosing ϱ, and using a standard interpolation-type estimate, we get

$$\int_{S_{z'}} (f - f_{z'}) \operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) \lesssim h_{z'}^s \|f - f_{z'}\|_{L^2(S_{z'})} \big\| \operatorname{tr}_\Omega(\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) \big\|_{\mathbb{H}^s(\Omega)}.$$

Consequently,

$$\sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} \int_{\mathcal{C}_{z'}} y^\alpha \nabla e_{\mathscr{T}_\mathcal{Y}} \nabla (\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}) \le \sum_{z'\in\mathcal{N}(\mathscr{T}_\Omega)} C_{\operatorname{tr}_\Omega} d_s h_{z'}^s \|f - f_{z'}\|_{L^2(S_{z'})} |||\psi_{z'} - \mathcal{M}_{z'}\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim \operatorname{osc}_{\mathscr{T}_\Omega}(f) \|\nabla w\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})},$$

where we applied the trace inequality (2.9), the estimate (2.10) on $C_{\operatorname{tr}_\Omega}$, the stability assumption (5.31) of $\mathcal{M}_{z'}$, the bound $|||\psi_{z'}|||_{\mathcal{C}_{z'}} \lesssim |||w|||_{\mathcal{D}_{z'}}$, and the finite overlapping property of the stars $\mathcal{C}_{z'}$.

Collecting the above estimates, we obtain the asserted bound (5.38). □

Remark 5.39 *(Role of oscillation).* Definition 5.21 of $\mathcal{W}(\mathcal{C}_{z'})$ is meant to provide enough degrees of freedom for existence of the operator $\mathcal{M}_{z'}$ satisfying (5.29)–(5.31). This leads to a *solution free* oscillation term (5.32). Otherwise, if we were not able to impose (5.30), then the oscillation (5.32) should be supplemented by the term

$$\mathrm{osc}_{z'}(V_{\mathscr{T}_{\mathcal{Y}}}) := \left\| y^{\alpha} \nabla V_{\mathscr{T}_{\mathcal{Y}}} - \boldsymbol{\sigma}_{z'} \right\|_{L^2(y^{-\alpha}, \mathcal{C}_{z'})},$$

with $\boldsymbol{\sigma}_{z'}$ being the local average of $y^{\alpha} \nabla V_{\mathscr{T}_{\mathcal{Y}}}$, for (5.38) to be valid. This term cannot be guaranteed to be of higher order due to the presence of the weight $y^{-\alpha}$. In fact, computations show that, for $s > \frac{1}{2}$ (or $\alpha < 0$), the magnitude of $\mathrm{osc}_{z'}(V_{\mathscr{T}_{\mathcal{Y}}})$ is of lower order than the actual error estimator $\mathscr{E}_{z'}$ unless a stronger mesh grading in the extended direction is employed to control this term. This can be achieved, for instance, by taking the largest grading parameter γ in (3.5) for $\pm\alpha$, namely $\gamma > 3/(1 - |\alpha|)$. Numerical experiments show no degradation of the convergence rate, expressed in terms of degrees of freedom, for the ensuing meshes $\mathscr{T}_{\mathcal{Y}}$ with stronger anisotropic refinement.

Remark 5.40 *(Construction of $\mathcal{M}_{z'}$).* The construction of the operator $\mathcal{M}_{z'}$ is an open problem. We design the local space $\mathcal{W}(\mathcal{C}_{z'})$ in order to provide enough degrees of freedom for the existence of the operator $\mathcal{M}_{z'}$ satisfying (5.29)–(5.31). The numerical experiments of Section 6 provide consistent computational evidence that the upper bound (5.38) is valid without $\mathrm{osc}_{z'}(V_{\mathscr{T}_{\mathcal{Y}}})$, and thus indirect evidence of the existence of $\mathcal{M}_{z'}$ with the requisite properties (5.29)–(5.31).

