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An auxiliary space multigrid preconditioner for the weak Galerkin method

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ABSTRACT

In this paper, we construct an auxiliary space multigrid preconditioner for the weak Galerkin method for second-order diffusion equations, discretized on simplicial 2D or 3D meshes. The idea of the auxiliary space multigrid preconditioner is to use an auxiliary space as a "coarse" space in the multigrid algorithm, where the discrete problem in the auxiliary space can be easily solved by an existing solver. In our construction, we conveniently use the H^1 conforming piecewise linear finite element space as an auxiliary space. The main technical difficulty is to build the connection between the weak Galerkin discrete space and the H^1 conforming piecewise linear finite element space. We successfully constructed such an auxiliary space multigrid preconditioner for the weak Galerkin method, as well as the reduced system of the weak Galerkin method involving only the degrees of freedom on edges/faces. The preconditioned systems are proved to have condition numbers independent of the mesh size. Numerical experiments further support the theoretical results.

1. Introduction

Consider the second-order elliptic equation

$$\begin{aligned} -\nabla\cdot(\mathbb{A}\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$. Assume that $\mathbb{A}$ is a symmetric, uniformly positive definite, and uniformly bounded-above diffusion matrix. Namely, there exist positive constants α and β such that

$$\alpha\xi^T\xi \le \xi^T\mathbb{A}(x)\xi \le \beta\xi^T\xi \quad \text{for all } \xi\in\mathbb{R}^d \text{ and } x\in\Omega. \tag{1.2}$$

The goal of this paper is to construct and analyze an auxiliary space multigrid preconditioner for the weak Galerkin finite element discretization of Problem (1.1).

The weak Galerkin method was recently introduced in [1] for second order elliptic equations. It is an extension of the standard Galerkin finite element method where classical derivatives were substituted by weakly defined derivatives on functions with discontinuity. Optimal order of *a priori* error estimates has been observed and established for various weak

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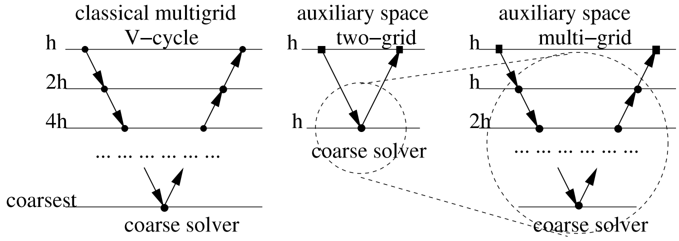

Fig. 1.1. Illustration of auxiliary space multigrid. We use black rectangles and black dots to denote different types of discretization spaces. The dashed circles show how to derive an auxiliary space "multi"-grid method by using a classical multigrid as a coarse solver in the two-grid auxiliary space multigrid framework.

Galerkin discretization schemes for second order elliptic equations [1–3]. An *a posteriori* error estimator was given in [4]. Numerical implementations of weak Galerkin were discussed in [2,5] for some model problems.

The weak Galerkin method has already demonstrated many nice properties in various cases [1–3,6]. Thus we are motivated to study fast solvers and preconditioning techniques for the weak Galerkin method.

The main results of this paper are:

- We develop a fast auxiliary space preconditioner for weak Galerkin methods using Raviart–Thomas element and Brezzi–Douglas–Marini element on triangular grids.
- We consider both the original system and the reduced system. The original weak Galerkin discretization of (1.1) involves degrees of freedom both on the interior of each mesh element and on mesh edges/faces. The reduced system, which only involves degrees of freedom on edges/faces, is to our knowledge first rigorously constructed and analyzed here.

Recently, Li and Xie announced an auxiliary space multigrid preconditioning method for the weak Galerkin finite element method, and the result was posted on ArXiv [7]. This result became known to us after the bulk portion of the present paper was developed. Following a thorough comparison, we conclude that our results are more general, and the two approaches are different in analysis. Although the multigrid algorithm is essentially the same, the analysis in [7] only works for one type of weak Galerkin elements which uses the Brezzi–Douglas–Marini element for the flux, while our work uses a more general theoretical framework and applies to a larger selection of weak Galerkin elements. Non-trivial and necessary technique tools are developed in our proof for the general case. In addition, our result offers a new feature by covering the weak Galerkin method in the reduced system.

We shall briefly introduce the auxiliary space preconditioner constructed in [8]. A classical geometric multigrid method constructs discrete spaces on different mesh levels using the same type of discretization. For example, in the classical multigrid method for H^1 conforming piecewise linear (P_1) finite element approximation, one uses a set of nested meshes with characteristic mesh sizes $h, 2h, 4h, \dots$, from the finest mesh to the coarsest. An illustration of V-cycle multigrid is given in Fig. 1.1. The auxiliary space multigrid method can be essentially understood as a two-level method involving a "fine" level and a "coarse" level, while the "fine" space and "coarse" space are not necessarily using the same type of discretization or the same type of meshes. This gives great freedom in choosing the "coarse" space, which is also called an auxiliary space. Here we use the weak Galerkin discretization for the "fine" level, and the H^1 conforming piecewise linear finite element discretization for the "coarse" level. Both the "fine" level and the "coarse" level are discretized on the same mesh, as shown in Fig. 1.1. In the figure, we conveniently use black rectangles and black dots to denote different type of discretization spaces on different levels. Because the fast solvers for the H^1 conforming piecewise linear finite element discretization have been thoroughly studied, one can use any existing solvers/preconditioners as a "coarse" solver. For example, one may use a classical multigrid method as a "coarse" solver and consequently achieves a true "multi"-grid effect (see Fig. 1.1).

The weak Galerkin method is often compared with the hybridizable discontinuous Galerkin (HDG) method [9]. Indeed, these two methods share some common tools, for example, the discrete/weak gradient [1,9,10] etc., and are identical when $\mathbb{A}$ is piecewise constant. However, we emphasize that they are constructed by fundamentally different concepts. As a result, the weak Galerkin method differs from HDG for the model problem (1.1) when the coefficient matrix $\mathbb{A}$ is not equal to a constant times the identity matrix. More differences between the weak Galerkin method and HDG have been stated in [2]. We shall mention that recently an auxiliary space multigrid preconditioner has been developed for HDG [11]. Given the similarity of HDG and WG, it is not surprising that the multigrid preconditioner to be developed in this paper share some similarities with the preconditioner proposed in [11]. For example, both use the H^1 conforming piecewise linear finite element space as the auxiliary space, and both use the same type of projection operator as the intergrid transfer operator. However, we point out that they also have significant differences. First, the multigrid preconditioner in [11] is designed for a linear system involving only the Lagrange multiplier on edges/faces, while we develop multigrid preconditioners for both the original system and a reduced system. Second, although both multigrid preconditioners are proved to be optimal, our analysis requires less regularity than the analysis in [11]. To be more specific, the analysis in [11] requires that the solution to System (1.1) satisfy $u \in H^{1+s}(\Omega)$ for $s > 1/2$, while we only require $s > 0$. Finally, many techniques in our proof are different from those in [11]. For example, we explicitly construct an averaging operator with nice properties, which has greatly simplified the proof.

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The rest of the paper is organized as follows. In Section 2, we give a brief introduction of the weak Galerkin method, and in Section 3, we construct the auxiliary space multigrid preconditioner for the weak Galerkin discretization, and prove that the condition number of the preconditioned system does not depend on the mesh size. After that, we consider a reduced system of the weak Galerkin discretization in Section 4 and construct an auxiliary space multigrid solver/preconditioner for the reduced system, again with an optimal condition number estimate. Finally in Section 5, we present supporting numerical results.

2. A weak Galerkin finite element scheme

In this section, we give a brief introduction to the weak Galerkin method. Related notation, definitions, and several important inequalities will be stated.

Let $D \subseteq \Omega$ be a polygon or polyhedron, we use the standard definition of Sobolev spaces $H^s(D)$ and $H_0^s(D)$ with $s \geq 0$ (e.g., see [12,13] for details). The associated inner-product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{r,D}$, $0 \leq r \leq s$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square-integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$. For $s < 0$, the space $H^s(D)$ is defined to be the dual of $H_0^{|s|}(D)$.