6. Numerical experiments

Here we explore computationally the performance and limitations of the a posteriori error estimator introduced in Section 5.4 with a series of test cases. To do so, we start by formulating an adaptive finite element method (AFEM) based on iterations of the loop

$$\mathsf{SOLVE} \rightarrow \mathsf{ESTIMATE} \rightarrow \mathsf{MARK} \rightarrow \mathsf{REFINE}. \tag{6.1}$$

6.1. Design of AFEM

The modules in (6.1) are as follows:

- SOLVE: Given a mesh $\mathscr{T}_{\mathcal{Y}}$ we compute the Galerkin solution of (3.2):
$$V_{\mathscr{T}_{\mathcal{Y}}} = \mathsf{SOLVE}(\mathscr{T}_{\mathcal{Y}}).$$
- ESTIMATE: Given $V_{\mathscr{T}_{\mathcal{Y}}}$ we calculate the local error indicators (5.24) and the local oscillations (5.32) to construct the total error indicator of (5.35):
$$\left\{\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_{\mathcal{Y}}}, S_{z'})\right\}_{S_{z'} \in \mathscr{K}_{\mathscr{T}_\Omega}} = \mathsf{ESTIMATE}(V_{\mathscr{T}_{\mathcal{Y}}}, \mathscr{T}_{\mathcal{Y}}).$$
- MARK: Using the so-called Dörfler marking [13] (bulk chasing strategy) with parameter $0 < \theta \le 1$, we select a set
$$\mathscr{M} = \mathsf{MARK}\left(\left\{\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_{\mathcal{Y}}}, S_{z'})\right\}_{S_{z'} \in \mathscr{K}_\Omega}, V_{\mathscr{T}_{\mathcal{Y}}}\right) \subset \mathscr{K}_{\mathscr{T}_\Omega}$$
of minimal cardinality that satisfies
$$\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_{\mathcal{Y}}}, \mathscr{M}) \ge \theta \tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_{\mathcal{Y}}}, \mathscr{K}_{\mathscr{T}_\Omega}).$$
- REFINE: We generate a new mesh $\mathscr{T}'_\Omega$ by bisecting all the elements $K \in \mathscr{T}_\Omega$ contained in $\mathscr{M}$ based on the newest-vertex bisection method; see [32,31]. We choose the truncation parameter as $\mathcal{Y} = 1 + \frac{1}{3}\log(\#\mathscr{T}'_\Omega)$ to balance the approximation and truncation errors; see [29, Remark 5.5]. We construct the mesh $\mathcal{I}'_{\mathcal{Y}}$ by the rule (3.5), with a number of degrees of freedom M sufficiently large so that condition (5.1) holds. This is attained by first creating a partition $\mathcal{I}'_{\mathcal{Y}}$ with $M \approx (\#\mathscr{T}'_\Omega)^{1/n}$ and checking (5.1). If this condition is violated, we increase the number of points until we get the desired result. The new mesh
$$\mathscr{T}'_{\mathcal{Y}} = \mathsf{REFINE}(\mathscr{M}),$$
is obtained as the tensor product of $\mathscr{T}'_\Omega$ and $\mathcal{I}'_{\mathcal{Y}}$.

6.2. Implementation

The AFEM (6.1) is implemented within the MATLAB© software library *i*FEM [11]. The stiffness matrix of the discrete system (3.8) is assembled exactly, and the forcing boundary term is computed by a quadrature formula which is exact for polynomials of degree 4. The resulting linear system is solved by using the multigrid method with line smoother introduced and analyzed in [12].

To compute the solution $\eta_{z'}$ to the discrete local problem (5.22), we loop around each node $z' \in \mathcal{N}(\mathscr{T}_\Omega)$, collect data about the cylindrical star $\mathcal{C}_{z'}$ and assemble the small linear system (5.22) which is solved by the built-in *direct solver* of MATLAB©. All integrals involving only the weight and discrete functions are computed exactly, whereas those also involving data functions are computed element-wise by a quadrature formula which is exact for polynomials of degree 7.

For convenience, in the MARK step we change the estimator from star-wise to element-wise as follows: We first scale the nodal-wise estimator as $\mathscr{E}_{z'}^2/(\#S_{z'})$ and then, for each element $K \in \mathscr{T}_\Omega$, we compute

$$\mathscr{E}_K^2 := \sum_{z' \in K} \mathscr{E}_{z'}^2.$$

The scaling is introduced so that $\sum_{K \in \mathscr{T}_\Omega} \mathscr{E}_K^2 = \sum_{z' \in \mathcal{N}(\mathscr{T}_\Omega)} \mathscr{E}_{z'}^2$. The cell-wise data oscillation is now defined as

$$\mathrm{osc}_K(f)^2 := d_s h_K^{2s} \|f - \bar{f}_K\|_{L^2(K)}^2,$$

where $\bar{f}_K$ denotes the average of f over the element K. This quantity is computed using a quadrature formula which is exact for polynomials of degree 7.

Unless specifically mentioned, all computations are done without explicitly enforcing the mesh restriction (5.1), which shows that this is nothing but an artifact in our proofs; see Remark 5.20. How to remove this assumption is currently under investigation. Nevertheless, computations (not shown here for brevity) show that optimality is still retained if one imposes (5.1).

For the cases where the exact solution to problem (3.8) is available, the error is computed by using the identity

$$\|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})}^2 = d_s \int_\Omega f \,\mathrm{tr}_\Omega (v - V_{\mathscr{T}_\mathcal{Y}}),$$

which follows from Galerkin orthogonality and integration by parts. Thus, we avoid evaluating the singular/degenerate weight y^α and reduce the computational cost. The right hand side of the equation above is computed by a quadrature formula which is exact for polynomials of degree 7. On the other hand, if the exact solution is not available, we introduce the energy

$$E(w) = \frac{1}{2} \int_{\mathcal{C}_\mathcal{Y}} y^\alpha |\nabla w|^2 - d_s \langle f, \mathrm{tr}_\Omega\, w \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)},$$

where $w \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ and $f \in \mathbb{H}^{-s}(\Omega)$. Consequently, Galerkin orthogonality implies

$$E(V_{\mathscr{T}_\mathcal{Y}}) - E(v) = \frac{1}{2} \|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})}^2,$$

where $v \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ and $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ denote the solution to problems (3.2) and (3.8), respectively. We remark that for a discrete function W the energy $E(W)$ can be computed simply using a matrix–vector product. Then, an approximation of the error in the energy norm can be obtained by computing

$$\left(2\left(E(V_{\mathscr{T}_\mathcal{Y}}) - E(V_{\mathscr{T}_\mathcal{Y}^*})\right)\right)^{\frac{1}{2}},$$

where $V_{\mathscr{T}_\mathcal{Y}^*}$ is the Galerkin approximation to v on a very fine grid $\mathscr{T}_\mathcal{Y}^*$, which is obtained by a uniform refinement of the last adaptive mesh.

We also compute the *effectivity index* of our a posterior error estimator defined as the ratio of the estimator and the true error, i.e.,

$$\frac{\tau_{\mathscr{T}_\Omega}(V_{\mathscr{T}_\mathcal{Y}}, \mathscr{K}_{\mathscr{T}_\Omega})}{\|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})}},$$

as well as the *aspect ratio* of elements T

$$\frac{h_K}{h_I} \quad \forall T = K \times I \in \mathscr{T}_\mathcal{Y}.$$

6.3. Smooth and compatible data

The purpose of this numerical example is to show how the error estimator (5.23)–(5.24) based on the solution of local problems on anisotropic cylindrical stars allows us to recover optimality of our AFEM. We recall that adaptive isotropic refinement cannot be optimal; see Section 4.2. We consider $\Omega = (0, 1)^2$ and $f(x^1, x^2) = \sin(2\pi x^1)\sin(2\pi x^2)$. Then, the exact solution to (1.1) is given by $u(x^1, x^2) = 8^{-s}\pi^{2s}\sin(2\pi x^1)\sin(2\pi x^2)$.

The asymptotic relation $\|\nabla(\mathscr{U} - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \approx \#(\mathscr{T}_\mathcal{Y})^{-1/3}$ is shown in Fig. 3 which illustrates the quasi-optimal decay rate of our AFEM driven by the error estimator (5.23)–(5.24) for all choices of the parameter s considered. These examples show that anisotropy in the extended dimension is essential to recover optimality of our AFEM.