The above definition/notation can easily be extended to vector-valued and matrix-valued functions. The norm, semi-norms, and inner-product for such functions shall follow the same naming convention. In addition, all these definitions can be transferred from a polygonal/polyhedral domain D to an edge/face e, a domain with lower dimension. Similar notation system will be employed. For example, $\|\cdot\|_{s,e}$ and $\|\cdot\|_e$ would denote the norm in $H^s(e)$ and $L^2(e)$, etc. We also define the $H(\mathrm{div})$ space as follows

$$H(\mathrm{div},\Omega) = \{\mathbf{q} : \mathbf{q} \in [L^2(\Omega)]^d, \nabla\cdot\mathbf{q} \in L^2(\Omega)\}.$$

Using the notation defined above, the variational form of Eq. (1.1) can be written as: Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$(\mathbb{A}\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega). \tag{2.1}$$

It is well known that Eq. (2.1) admits a unique solution. In addition, we assume that the solution to (2.1) has H^{1+s} regularity [14,15], where $0 < s \leq 1$. In other words, the solution u is in $H^{1+s}(\Omega)$ and there exists a constant C independent of u such that

$$\|u\|_{1+s} \leq C\|f\|_0. \tag{2.2}$$

Next, we present the weak Galerkin method for solving (2.1). Let $\mathcal{T}_h$ be a shape-regular, quasi-uniform triangular/tetrahedral mesh on the domain Ω, with characteristic mesh size h. For each triangle/tetrahedron $K \in \mathcal{T}_h$, denote by K_0 and ∂K the interior and the boundary of K, respectively. Geometrically, K_0 is identical to K. Therefore, later in the paper, we often identify these two if it causes no ambiguity. The boundary ∂K consists of three edges in two-dimension, or four triangles in three-dimension. Denote by $\mathcal{E}_h$ the collection of all edges/faces in $\mathcal{T}_h$. For simplicity, throughout the paper, we use "$\lesssim$" to denote "less than or equal to up to a general constant independent of the mesh size or functions appearing in the inequality".

Let j be a non-negative integer. On each $K \in \mathcal{T}_h$, denote by $P_j(K_0)$ the set of polynomials with degree less than or equal to j. Likewise, on each $e \in \mathcal{E}_h$, $P_j(e)$ is the set of polynomials of degree no more than j. Following [1], we define a weak discrete space on mesh $\mathcal{T}_h$ by

$$V_h = \{v : v|_{K_0} \in P_j(K_0) \text{ for } K \in \mathcal{T}_h;\ v|_e \in P_l(e) \text{ for } e \in \mathcal{E}_h, \text{ and } v|_e = 0 \text{ for } e \in \mathcal{E}_h \cap \partial\Omega\}, \quad \text{where } l = j \text{ or } j+1.$$

Observe that the definition of V_h does not require any continuity of $v \in V_h$ across interior edges/faces. A function in V_h is characterized by its value on the interior of each mesh element plus its value on edges/faces. Therefore, it is convenient to represent functions in V_h with two components, $v = \{v_0, v_b\}$, where v_0 denotes the value of v on all K_0 and v_b denotes the value of v on $\mathcal{E}_h$. The polynomial space $P_l(e)$ consists of two choices: $l = j$ or $j+1$ and the corresponding weak function space will sometimes be abbreviated as $W_{j,j}$ or $W_{j,j+1}$, respectively.

The weak Galerkin method seeks an approximation $u_h \in V_h$ to the solution of problem (2.1). To this end, we first introduce a discrete gradient operator, which is defined element-wisely on each $K \in \mathcal{T}_h$. For the choices of V_h given above, i.e., using $W_{j,j}$ or $W_{j,j+1}$, suitable definitions of the weak gradient involve the Raviart–Thomas (RT) element [16] and the Brezzi–Douglas–Marini (BDM) element [17], respectively. Let K be either a triangle or a tetrahedron and denote by $\widehat{P}_k(K)$ the set of homogeneous polynomials of order k in the variable $\mathbf{x} = (x_1,\ldots,x_d)^T$. Define the BDM element by $G_j(K) = \left[P_{j+1}(K)\right]^d$ and the RT element by $G_j(K) = \left[P_j(K)\right]^d + \widehat{P}_j(K)\mathbf{x}$ for $j \geq 0$. Then, define a discrete space

$$\Sigma_h = \{\mathbf{q} \in (L^2(\Omega))^d : \mathbf{q}|_K \in G_j(K) \text{ for } K \in \mathcal{T}_h\}.$$

Here in the definition of V_h and Σ_h, the RT element is paired with $W_{j,j}$ while the BDM element is paired with $W_{j,j+1}$. Note that Σ_h is not necessarily a subspace of $H(\mathrm{div},\Omega)$, since it does not require any continuity in the normal direction across mesh edges/faces.

Definition 2.1 (*Discrete Weak Gradient*)**.** The discrete weak gradient of v_h denoted by $\nabla_w v_h$ is defined as the unique polynomial $(\nabla_w v_h)|_K \in G_j(K)$ satisfying the following equation

$$(\nabla_w v_h, \mathbf{q})_K = -(v_0, \nabla \cdot \mathbf{q})_K + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial K} \quad \text{for all } \mathbf{q} \in G_j(K), \tag{2.3}$$

where $\mathbf{n}$ is the unit outward normal on ∂K.

Clearly, such a discrete weak gradient is always well-defined. Furthermore, if $v \in H^1(K)$, i.e. $v_b = v_0|_{\partial K}$, and $\nabla v \in G_j(K)$, then one has $\nabla_w v = \nabla v$. In this paper we only consider the $W_{j,j} - RT$ and $W_{j,j+1} - BDM$ pairs on simplicial elements in the discretization, for which the weak gradient operator has been introduced in [10] to design a domain decomposition preconditioner of the saddle point system arising from the mixed method discretization of elliptic equations. But there are many other different choices of discrete spaces in the weak Galerkin method, defined on either simplicial meshes or other types of meshes including general polytopal meshes [2]. Extension of the multigrid preconditioner to other weak Galerkin discretizations will be considered in the future work.

We define an L^2 projection from $H_0^1(\Omega)$ onto V_h by setting $Q_h v \equiv \{Q_0 v,\ Q_b v\}$, where $Q_0 v|_{K_0}$ is the local L^2 projection of v to $P_j(K_0)$, for $K \in \mathcal{T}_h$, and $Q_b v|_e$ is the local L^2 projection to $P_l(e)$, for $e \in \mathcal{E}_h$. We also introduce $\mathbb{Q}_h$ the L^2 projection onto Σ_h. It is not hard to see the following operator identity [1]:

$$\mathbb{Q}_h \nabla u = \nabla_w Q_h u, \quad \text{for all } u \in H_0^1(\Omega) \tag{2.4}$$

For the $W_{j,j} - RT$ and $W_{j,j+1} - BDM$ pairs, it follows directly from (2.4) that the discrete weak gradient is a good approximation to the classical gradient, as summarized in the following lemma [1].

Lemma 2.2. *For any $v_h = \{v_0,\ v_b\} \in V_h$ and $K \in \mathcal{T}_h$, $\nabla_w v_h|_K = 0$ if and only if $v_0 = v_b =$ constant on K. Furthermore, for any $v \in H^{m+1}(\Omega)$, where $0 \le m \le j+1$, we have*

$$\|\nabla_w (Q_h v) - \nabla v\| \lesssim h^m \|v\|_{m+1}.$$

In particular, for $v \in H^1(\Omega)$, the L^2-projection Q_h is energy stable, i.e,

$$\|\nabla_w (Q_h v)\| \lesssim \|\nabla v\| \quad \text{for } v \in H^1(\Omega). \tag{2.5}$$

Now we are able to present the weak Galerkin finite element formulation for (2.1): Find $u_h = \{u_0,\ u_b\} \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_0) \quad \text{for all } v_h = \{v_0,\ v_b\} \in V_h, \tag{2.6}$$

where the bilinear form $a_h(\cdot,\cdot)$ on $V_h \times V_h$ is defined by

$$a_h(u_h, v_h) := (\mathbb{A} \nabla_w u_h,\ \nabla_w v_h). \tag{2.7}$$

Remark 2.1. When $\mathbb{A} = cI$ where c is piecewise constant, it is known [1,6,18] that, by setting the weak gradient as the flux and u_b as a Lagrange multiplier, the weak Galerkin formulation (2.6) is equivalent to the hybridized mixed finite element method using either the RT or the BDM element in the discretization [17,19]. But in general, these two methods are different.

The well-posedness and error estimates of the weak Galerkin formulation (2.6) have been discussed in [1,6]. To state these results, we first define a few discrete inner-products and norms. For any $v_h = \{v_0, v_b\}$ and $\phi_h = \{\phi_0, \phi_b\}$ in V_h, define a discrete L^2 inner-product by

$$((v_h, \phi_h)) \triangleq \sum_{K \in \mathcal{T}_h} \left[(v_0, \phi_0)_K + h(v_0 - v_b, \phi_0 - \phi_b)_{\partial K} \right].$$

It is not hard to see that $((v_h, v_h)) = 0$ implies $v_h \equiv 0$. Hence, the inner-product is well-defined. Notice that the inner-product $((\cdot,\cdot))$ is also well-defined for any $v \in H^1(\Omega)$, for which $v_b|_e = v|_e$ is the trace of v on the edge e. In this case, the inner-product $((\cdot,\cdot))$ is identical to the standard L^2 inner-product.