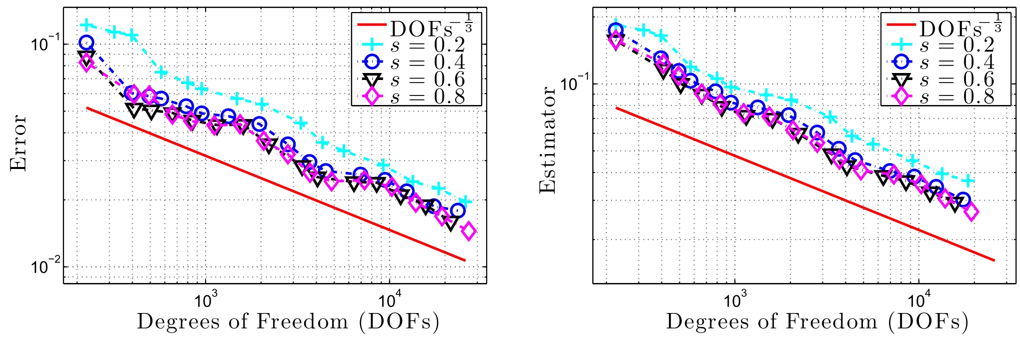


Fig. 3. Computational rate of convergence for our anisotropic AFEM on the smooth and compatible right hand side of Section 6.3 for $n = 2$ and $s = 0.2$, 0.4, 0.6 and $s = 0.8$. The left panel shows the decrease of the error with respect to the number of degrees of freedom, whereas the right one that for the total error estimator. In all cases we recover the optimal rate $\#(\mathscr{T}_{\mathcal{Y}})^{-1/3}$. The aspect ratios, averaged over x', of the cells on the bottom layer $[0, y_1]$ in the finest mesh are: 1.65×10^{11}, 6.30×10^4, 396 and 36.2, respectively. The average effectivity indices are 1.47, 1.61, 1.61, 1.62, respectively.

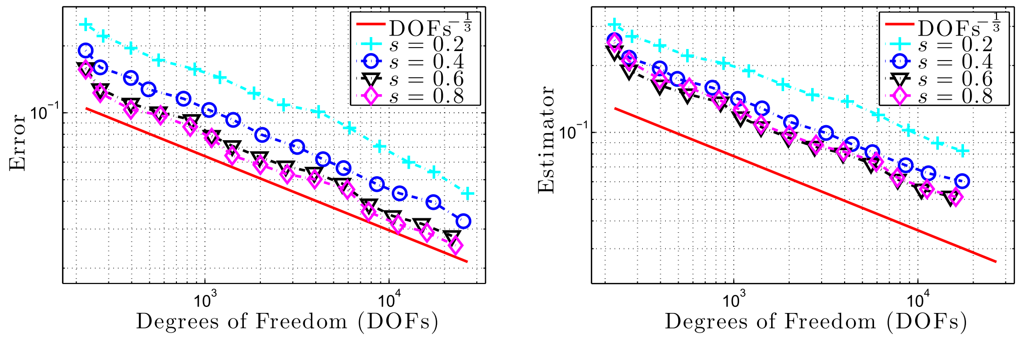


Fig. 4. Computational rate of convergence for our anisotropic AFEM on the smooth but incompatible right hand side of Section 6.4 for $n = 2$ and $s = 0.2$, 0.4, 0.6 and $s = 0.8$. The left panel shows the decrease of the error with respect to the number of degrees of freedom, whereas the right one that for the total error indicator. In all cases we recover the optimal rate $\#(\mathscr{T}_{\mathcal{Y}})^{-1/3}$. Notice that, for $s < \frac{1}{2}$, the right hand side $f = 1 \notin \mathbb{H}^{1-s}(\Omega)$ and so a quasi-uniform mesh in Ω does not deliver the optimal rate of convergence [29, §6.3]. The aspect ratios, averaged over x', of the cells on the bottom layer $[0, y_1]$ in the finest mesh are: 2.09×10^{11}, 6.03×10^4, 387 and 33.4, respectively. The average effectivity indices are 1.34, 1.41, 1.51, 1.67, respectively.

6.4. Smooth but incompatible data

The numerical example presented in [29, §6.3] shows that the results of Theorem 3.9 are sharp in the sense that the regularity $f \in \mathbb{H}^{1-s}(\Omega)$ is indeed necessary to obtain an optimal rate of convergence with a quasi-uniform mesh in the x'-direction. The heuristic explanation for this is that a certain compatibility between the data and the boundary condition is necessary. The results of [29, §6.3] also show that, in some simple cases, one can guess the nature of the singularity that is introduced by this incompatibility and a priori design a mesh that captures it, thus recovering the optimal rate of convergence. Evidently, this is not possible in all cases and here we show that the a posteriori error estimator (5.23)–(5.24) automatically produces a sequence of meshes that yield the optimal rate of convergence.

We consider $\Omega = (0, 1)^2$ and $f = 1$. Owing to definition (2.3), we have that a function in $\mathbb{H}^{1-s}(\Omega)$, for $1 - s > \frac{1}{2}$, must have a vanishing trace on the boundary. Therefore, for $s < \frac{1}{2}$, we have that $f \notin \mathbb{H}^{1-s}(\Omega)$ and we cannot invoke the results of Theorem 3.9. Nevertheless, as the results of Fig. 4 show, we recover the optimal rate of convergence. Due to the aforementioned incompatibility the solution must have a boundary layer, which an adaptive grid resolves by having a larger density near the boundary, as it is illustrated in Fig. 5. This is the reason why a quasi-uniform mesh in the x'-direction will not produce an optimal rate of convergence.

6.5. L-shaped domain with compatible data

The result of Theorem 3.9 relies on the assumption that the Laplace operator $-\Delta_{x'}$ on Ω supplemented with Dirichlet boundary conditions possesses optimal smoothing properties, i.e., (3.1). As it is well known [18], (3.1) holds when Ω is a convex polyhedron. Let us then consider the case when this condition is not met.

We consider the classical L-shaped domain $\Omega = (-1, 1)^2 \setminus (0, 1) \times (-1, 0)$ and $f(x^1, x^2) = \sin(2\pi x^1)\sin(\pi x^2)$, which is a smooth and compatible right hand side, i.e., $f = 0$ on $\partial\Omega$, for all values of $s \in (0, 1)$. The results of Fig. 6 show the experimental rates of convergence and confirm that our AFEM is able to capture the singularity that the reentrant corner in the domain introduces and recover the optimal rate of convergence. As the exact solution of this problem is not known, we compute the rate of convergence by comparing the computed solution with one obtained on a very fine mesh.

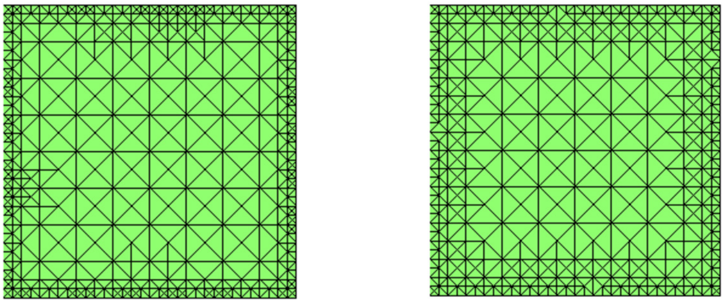


Fig. 5. Adaptive grids for the smooth but incompatible right hand side of Section 6.4. The left panel shows the mesh for $s = 0.2$, the right that for $s = 0.4$.