Define on each $K \in \mathcal{T}_h$

$$\|v_h\|_{0,h,K}^2 = \|v_0\|_K^2 + h\|v_0 - v_b\|_{\partial K}^2,$$
$$\|v_h\|_{1,h,K}^2 = \|v_0\|_{1,K}^2 + h^{-1}\|v_0 - v_b\|_{\partial K}^2,$$
$$|v_h|_{1,h,K}^2 = |v_0|_{1,K}^2 + h^{-1}\|v_0 - v_b\|_{\partial K}^2.$$

Using the above quantities, we define the following discrete norms and semi-norms on the discrete space V_h

$$\|v_h\|_{0,h} := \left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{0,h,K}^2 \right)^{1/2},$$

$$\|v_h\|_{1,h} := \left(\sum_{K\in\mathcal{T}_h} \|v_h\|_{1,h,K}^2\right)^{1/2},$$

$$|v_h|_{1,h} := \left(\sum_{K\in\mathcal{T}_h} |v_h|_{1,h,K}^2\right)^{1/2}.$$

It is clear that $\|v_h\|_{0,h}^2 = ((v_h, v_h))$. Moreover, we point out that the above norms and semi-norms are also well-defined for functions in $H^1(\Omega)$. In this case they are identical to the usual L^2-norm, H^1-norm, and H^1-seminorm, respectively. Similar norms are also used in the analysis of HDG method [9].

With the aid of the above defined norms, we state an additional estimate of the L^2 projection Q_h, which was proved in [6].

Lemma 2.3. *For any $v \in H^m(\Omega)$ with $\frac{1}{2} < m \le j+1$, we have*

$$\|v - Q_h v\|_{0,h} \lesssim h^m \|v\|_m. \tag{2.8}$$

The following three Lemmas have also been proved in [6]. First, we have the equivalence between $\|\nabla_w(\cdot)\|$ and the $|\cdot|_{1,h}$ semi-norm.

Lemma 2.4. *For any $v_h = \{v_0, v_b\} \in V_h$, we have*

$$|v_h|_{1,h} \lesssim \|\nabla_w v_h\| \lesssim |v_h|_{1,h}. \tag{2.9}$$

Moreover, the discrete semi-norms satisfy the usual inverse inequality, as stated in the following lemma.

Lemma 2.5. *For any $v_h = \{v_0, v_b\} \in V_h$, we have*

$$|v_h|_{1,h} \lesssim h^{-1}\|v_h\|_{0,h}. \tag{2.10}$$

Consequently, by combining (2.9) and (2.10), we have

$$\|\nabla_w v_h\| \lesssim h^{-1}\|v_h\|_{0,h}. \tag{2.11}$$

Next, the discrete semi-norm $\|\nabla_w(\cdot)\|$, which is equivalent to $|\cdot|_{1,h}$ as shown in Lemma 2.4, satisfies a Poincaré-type inequality.

Lemma 2.6. *The Poincaré-type inequality holds true for functions in V_h. In other words, we have the following estimate:*

$$\|v_h\|_{0,h} \lesssim \|\nabla_w v_h\| \quad \text{for all } v_h \in V_h. \tag{2.12}$$

Following the above lemmas and (1.2), it is clear that Eq. (2.6) admits a unique solution. This, together with error estimates for the weak Galerkin method, has been proved in [1].

Theorem 2.7. *Assume Problem (2.1) has H^{1+s} regularity, where $0 < s \le 1$. The weak Galerkin problem (2.6) admits a unique solution. Let $u \in H_0^1(\Omega) \cap H^{m+1}(\Omega)$, $0 \le m \le j+1$ be the solution to (2.1) and $u_h = \{u_{h,0}, u_{h,b}\}$ be the solution to (2.6), then we have*

$$\|\nabla_w(Q_h u - u_h)\| \lesssim h^m \|u\|_{m+1}, \tag{2.13}$$

$$\|Q_0 u - u_{h,0}\| \lesssim h^{m+s}\|u\|_{m+1} + h^{1+s}\|f - Q_0 f\|. \tag{2.14}$$

Remark 2.2. Theorem 2.7 is only stated for homogeneous Dirichlet boundary value problems. Similar results hold for problems with non-homogeneous Dirichlet boundary or Neumann boundary conditions [1,6].

At the end of this section, we state a scaled trace theorem. Let K be an element with e as an edge. It is well known that for any function $g \in H^1(K)$ one has

$$\|g\|_e^2 \lesssim h^{-1}\|g\|_K^2 + h\|\nabla g\|_K^2. \tag{2.15}$$

3. An auxiliary space multigrid preconditioner

In this section, we construct an auxiliary space multigrid method for the weak Galerkin formulation (2.6). The auxiliary space multigrid method was introduced by J. Xu in [8]. Its main idea is to use an auxiliary space as a "coarse" space in the multigrid algorithm, where the discrete problem in the auxiliary space can be easily solved by an existing solver. In our

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construction, we will use the H^1 conforming piecewise linear finite element space as an auxiliary space. The main technical difficulty is to build the connection between the weak Galerkin discrete space V_h and the H^1 conforming piecewise linear finite element space.

Define the auxiliary space $\mathcal{V}_h \subset H_0^1(\Omega)$ to be the H^1 conforming piecewise linear finite element space on mesh $\mathcal{T}_h$. The spaces V_h and $\mathcal{V}_h$ are equipped with inner-products $((\cdot, \cdot))$ and $(\cdot, \cdot)$, and induced norms $\|\cdot\|_{0,h}$ and $\|\cdot\|$, respectively. Define linear operators $A: V_h \to V_h$ and $\mathcal{A}: \mathcal{V}_h \to \mathcal{V}_h$ by

$$
\begin{aligned}
&((Au,\ v)) = (\mathbb{A}\nabla_w u,\ \nabla_w v) \quad \text{for all } v \in V_h,\\
&(\mathcal{A}u,\ v) = (\mathbb{A}\nabla u,\ \nabla v) \quad \text{for all } v \in \mathcal{V}_h.
\end{aligned} \tag{3.1}
$$

By the Poincaré inequality and Lemma 2.6, it is clear that operators A and $\mathcal{A}$ are symmetric and positive definite with respect to $((\cdot, \cdot))$ and $(\cdot, \cdot)$, respectively. Hence we can define the A-norm and $\mathcal{A}$-norm on V_h and $\mathcal{V}_h$, respectively, by

$$
\begin{aligned}
&\|v\|_A = ((Av,\ v))^{1/2} = (\mathbb{A}\nabla_w v,\ \nabla_w v)^{1/2} \quad \text{for all} v \in V_h,\\
&\|w\|_{\mathcal{A}} = (\mathcal{A}w,\ w)^{1/2} = (\mathbb{A}\nabla w,\ \nabla w)^{1/2} \quad \text{for all} w \in \mathcal{V}_h.
\end{aligned}
$$

It is well-known that the spectral radius and condition number of operator $\mathcal{A}$ is $O(h^{-2})$ [20]. We have similar estimate for the operator A. Note that the authors of [7] also give a proof of the order of the condition number. But our proof is different from theirs and seems to be easier.

Lemma 3.1. *The spectral radius of operator A, denoted by $\rho_A = \lambda_{\max}(A)$, and the condition number of operator A, denoted by $\kappa(A)$, are both of order h^{-2}.*

Proof. By the definition of A and Lemma 2.5, for all $v \in V_h$,

$$((Av, v)) \lesssim \|\nabla_w v\|^2 \lesssim h^{-2}\|v\|_{0,h}^2 = h^{-2}((v, v)).$$

Because A is symmetric and positive definite with respect to $((\cdot, \cdot))$, the above inequality implies that $\lambda_{\max}(A) \lesssim h^{-2}$. The discrete Poincaré inequality (2.12) implies $\lambda_{\min}(A) \gtrsim 1$. Therefore $\kappa(A) = \lambda_{\max}(A)/\lambda_{\min}(A) \lesssim h^{-2}$.