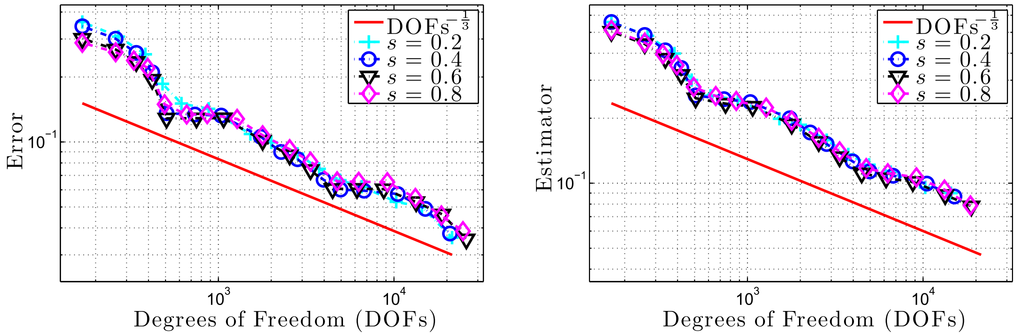


Fig. 6. Computational rate of convergence for our anisotropic AFEM on the smooth and compatible right hand side over an L-shaped domain of Section 6.5 for $n = 2$ and $s = 0.2$, 0.4, 0.6 and $s = 0.8$. The left panel shows the decrease of the error with respect to the number of degrees of freedom, whereas the right one that for the total error indicator. In all cases we recover the optimal rate $\#(\mathscr{T}_{\mathcal{Y}})^{-1/3}$. As the exact solution of this problem is not known, we compute the rate of convergence by comparing the computed solution with one obtained on a very fine mesh. The aspect ratios, averaged over x', of the cells on the bottom layer $[0, y_1]$ in the finest mesh are: 2.56×10^{11}, 1.20×10^{5}, 667 and 56.7, respectively. The average effectivity indices are 1.73, 1.77, 1.73, 1.74, respectively.

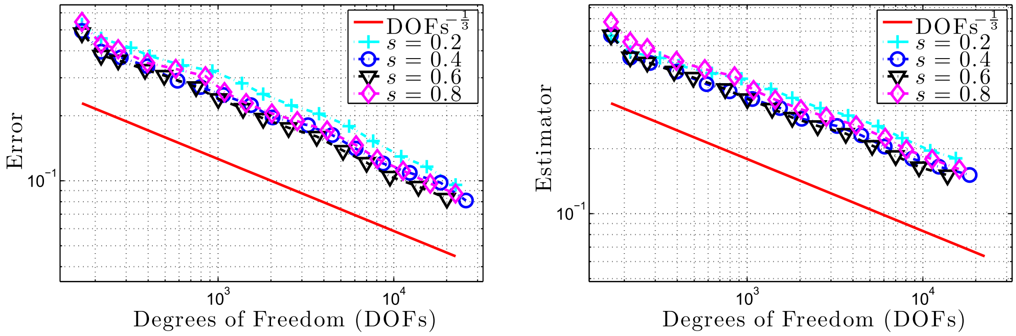


Fig. 7. Computational rate of convergence for our anisotropic AFEM on the smooth but incompatible right hand side over an L-shaped domain of Section 6.6 for $n = 2$ and $s = 0.2$, 0.4, 0.6 and $s = 0.8$. The left panel shows the decrease of the error with respect to the number of degrees of freedom, whereas the right one that for the total error indicator. In all cases we recover the optimal rate $\#(\mathscr{T}_{\mathcal{Y}})^{-1/3}$. As the exact solution of this problem is not known, we compute the rate of convergence by comparing the computed solution with one obtained on a very fine mesh. The aspect ratios, averaged over x', of the cells on the bottom layer $[0, y_1]$ in the finest mesh are: 2.24×10^{11}, 1.02×10^{5}, 632 and 55.6, respectively. The average effectivity indices are 1.36, 1.40, 1.44, 1.49, respectively.

6.6. L-shaped domain with incompatible data

Let us combine the singularity introduced by the data incompatibility of Section 6.4 and the reentrant corner explored in Section 6.5. To do so, we again consider the L-shaped domain $\Omega = (-1, 1)^2 \setminus (0, 1) \times (-1, 0)$ and $f = 1$. As the results of Fig. 7 show, we again recover the optimal rate of convergence for all possible cases of s. As the exact solution of this problem is not known, we compute the rate of convergence by comparing the computed solution with one obtained on a very fine mesh.