To derive a lower bound for $\lambda_{\max}(A)$, we first consider functions in V_h with the form $v = \{0, v_b\}$. In other words, $v_0 \equiv 0$. Then, by the definition of discrete norms, Lemma 2.4 and the fact that $\mathbb{A}$ is uniformly positive definite, for such function v we have

$$
\begin{aligned}
((Av, v)) &\gtrsim \|\nabla_w v\|^2 \gtrsim |v|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} h^{-1}\|v_b\|_{\partial K}^2\\
&= h^{-2}\sum_{K\in\mathcal{T}_h} h\|v_b\|_{\partial K}^2 = h^{-2}\|v\|_{0,h}^2 = h^{-2}((v, v)).
\end{aligned}
$$

Therefore, we must have $\lambda_{\max}(A) \gtrsim h^{-2}$. This implies the spectral radius $\rho_A = \lambda_{\max}(A) = O(h^{-2})$.

To get $\lambda_{\min}(A) \lesssim 1$, we chose the eigenfunction w of the smallest eigenvalue, λ_1, of $-\Delta$ with homogeneous Dirichlet boundary condition which satisfies $1 = \|\nabla w\| = \sqrt{\lambda_1}\|w\|$. It is well known that $\lambda_1 = O(1)$. We then project w to V_h using the L^2-projection, i.e., $w_h = Q_h w$. We estimate the norm of w_h as follows: when h is sufficiently small, by the triangle inequality and Lemma 2.3 one has

$$\|w_h\| \geq \|w\| - \|w - w_h\| \gtrsim \|w\| - Ch\|\nabla w\| = \|w\| - Ch \gtrsim \|w\|,$$

where C is a positive, general constant. By the above inequality and the stability of Q_h in the energy norm, c.f. (2.5), we have

$$\|w_h\|_A \lesssim \|\nabla w\| = \sqrt{\lambda_1}\|w\| \lesssim \|w_h\|.$$

This completes the proof of the lemma. □

Remark 3.1. By the triangle inequality, the trace inequality (2.15) and the inverse inequality, the norm $\|v_h\|_{0,h}$ is equivalent to $\left(\sum_{K\in\mathcal{T}_h}(\|v_0\|_K^2 + h\|v_b\|_{\partial K}^2)\right)^{1/2}$ in V_h. In practice, Eq. (2.6) can be written as a linear algebraic system by using the canonical bases of V_h, i.e. Lagrange bases of $P_j(K)$ and $P_l(e)$ on each K and e. Using the standard scaling argument and the equivalent norm of $\|\cdot\|_{0,h}$, it is not hard to see that for any $v_h \in V_h$, one has $\|v_h\|_{0,h}^2 \approx h^d\|\underline{v_h}\|_{l^2}^2$, where $\underline{v_h}$ is the vector representation of v_h under the canonical bases and $\|\cdot\|_{l^2}$ is the Euclidean norm of vectors. Then, the stiffness matrix in the linear algebraic system resulting from (2.6), i.e., the matrix representation of $((A\cdot, \cdot))$, also has condition number of order $O(h^{-2})$. Thus it is not easy to solve Eq. (2.6) without efficient preconditioning.

Next, we introduce the auxiliary space multigrid method for solving Eq. (2.6). The idea is to construct a multigrid method using V_h as the "fine" space and $\mathcal{V}_h$ as the "coarse" space. Since $\mathcal{A}$ is the discrete Laplacian on the conforming piecewise linear finite element space, the "coarse" problem in $\mathcal{V}_h$ can be solved by many efficient, off-the-shelf solvers such as the standard multigrid solver or a domain decomposition solver. Denote $\mathcal{B}: \mathcal{V}_h \to \mathcal{V}_h$ to be such a "coarse" solver. It can be either an exact solver or an approximate solver that satisfies certain conditions, which will be given later. Next, on the fine space, we need a "smoother" $R : V_h \to V_h$, which is symmetric and positive definite. For example, R can be a Jacobi or symmetric Gauss–Seidel smoother. Finally, to connect the "coarse" space with the "fine" space, we need a "prolongation" operator $\Pi : \mathcal{V}_h \to V_h$. A "restriction" operator $\Pi^t : V_h \to \mathcal{V}_h$ is consequently defined by

$$(\Pi^t v,\ w) = ((v,\ \Pi w)) \quad \text{for } v \in V_h \text{ and } w \in \mathcal{V}_h.$$

Then, the auxiliary space multigrid preconditioner $B : V_h \to V_h$, following the definition in [8,20], is given by

$$\text{Additive} \quad B = R + \Pi \mathcal{B} \Pi^t, \tag{3.2}$$

$$\text{Multiplicative} \quad I - BA = (I - RA)(I - \Pi \mathcal{B} \Pi^t)(I - RA). \tag{3.3}$$

Both the additive and the multiplicative versions define symmetric multigrid solvers/preconditioners. Readers may refer to [21] for the equivalence between symmetric solvers and preconditioners for symmetric problems. Non-symmetric multiplicative multigrid solver can similarly be defined but it cannot be used as a preconditioner. Thus we restrict our attention to the symmetric version.

According to [8], the following theorem holds.

Theorem 3.2. *Assume that for all $v \in V_h$, $w \in \mathcal{V}_h$,*

$$\rho_A^{-1}((v,\ v)) \lesssim ((Rv,\ v)) \lesssim \rho_A^{-1}((v,\ v)), \tag{3.4}$$

$$(\mathcal{A}w,\ w) \lesssim (\mathcal{B}\mathcal{A}w,\ \mathcal{A}w) \lesssim (\mathcal{A}w,\ w), \tag{3.5}$$

$$\|\Pi w\|_A \lesssim \|w\|_{\mathcal{A}} \quad (\textit{stability of } \Pi), \tag{3.6}$$

and furthermore, assume that there exists a linear operator $P : V_h \to \mathcal{V}_h$ such that

$$\|Pv\|_{\mathcal{A}} \lesssim \|v\|_A \quad (\textit{stability of } P), \tag{3.7}$$

$$\|v - \Pi P v\|_{0,h}^2 \lesssim \rho_A^{-1} \|v\|_A^2 \quad (\textit{approximability}). \tag{3.8}$$

Then the preconditioner B defined in (3.2) or (3.3) satisfies

$$\kappa(BA) \lesssim O(1).$$

Remark 3.2. Theorem 3.2 states that B is a good preconditioner for A as the condition number of BA is uniformly bounded. We thus can use the preconditioned conjugate gradient (PCG) method with B being an effective preconditioner for solving the linear algebraic equation system associate to $Au = f$. According to [21], Theorem 3.2 also implies that $I - \omega BA$, where $0 < \omega < 2/\rho_{BA}$, defines an efficient iterative solver.

Remark 3.3. Operator P is only needed in the theoretical analysis. In the implementation, one only needs $\mathcal{B}$, R and Π. It is also well-known that the matrix representation of the restriction operator Π^t is just the transpose of the matrix representation of the prolongation operator Π.

Now we shall construct an auxiliary space preconditioner which satisfies all conditions in Theorem 3.2, namely, inequalities (3.4)–(3.8). It is straight forward to pick $\mathcal{B}$ that satisfies condition (3.5). For example, $\mathcal{B}$ can be either the direct solver, for which $\mathcal{B} = \mathcal{A}^{-1}$, or one step of classical multigrid iteration [20] which satisfies condition (3.5).

The smoother R is also easy to define. In view of Remark 3.1, a Jacobi or a symmetric Gauss–Seidel smoother [20] will satisfy condition (3.4). Hence it remains to construct operators Π and P that satisfy the conditions (3.6)–(3.8).

The operator Π is actually easy to choose, and we simply define $\Pi = Q_h = \{Q_0,\ Q_b\}$. Note when V_h consists of $W_{j,j}$ elements or $W_{j,j+1}$ elements with $j \geq 1$, it is clear that for all $w \in \mathcal{V}_h$ and $K \in \mathcal{T}_h$, $(\Pi w)|_K = \{w|_{K_0},\ w|_{\partial K}\}$ which is just the natural inclusion of $\mathcal{V}_h$ into V_h. The stability of Π in the energy norm follows immediately from (2.5) and the boundedness of the diffusion coefficient $\mathbb{A}$, as shown in the following lemma.

Lemma 3.3. *Let $\Pi = Q_h = \{Q_0,\ Q_b\}$. Then Π satisfies condition (3.6), i.e.,*

$$\|\Pi w\|_A \lesssim \|w\|_{\mathcal{A}}, \quad \textit{for all } w \in \mathcal{V}_h.$$

Next, we construct an operator P that satisfies (3.7) and (3.8).

Definition 3.4. Let $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_k \leq 1$ satisfy $\sum_{i=1}^k \alpha_i = 1$, and let $\{c_1, c_2, \ldots, c_k\}$ be a sequence of numbers. The value $\sum_{i=1}^k \alpha_i c_i$ is called a convex combination of $\{c_1, c_2, \ldots, c_k\}$.

A function in $\mathcal{V}_h$ is completely determined by its value on mesh vertices. Let $v = \{v_0, v_b\} \in V_h$. To define Pv, one only needs to specify its value on all mesh vertices. Hence we can define P as follows: on each mesh vertex $\mathbf{x}$, the value of $Pv(\mathbf{x})$ is a prescribed convex combination of the values of $v_0(\mathbf{x})$ and $v_b(\mathbf{x})$ on all mesh elements and edges/faces that have $\mathbf{x}$ as

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a vertex. Moreover, to preserve the homogeneous boundary condition, when $\mathbf{x} \in \partial\Omega$, the convex combination shall be constructed such that it only depends on the value of $v_b(\mathbf{x})$ on boundary edges/faces that have $\mathbf{x}$ as a vertex. Of course, for problems with the homogeneous Dirichlet boundary condition, one can simply set $Pv(\mathbf{x}) = 0$ on boundary vertices. But the current set-up would allow easy extension to non-homogeneous boundary conditions.

Lemma 3.5. *Operator P satisfies*

$$\|v - Pv\|_{0,h}^2 + h^2|v - Pv|_{1,h}^2 \lesssim h^2|v|_{1,h}^2, \quad \text{for all } v \in V_h. \tag{3.9}$$

Proof. For each $K \in \mathcal{T}_h$, denote by $V(K)$ the vertices of K. For each $K \in \mathcal{T}_h$ and $v = \{v_0, v_b\} \in V_h$, denote by $I_{h,K}v_0$ the nodal value interpolation of v_0 into $P_1(K)$, i.e., $I_{h,K}v_0 \in P_1(K)$ and is identical to v_0 on $V(K)$. By the approximation property of nodal value interpolations, the scaling argument, the definition of P, the triangle inequality, and the finite overlapping property of quasi-uniform meshes, we have

$$\begin{aligned}
\|v - Pv\|_{0,h}^2 &= \sum_{K\in\mathcal{T}_h} \left(\|v_0 - Pv\|_K^2 + h\|v_0 - v_b\|_{0,\partial K}^2\right) \\
&\lesssim \sum_{K\in\mathcal{T}_h} \left(\|v_0 - I_{h,K}v_0\|_K^2 + \sum_{\mathbf{x}\in V(K)} h^2|I_{h,K}v_0(\mathbf{x}) - Pv(\mathbf{x})|^2 + h\|v_0 - v_b\|_{0,\partial K}^2\right) \\
&\lesssim \sum_{K\in\mathcal{T}_h} \left(h^2|v_0|_{1,K}^2 + \sum_{\mathbf{x}\in V(K)} h^2|v_0(\mathbf{x}) - v_b(\mathbf{x})|^2 + h\|v_0 - v_b\|_{0,\partial K}^2\right) \\
&\lesssim \sum_{K\in\mathcal{T}_h} \left(h^2|v_0|_{1,K}^2 + h\|v_0 - v_b\|_{0,\partial K}^2\right) \\
&= h^2|v|_{1,h}^2.
\end{aligned}$$

Combining the above with the inverse inequality (2.10) completes the proof of the lemma. □

Lemma 3.6. *The operator P satisfies the properties (3.7) and (3.8).*

Proof. By using inequalities (3.9) and (2.9), for all $v \in V_h$, we have

$$\|Pv\|_{\mathcal{A}}^2 \lesssim |Pv|_{1,h}^2 \lesssim |v - Pv|_{1,h}^2 + |v|_{1,h}^2 \lesssim |v|_{1,h}^2 \lesssim \|v\|_A^2.$$

This completes the proof of Inequality (3.7).

We then estimate $\|Pv - \Pi Pv\|_{0,h}$. When $j \geq 1$, $\|Pv - \Pi Pv\|_{0,h} = 0$ since Π is the natural inclusion. We only need to consider the case $j = 0$. Since ΠPv is the average of Pv, we get

$$\|Pv - \Pi Pv\|_K \lesssim h|Pv|_1, \qquad \|Pv - \Pi Pv\|_{\partial K} \lesssim h|Pv|_{1,\partial K},$$

by the average type Poincaré inequality. By the scaled trace inequality (2.15) and the fact $|Pv|_{2,K} = 0$ for a piecewise linear function, we can bound $h^{1/2}|Pv|_{1,\partial K} \lesssim |Pv|_{1,K}$. Therefore, we obtain

$$\|Pv - \Pi Pv\|_{0,h} \lesssim h|Pv|_1 = h|Pv|_{1,h} \lesssim h|v|_{1,h}.$$

Then, by the triangle inequality and the coercivity of operator A, for all $v \in V_h$, we have

$$\|v - \Pi Pv\|_{0,h} \lesssim \|v - Pv\|_{0,h} + \|Pv - \Pi Pv\|_{0,h} \lesssim h|v|_{1,h} \lesssim h\|v\|_A.$$

Combining the above with the estimate $\rho_A = O(h^{-2})$ (see Lemma 3.1), this completes the proof of Inequality (3.8). □

Remark 3.4. In the proof of Lemma 3.6, one may also use Lemma 2.3 and the Poincaré inequality to estimate $\|Pv - \Pi Pv\|_{0,h}$, i.e.,

$$\|Pv - \Pi Pv\|_{0,h} \lesssim h\|Pv\|_1 \lesssim h|Pv|_1.$$

This requires the Poincaré inequality for Pv, which is not true for non-homogeneous Dirichlet boundary problems. The current approach avoids such difficulty and can thus be easily extended to non-homogeneous Dirichlet boundary problems or Neumann boundary problems.

By now, all conditions in Theorem 3.2 have been verified for the given multigrid construction. We summarize these in the following theorem.

Theorem 3.7. *Suppose we have a smoother R and an auxiliary solver $\mathcal{B}$ satisfying the property: for all $v \in V_h$, $w \in \mathcal{V}_h$,*

$$\begin{aligned}
&\rho_A^{-1}((v, v)) \lesssim ((Rv, v)) \lesssim \rho_A^{-1}((v, v)), \\
&(\mathcal{A}w, w) \lesssim (\mathcal{B}\mathcal{A}w, \mathcal{A}w) \lesssim (\mathcal{A}w, w).
\end{aligned}$$

Let $B = R + \Pi \mathcal{B} \Pi^t$ *or defined implicitly by the relation* $I - BA = (I - RA)(I - \Pi \mathcal{B} \Pi^t)(I - RA)$*. Then* B *is symmetric and positive definite and* $\kappa(BA) \lesssim O(1)$*.*

4. Reduced system and its multigrid preconditioner

By using the Schur complement, the weak Galerkin problem (2.6) can be reduced to a system involving only the degrees of freedom on mesh edges/faces. In this section, we present such a reduced system and construct an auxiliary space multigrid preconditioner for the reduced system.

4.1. Reduced system

Let

$$V_0 = \{v | v = \{v_0, 0\} \in V_h\},$$
$$V_b = \{v | v = \{0, v_b\} \in V_h\},$$

be two subspaces of V_h. Clearly one has $V_h = V_0 + V_b$. For any function $v = \{v_0, v_b\} \in V_h$, it is convenient to extend the notation of v_0 and v_b so that, without ambiguity, $v_0 \in V_0$ and $v_b \in V_b$. Functions in V_0 and V_b will also often be referred to as v_0 and v_b, respectively.

Then Eq. (2.6) can be rewritten into

$$\begin{aligned} &a_h(u_0, v_b) + a_h(u_b, v_b) = 0, \quad \text{for all } v_b \in V_b, \\ &a_h(u_0, v_0) + a_h(u_b, v_0) = (f, v_0) \quad \text{for all } v_0 \in V_0. \end{aligned} \tag{4.1}$$

By choosing a basis of V_h, we can obtain a matrix form of (4.1). Let $\mathbf{v}$ be the vector representation of a weak function $v \in V_h$ and $\mathbf{M}$ be the matrix representation of an operator M relative to the chosen basis. We can write the matrix form of (4.1) as follows

$$\begin{pmatrix} \mathbf{A}_b & \mathbf{A}_{b0} \\ \mathbf{A}_{0b} & \mathbf{A}_0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_b \\ \mathbf{u}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}. \tag{4.2}$$

Note that $\mathbf{A}_0$ is block-diagonal. We can thus solve $\mathbf{u}_0$ from the second equation and substitute into the first equation to obtain the Schur complement equation

$$(\mathbf{A}_b - \mathbf{A}_{b0}\mathbf{A}_0^{-1}\mathbf{A}_{0b})\mathbf{u}_b = -\mathbf{A}_{b0}\mathbf{A}_0^{-1}\mathbf{f}. \tag{4.3}$$

After $\mathbf{u}_b$ is obtained by solving (4.3), the interior part $\mathbf{u}_0 = \mathbf{A}_0^{-1}(\mathbf{f} - \mathbf{A}_{0b}\mathbf{u}_b)$ can be computed element-wise.