We display some meshes in Fig. 8. As expected, when $s < \frac{1}{2}$ the data $f = 1$ is incompatible with the equation and this causes a boundary layer on the solution. To capture it, the AFEM refines near the boundary. In contrast, when $s > \frac{1}{2}$ the refinement is only near the reentrant corner, since there is no boundary layer anymore.

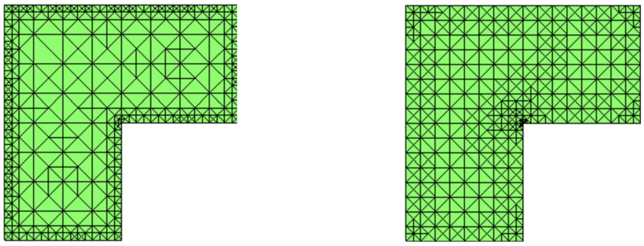


Fig. 8. Adaptively graded mesh for an L-shaped domain with incompatible data (see Section 6.6): $s = 0.2$ (left) and $s = 0.8$ (right). As expected, when $s < \frac{1}{2}$ the data $f = 1$ is incompatible with the equation and this causes a boundary layer on the solution. To capture it, our AFEM refines near the boundary. In contrast, when $s > \frac{1}{2}$ the refinement is only near the reentrant angle, since there is no boundary layer anymore.

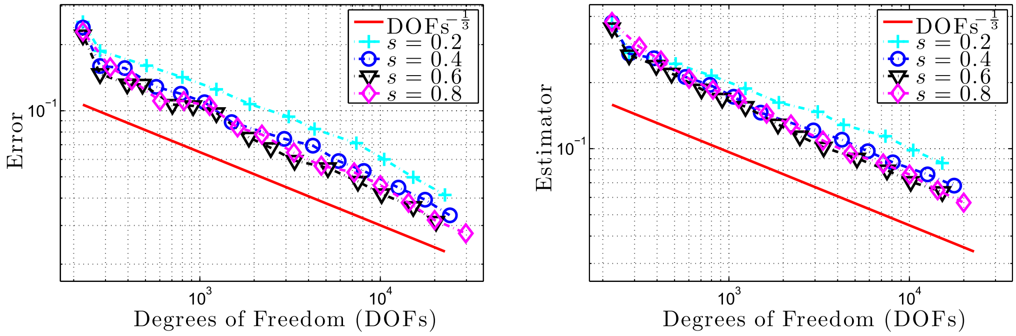


Fig. 9. Computational rate of convergence for our anisotropic AFEM on the problem with discontinuous coefficients described in Section 6.7 for $n = 2$ and $s = 0.2$, 0.4, 0.6 and $s = 0.8$. The left panel shows the decrease of the error with respect to the number of degrees of freedom, whereas the right one that for the total error indicator. In all cases we recover the optimal rate $\#(\mathscr{T}_{\mathcal{Y}})^{-1/3}$. As the exact solution of this problem is not known, we compute the rate of convergence by comparing the computed solution with one obtained on a very fine mesh. The aspect ratios, averaged over x', of the cells on the bottom layer $[0, y_1]$ in the finest mesh are: 3.74×10^{11}, 1.21×10^5, 719 and 69.1, respectively. The average effectivity indices are 1.55, 1.64, 1.69, 1.71, respectively.

6.7. Discontinuous coefficients

In this example we compute the fractional powers of a general elliptic operator with piecewise discontinuous coefficients. We invoke the formulas derived by Kellog [19] (see also [32, §5.4]) to construct an exact solution to the particular case: $\Omega = (-1, 1)^2$, $f(x^1, x^2) = ((x^1)^2 - 1)((x^2)^2 - 1)$ and

$$\mathcal{L}w = -\mathrm{div}_{x'}(A \nabla_{x'} w),$$

with

$$A = \begin{cases} \varrho \mathbb{I}, & x^1 x^2 > 0, \\ \mathbb{I}, & x^1 x^2 \le 0, \end{cases}$$

where $\mathbb{I}$ denotes the identity tensor and $\varrho = 161.4476387975881$. This problem is well known as a pathological case, where the solution is *barely in* $H_0^1(\Omega)$.