The reduced system (4.3) involves less degrees of freedom than the original weak Galerkin system (4.2). Indeed, the difference between these two degrees of freedom is exactly $\dim(V_0)$, which is equal to $(j+1)(j+2)/2$ times the total number of mesh triangles in two-dimension, and $(j+1)(j+2)(j+3)/6$ times the total number of mesh tetrahedron in three-dimension. More importantly, the Schur complement $\mathbf{A}_b - \mathbf{A}_{b0}\mathbf{A}_0^{-1}\mathbf{A}_{0b}$ is also a SPD matrix and has the same sparsity as $\mathbf{A}_b$. Therefore solving the reduced system (4.3) is more efficient than solving the original system (4.2) provided a good preconditioner for (4.3) is available. In the rest of this section, we will construct a fast auxiliary multigrid preconditioner for (4.3). Note that the algorithm is implemented in the matrix formulation. The analysis, however, is given in the operator form. In the following we will introduce corresponding operators.

We first introduce an $a_h(\cdot, \cdot)$-orthogonal projector P_0 from V_b to V_0 as follows: For $v_b \in V_b$, define $P_0 v_b \in V_0$ such that

$$a_h(P_0 v_b, \zeta_0) = a_h(v_b, \zeta_0) \quad \text{for all } \zeta_0 \in V_0.$$

It is not hard to see that $\|(I - P_0)v_b\|_{0,h} = \|\{-P_0 v_b, v_b\}\|_{0,h}$ is a well-defined norm on V_b. In the following analysis we shall always equip V_b with this new norm and V_0 with the inherited norm $\|\cdot\|_{0,h}$. By the trace inequality, the inverse inequality and the definition of $\|\cdot\|_{0,h}$, one has

$$\|P_0 v_b\|_{0,h} \lesssim \|P_0 v_b\| \lesssim \|\{-P_0 v_b, v_b\}\|_{0,h} = \|(I - P_0)v_b\|_{0,h},$$

which implies that $P_0 : V_b \to V_0$ is a bounded linear operator under the newly assigned norms. Denote by V_0' and V_b' the space of bounded linear functionals on V_0 and V_b, respectively. Then the bounded linear operator P_0 induces a bounded dual operator $P_0' : V_0' \to V_b'$, i.e., for $F \in V_0'$, $\langle P_0' F, v_b\rangle \triangleq \langle F, P_0 v_b\rangle$ for all $v_b \in V_b$. In particular, let F be defined by $\langle F, \cdot\rangle = (f, \cdot)$ for $f \in L^2(\Omega)$, then one has $\langle P_0' F, v_b\rangle = (f, P_0 v_b)$.

We claim, and will prove later, that the operator form of the Schur complement equation (4.3) is

$$a_h((I - P_0)u_b, v_b) = -\langle P_0' F, v_b\rangle, \quad \text{for all } v_b \in V_b. \tag{4.4}$$

Note that by the property of the projection P_0, Eq. (4.4) can also be written into the symmetric form $a_h((I - P_0)u_b, (I - P_0)v_b) = -\langle P_0' F, v_b\rangle$ for all $v_b \in V_b$.

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To prove this, we first define a linear operator $A_0^{-1} : L^2(\Omega) \to V_0$ by: for a function $g \in L^2(\Omega)$, one has $A_0^{-1}g \in V_0$ such that

$$a_h(A_0^{-1}g,\ v_0) = (g,\ v_0) \quad \text{for all } v_0 \in V_0.$$

The well-posedness of A_0^{-1} follows directly from the coercivity of $a_h(\cdot,\cdot)$ on V_h, and consequently on its subspace V_0. Moreover, the restriction of A_0^{-1} to V_0 is symmetric and positive definite. Noticing that $\nabla_w v_0$ is locally defined on each mesh element, it is clear that A_0^{-1} is also locally defined on each mesh element.

Denote by $\nabla_h\cdot$ the piecewise divergence operator on Σ_h, and by $\mathbb{Q}_h : L^2(\Omega)^d \to \Sigma_h$ the L^2 projection. Using the above notation and the definition of ∇_w, the second equation in (4.1) implies that for all $v_0 \in V_0$,

$$\begin{aligned}(\mathbb{A}\nabla_w u_0,\ \nabla_w v_0) &= (f,\ v_0) - (\mathbb{A}\nabla_w u_b,\ \nabla_w v_0)\\ &= (f,\ v_0) + (\nabla_h\cdot(\mathbb{Q}_h\mathbb{A}\nabla_w u_b),\ v_0),\end{aligned} \tag{4.5}$$

which leads to

$$u_0 = A_0^{-1}(f + \nabla_h\cdot(\mathbb{Q}_h\mathbb{A}\nabla_w u_b)). \tag{4.6}$$

Next, we note that the projection P_0 is identical to $-A_0^{-1}\nabla_h\cdot(\mathbb{Q}_h\mathbb{A}\nabla_w)$ on V_b, as shown in the following lemma.

Lemma 4.1. *The orthogonal operator $P_0 : V_b \to V_0$*

$$P_0 v_b = -A_0^{-1}\nabla_h\cdot(\mathbb{Q}_h\mathbb{A}\nabla_w v_b) \quad \textit{for all } v_b \in V_b.$$

Proof. By the definition of weak gradient ∇_w and A_0^{-1}, we have

$$\begin{aligned}(\mathbb{A}\nabla_w v_b,\ \nabla_w \zeta_0) &= -(\nabla_h\cdot(\mathbb{Q}_h\mathbb{A}\nabla_w v_b),\ \zeta_0)\\ &= -(\mathbb{A}\nabla_w A_0^{-1}\nabla_h\cdot(\mathbb{Q}_h\mathbb{A}\nabla_w v_b),\ \nabla_w\zeta_0).\end{aligned}$$

By the definition of P_0, we then complete the proof of the lemma. □

Remark 4.1. The operator P_0 corresponds to the matrix $\mathbf{A}_0^{-1}\mathbf{A}_{0b}$.

Now, by (4.6) and Lemma 4.1, one has $u_0 = A_0^{-1}f - P_0u_b$. Substituting this into the first equation of (4.1) gives

$$\begin{aligned}a_h((I-P_0)u_b,\ v_b) &= -a_h(A_0^{-1}f,\ v_b) = -a_h(A_0^{-1}f,\ P_0v_b)\\ &= -(f,\ P_0v_b) = -\langle P_0'F,\ v_b\rangle.\end{aligned}$$

This completes the derivation of the reduced problem (4.4) from the original problem (2.6). Here we emphasize again that $P_0'F \in V_b'$ is bounded in the sense that

$$|\langle P_0'F,\ v_b\rangle| \lesssim \|(I-P_0)v_b\|_{0,h}. \tag{4.7}$$

We will further reformulate the reduced system (4.4). To this end, we define a subspace of V_h as $V_r = \{v_r \mid v_r = (I-P_0)v_b = \{-P_0v_b, v_b\}$ for all $v_b \in V_b\}$, which is just the graph of V_b under $I - P_0$. The space V_r inherits the norm $\|\cdot\|_{0,h}$ from V_h, and hence V_r and V_b (equipped with the norm $\|(I-P_0)\cdot\|_{0,h}$) are clearly isomorphic under the mapping $I - P_0 : V_b \to V_r$. Moreover, the right-hand side of Eq. (4.4) can be written into

$$\begin{aligned}-\langle P_0'F,\ v_b\rangle &= -\langle\{0, P_0'F\},\ \{-P_0v_b,\ v_b\}\rangle = -\langle\{0, P_0'F\},\ v_r\rangle\\ &\triangleq \langle\mathcal{F},\ v_r\rangle,\end{aligned}$$

where $\mathcal{F}$ is a bounded linear functional on V_r according to (4.7).

By using Lemma 4.1 and combining the above analysis, Eq. (4.4) can now be rewritten into: Find $u_r \in V_r$ such that

$$a_h(u_r,\ v_r) = \langle\mathcal{F}, v_r\rangle, \quad \text{for all } v_r \in V_r. \tag{4.8}$$

The well-posedness of (4.8) then follows from the continuity and coercivity of the bilinear form $a_h(\cdot,\cdot)$ restricted to V_r and the fact that $\mathcal{F}$ is a bounded linear functional on V_r.