On the other hand, as the results of [29, §7] show, the problem: find u such that

$$\mathcal{L}^s u = f, \quad \text{in } \Omega \qquad u|_{\partial\Omega} = 0$$

also admits an extension, which can be discretized with the techniques and ideas presented in [29, §7]. We can also write an a posteriori error estimator based on cylindrical stars and design an adaptive loop. Fig. 9 demonstrates that the associated decay rate is optimal: $(\mathscr{T}_{\mathcal{Y}})^{-1/3}$ for all the considered cases.

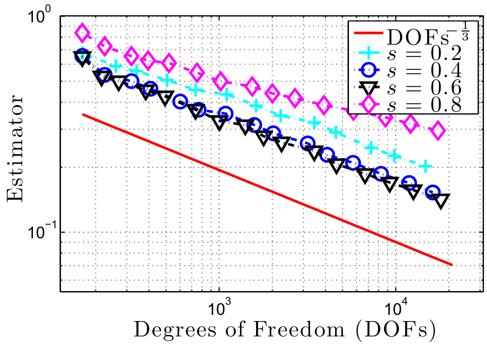
6.8. The role of oscillation

Let us explore the role of oscillation as discussed in Remark 5.39. We consider the same example as in Section 6.6, but we set in Definition 5.21 $\mathcal{P}_2(K) = \mathbb{P}_2(K)$. In this case the discrete local space $\mathcal{W}(\mathcal{C}_{z'})$ does not have enough degrees of freedom to impose (5.30). Consequently, in order to have (5.38) the oscillation (5.32) must be supplemented by the term

$$\mathrm{osc}_{z'}(V_{\mathscr{T}_{\mathcal{Y}}}) = \left\| y^{\alpha} \nabla V_{\mathscr{T}_{\mathcal{Y}}} - \boldsymbol{\sigma}_{z'} \right\|_{L^2(y^{-\alpha}, \mathcal{C}_{z'})}, \tag{6.2}$$

where $\boldsymbol{\sigma}_{z'}$ is the local average of $y^{\alpha} \nabla V_{\mathscr{T}_{\mathcal{Y}}}$.

Fig. 10 shows the experimental rates of convergence obtained by our AFEM for the total error where the oscillation term (5.32) is supplemented with (6.2). As we can see, especially for $s = 0.8$, the results are not optimal. To remedy this we



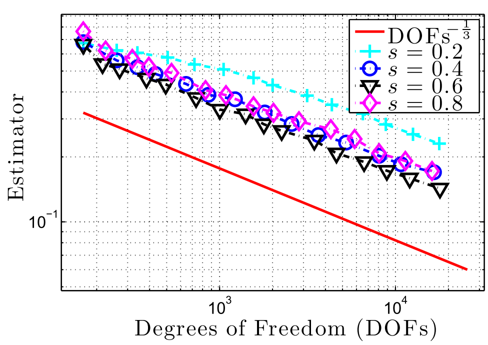


Fig. 10. Experimental rate of convergence for the example of Section 6.6 but with $\mathcal{P}_2(K) = \mathbb{P}_2(K)$ in Definition 5.21. In this case, the oscillation (5.32) must be supplemented by (6.2) to attain an upper bound. The expression (6.2) possesses a singularity and weight which cannot be captured with the mesh grading necessary for optimal approximation orders; this yields suboptimal decay rates (left figure). However, a graded mesh with $\gamma > 3/(1 - |\alpha|)$ in (3.5) gives optimal decay rates (right figure).

notice that a graded mesh is able to capture the singular behavior of (6.2). Indeed, the underlying weight in this expression is $y^{-\alpha}$, so that a grading parameter in (3.5) of $\gamma > 3/(1 - |\alpha|)$ would yield an optimal decay for (6.2). For this reason, setting $\gamma > 3/(1 - |\alpha|)$, we are able to obtain an optimal decay rates for both the error and the oscillation (6.2) and all values of s. The results shown in Fig. 10 confirm this.

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