4.2. Auxiliary space preconditioner for the reduced system

Now we are able to consider an auxiliary space multigrid preconditioner for the reduced system (4.8), using again the H^1 conforming piecewise linear finite element space as the auxiliary space. Denote by A_r the restriction of operator A, defined in (3.1), to the subspace V_r. That is, $A_r : V_r \to V_r$ is defined by

$$((A_r u,\ v)) = a_h(u,\ v) \quad \text{for all } v \in V_r.$$

To apply Theorem 3.2, we define a prolongation operator $\Pi_r : \mathcal{V}_h \to V_r$ and a linear operator $P_r : V_r \to \mathcal{V}_h$ by

$$\Pi_r = (I-P_0)Q_b \quad \text{and} \quad P_r = P|_{V_r}.$$

Lemma 4.2. *Both Π_r and P_r are stable in the energy norm, i.e.,*

$$\|\Pi_r v\|_A \lesssim \|v\|_{\mathcal{A}}, \quad \text{for all } v \in \mathcal{V}_h \tag{4.9}$$

$$\|P_r v_r\|_{\mathcal{A}} \lesssim \|v_r\|_A, \quad \text{for all } v_r \in V_r. \tag{4.10}$$

Proof. The stability of Π_r follows from the property of P_0 and the stability (2.5) of Q_h:

$$\begin{aligned}\|\Pi_r v\|_A^2 &= \|(I-P_0)Q_b v\|_A^2 = (\mathbb{A}\nabla_w(I-P_0)Q_b v,\ \nabla_w(Q_b v + Q_0 v))\\ &\lesssim \|\Pi_r v\|_A \|Q_h v\|_A \lesssim \|\Pi_r v\|_A \|v\|_{\mathcal{A}}.\end{aligned}$$

The stability of P_r simply follows from that of P. This completes the proof of the lemma. □

To verify the approximation property, we first explore the relation between $Q_h w$ and $\Pi_r w$ for $w \in \mathcal{V}_h$. It turns out that $Q_h w = \Pi_r w$ for all $w \in \mathcal{V}_h$ when the diffusion coefficient matrix $\mathbb{A}$ is piecewise constant.

Lemma 4.3. *When $\mathbb{A}$ is piecewise constant, we have for all $w \in \mathcal{V}_h$,*

$$Q_h w = \Pi_r w.$$

Proof. Recall that $\Pi_r w = (I - P_0)Q_b w$. Since P_0 is the orthogonal projection, we have

$$(\mathbb{A}\nabla_w \Pi_r w,\ \nabla_w \zeta_0) = (\mathbb{A}\nabla_w (I-P_0)Q_b w,\ \nabla_w \zeta_0) = 0 \quad \text{for all } \zeta_0 \in V_0.$$

On the other hand, using the relation (2.4) and the fact that both ∇w and $\mathbb{A}$ are piecewise constant,

$$\begin{aligned}(\mathbb{A}\nabla_w Q_h w,\ \nabla_w \zeta_0)_K &= (\mathbb{A}\mathbb{Q}_h \nabla w,\ \nabla_w \zeta_0)_K = (\mathbb{A}\nabla w,\ \nabla_w \zeta_0)_K\\ &= -(\nabla_h \cdot (\mathbb{A}\nabla w),\ \zeta_0)_K = 0, \quad \text{for all } \zeta_0 \in V_0.\end{aligned}$$

Therefore

$$a_h(\Pi_r w - Q_h w, \zeta_0) = 0, \quad \text{for all } \zeta_0 \in V_0. \tag{4.11}$$

The fact $\Pi_r w - Q_h w \in V_0$ and the orthogonality (4.11) imply $\Pi_r w = Q_h w$. □

Similar to the analysis in Section 3, we can establish the following results.

Lemma 4.4. *Suppose $\mathbb{A}$ is piecewise constant and the space V_r is non-trivial, i.e., the triangulation contains at least one interior vertex. Then the spectral radius of operator A_r, denoted by ρ_{A_r}, is of order h^{-2}.*

Proof. Recall that

$$\rho_{A_r} = \lambda_{\max}(A_r) = \max_{v\in V_r} \frac{((A_r v, v))}{((v, v))} = \max_{v\in V_r} \frac{((A v, v))}{((v, v))}.$$

Since $V_r \subset V_h$, we immediately get $\rho_{A_r} \le \rho_A \lesssim h^{-2}$.

To show the lower bound, we pick a hat function $w \in \mathcal{V}_h$. By the standard scaling argument,

$$\|w\| \lesssim h|\nabla w|. \tag{4.12}$$

We then chose $v = \Pi_r w \in V_r$ and estimate its norms. First

$$\|v\|_{0,h} = \|\Pi_r w\|_{0,h} = \|Q_h w\|_{0,h} \lesssim \|w\|. \tag{4.13}$$

Second, as ∇w is piecewise constant, $\mathbb{Q}_h \nabla w = \nabla w$ and

$$\|\nabla w\| = \|\mathbb{Q}_h \nabla w\| = \|\nabla_w Q_h w\| = \|\nabla_w \Pi_r w\| \lesssim (A_r v, v)^{1/2}. \tag{4.14}$$

Combining (4.12)–(4.14), we obtain

$$h^{-2}\|v\|_{0,h}^2 \lesssim (A_r v, v),$$

which implies $\rho_{A_r} \gtrsim h^{-2}$. □

Now we are able to derive the following approximation property.

Lemma 4.5. *Under the same assumptions as in Lemma 4.4, one has*

$$\|v_r - \Pi_r P_r v_r\|_{0,h} \lesssim \rho_{A_r}^{-1/2} \|v_r\|_A \quad \text{for all } v_r \in V_r.$$

Proof. By the triangle inequality and Eq. (3.9), one has

$$\begin{aligned}\|v_r - \Pi_r P_r v_r\|_{0,h} &= \|(I-P_0)v_b - \Pi_r P(I-P_0)v_b\|_{0,h}\\ &\lesssim \|(I-P_0)v_b - P(I-P_0)v_b\|_{0,h} + \|w - Q_h w\|_{0,h}\\ &\lesssim h|(I-P_0)v_b|_{1,h} + h\|w\|_1,\\ &\lesssim h|(I-P_0)v_b|_{1,h},\end{aligned}$$

where we conveniently denote $w = P(I-P_0)v_b \in \mathcal{V}_h$ and use $\Pi_r w = Q_h w$. In the last step, we have used

$$h\|w\|_1 \lesssim h|w|_1 = h|P(I-P_0)v_b|_{1,h} \lesssim h|(I-P_0)v_b|_{1,h}.$$

Combining the above and using Lemma 2.4 give

$$\|v_r - \Pi_r P_r v_r\|_{0,h} \lesssim h|(I-P_0)v_b|_{1,h} \lesssim h\|v_r\|_A.$$

According to Lemma 4.4, $\rho_{A_r} = O(h^{-2})$. This completes the proof of the lemma. □

Finally, we consider variable coefficient $\mathbb{A}$ which does not change too rapidly on each $K \in \mathcal{T}_h$, i.e., there exists a piecewise constant approximation $\bar{\mathbb{A}}$ such that $\mathbb{A}$ is spectrally equivalent to $\bar{\mathbb{A}}$ on each $K \in \mathcal{T}_h$. Then one has

$$(\mathbb{A}\nabla_w v, \nabla_w v) \lesssim (\bar{\mathbb{A}}\nabla_w v, \nabla_w v) \lesssim (\mathbb{A}\nabla_w v, \nabla_w v), \quad \text{for all } v \in V_h.$$

Therefore, a good preconditioner for the piecewise constant case will lead to a good preconditioner for the variable case.

We are able to claim that, the auxiliary space multigrid preconditioner for the reduced system (4.8) again yields a preconditioner system with condition number of $O(1)$.

Theorem 4.6. *Suppose we have a smoother R and auxiliary solver $\mathcal{B}$ satisfying the property: for all $v \in V_r$, $w \in \mathcal{V}_h$,*

$$\begin{aligned}&\rho_{A_r}^{-1}((v, v)) \lesssim ((Rv, v)) \lesssim \rho_{A_r}^{-1}((v, v)),\\ &(\mathcal{A}w, w) \lesssim (\mathcal{B}\mathcal{A}w, \mathcal{A}w) \lesssim (\mathcal{A}w, w).\end{aligned}$$

Let $B = R + \Pi\mathcal{B}\Pi^t$ or defined implicitly by the relation $I - BA_r = (I - RA_r)(I - \Pi\mathcal{B}\Pi^t)(I - RA_r)$. Then B is symmetric and positive definite and $\kappa(BA_r) \lesssim O(1)$.

5. Numerical results

In this section, we examine the effectiveness of the auxiliary space multigrid preconditioner using several numerical examples. The simulation is implemented using the MATLAB software package *i*FEM [22].

We use preconditioned Conjugate Gradient (PCG) method with the auxiliary space multigrid preconditioner. More precisely, we apply one step of the symmetric Gauss–Seidel smoothers as R and then one V-cycle of multigrid methods of the P_1 discretization with one pre-smoothing and one post-smoothing. In the auxiliary space preconditioner, we use the multiplicative version (3.3). It is known that the multiplicative version multigrid usually performs better than the corresponding additive version. The stopping criteria for PCG iterations are reached when the relative error of the residual in the preconditioned norm is less than 10^{-8}.

The matrix A for the lowest order weak Galerkin discretization, i.e., $P_0 - P_0$ element, is assembled and the matrix $\mathcal{A}$ for the auxiliary problem using $P1$ element is obtained through the triple product $\mathcal{A} = \Pi^t A \Pi$ where $\Pi : \mathcal{V}_h \to V_h$ is the simple average operator. By doing so, there is no need to repeat the assembling procedure to get $\mathcal{A}$ and the implementation is more algebraic. After that, the matrices in coarse levels are obtained by the triple product using the standard prolongation and restriction operators of linear elements on hierarchical meshes.

We report results for the original system and the reduced system, respectively. Note that after solving the reduced system, we can recover the interior part by a local solver with negligible time. Numerical solutions obtained by solving the original system or the reduced system are close within the stopping tolerance. Since the main purpose of these numerical results is to examine the efficiency of the auxiliary space preconditioner instead of testing the accuracy of the weak Galerkin approximation, in the report we omit the approximation error part. Because of this, there is no need to list the exact solution for each test problem.

Example 1. We first consider the Poisson equation defined on a circular mesh of the unit disk. The coarsest mesh is shown in Fig. 5.1(a). A sequence of meshes are obtained by several uniformly regular refinements, i.e., a triangle is divided into four congruent four triangles by connecting middle points of edges, of the coarsest mesh. Results are summarized in Table 5.1.

Example 2. Next, we consider a variable coefficient problem with an oscillating coefficient:

$$-\nabla \cdot (2(2 + \sin(10\pi x)\sin(10\pi y))\nabla u) = f$$

on $[0, 1] \times [0, 1]$. The coarsest mesh has size $h = 1/4$ and is shown in Fig. 5.1 (b). Fourth order quadrature is used when assembling the stiffness matrix. Results are summarized in Table 5.2.

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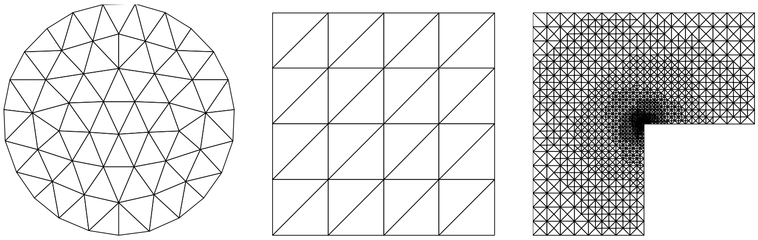


(a) Initial grid of Example 1. (b) Initial grid of Example 2. (c) An adaptive grid of Example 3.

Fig. 5.1. Meshes in Examples 1–3.

Table 5.1
PCG iteration steps and CPU time (in seconds) for Example 1. The left table is for the original system and the right table is for the reduced system.

Dof	Steps	Time	Dof	Steps	Time
3,446	13	0.052	2,086	8	0.02
13,692	13	0.11	8,252	8	0.053
54,584	13	0.41	32,824	8	0.17
217,968	13	1.8	130,928	8	0.66
871,136	13	8	522,976	8	2.9

Table 5.2
PCG iteration steps and CPU time (in seconds) for Example 2. The left table is for the original system and the right table is for the reduced system.

Dof	Steps	Time	Dof	Steps	Time
1,312	13	0.017	800	9	0.013
5,184	13	0.048	3,136	9	0.036
20,608	14	0.17	12,416	10	0.082
82,176	14	0.63	49,408	9	0.27
328,192	14	2.7	197,120	9	1.1

Example 3. We consider a test problem on an L-shaped domain obtained by subtracting $[0, 1] \times [-1, 0]$ from $(-1, 1) \times (-1, 1)$. The Poisson equation on such a domain has $H^{3/2}$-regularity. Adaptive finite element method based on *a posteriori* error estimator constructed in [4] is used. A sample adaptive mesh obtained by bisection refinement is shown in Fig. 5.1 (c). For bisection grids, we apply the coarsening algorithm developed in [23] to obtain a hierarchy of meshes. In Table 5.3, only results on some selected adaptive meshes are reported since the full list of adaptive meshes is long and the performance remains similar for all of these adaptive meshes.

Example 4. We consider the Poisson equation defined on the cube $\Omega = (-1, 1)^3$. The coarsest mesh is shown in Fig. 5.2(a). A sequence of meshes are obtained by several uniformly regular refinements, i.e., a tetrahedron is divided into 8 small tetrahedron by connecting middle points of edges, of the coarsest mesh. Results are summarized in Table 5.4.

Example 5. We consider the elliptic equation with jump coefficients [24,25]. Let $\Omega = (-1, 1)^3$ and the diffusion coefficient $a(x)I$ be defined such that $a(x)$ is equal to the constants $a_1 = a_2 = 1$ and $a_3 = \varepsilon$ on the three regions Ω_1, Ω_2 and Ω_3 respectively (see Fig. 5.2(b)), where

$$\Omega_1 = (-0.5, 0)^3, \Omega_2 = (0, 0.5)^3 \quad \text{and} \quad \Omega_3 = \Omega \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2).$$

A sequence of meshes are obtained by several uniformly regular refinements of the coarsest mesh.

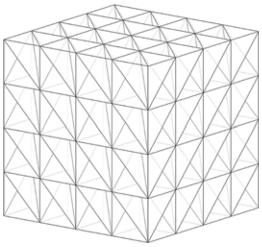
We choose $f = 1$ and impose the following boundary conditions: Dirichlet conditions

$$u_{\{-1\}\times[-1,1]\times[-1,1]} = 0, \qquad u_{\{1\}\times[-1,1]\times[-1,1]} = 1,$$

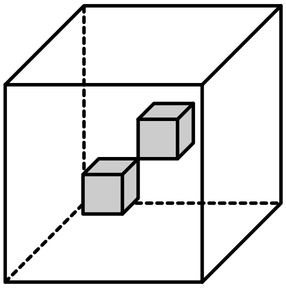
and homogeneous Neumann boundary conditions on the remaining boundary. We test the robustness of our solver as the coefficient ε changes. Only the reduced system is solved in this example. Results are summarized in Table 5.5.

From these experiments we may draw the following conclusions:

1. In all examples, the auxiliary space preconditioner works well for the linear system arising from discretization of the lowest order weak Galerkin method. The fluctuation of iteration steps of the PCG method applied to systems with different sizes is small which implies the condition number of the preconditioned system is uniformly bounded.



(a) Initial grid of Example 4.



(b) Domain of Example 5.

Fig. 5.2. The coefficients $a_1 = a_2 = 1$ in the gray domains Ω_1 and Ω_2, and $a_3 = \varepsilon$ in the rest of the domain.

Table 5.3
PCG iteration steps and CPU time (in seconds) for Example 3. The left table is for the original system and the right table is for the reduced system.

Dof	Steps	Time	Dof	Steps	Time
256	13	0.0099	160	8	0.0088
574	13	0.023	352	9	0.014
1,091	13	0.035	663	10	0.023
2,177	13	0.058	1317	10	0.036
4,398	13	0.11	2656	10	0.067
8,642	13	0.16	5206	9	0.091
10,742	13	0.2	6470	8	0.094

Table 5.4
PCG iteration steps and CPU time (in seconds) for Example 4. The left table is for the original system and the right table is for the reduced system.

Dof	Steps	Time	Dof	Steps	Time
1,248	16	0.018	864	11	0.058
9,600	18	0.1	6,528	12	0.054
75,264	18	1	50,688	13	0.41
595,968	19	10	399,360	13	4
4,743,168	19	100	3,170,304	13	36

Table 5.5
PCG iteration steps for Example 5. Only results for solving the reduced system are presented.

Dof	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-2}$	$\varepsilon = 1$	$\varepsilon = 10^2$	$\varepsilon = 10^4$
864	36	22	13	13	13
6,528	33	21	13	13	13
50,688	32	21	13	13	13
399,360	34	21	13	13	13
3,170,304	34	21	13	13	13

2. The solver for the reduced system is more efficient than the original system. The size of the reduced system is around two thirds of the original one and the time for solving the reduced system is around half of the original one. This shows the efficiency gained by working on the reduced system.
3. Although our theory is developed for quasi-uniform meshes, the third example indicates that our solver works well for adaptive grids and elliptic equations with less regularity.

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