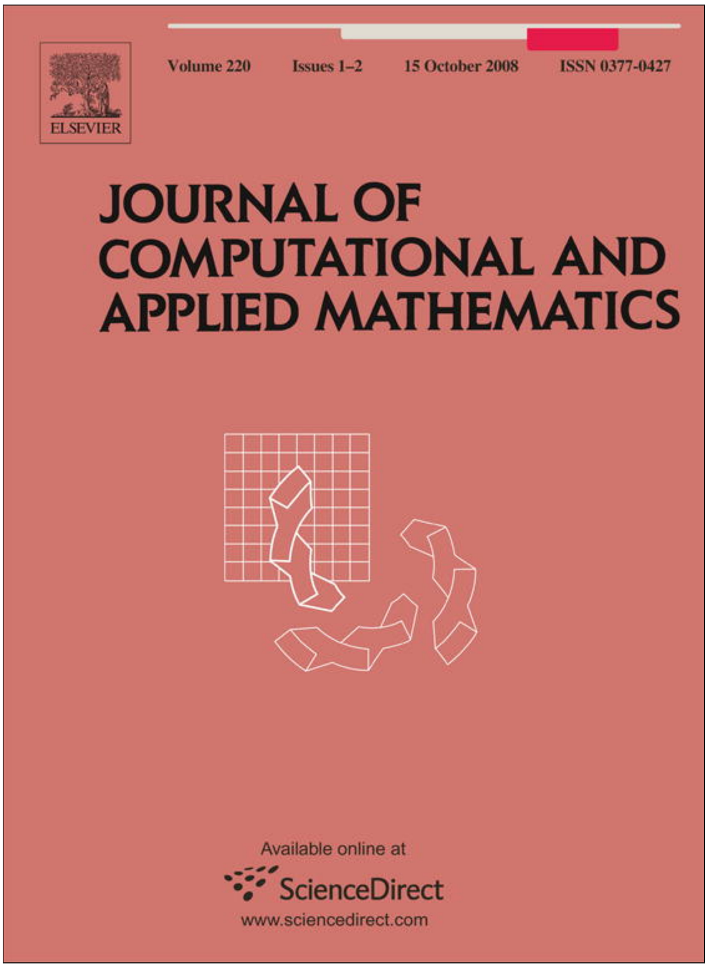
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Stability of a streamline diffusion finite element method for turning point problems

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Abstract

A one-dimensional singularly perturbed problem with a boundary turning point is considered in this paper. Let V^h be the linear finite element space on a suitable grid $\mathcal{T}_h$. A variant of streamline diffusion finite element method is proved to be almost uniform stable in the sense that the numerical approximation u_h satisfies $\|u - u_h\|_\infty \leqslant C|\ln \varepsilon| \inf_{v_h \in V^h} \|u - v_h\|_\infty$, where C is independent with the small diffusion coefficient ε and the mesh $\mathcal{T}_h$. Such stability result is applied to layer-adapted grids to obtain almost ε-uniform second order scheme for turning point problems.

MSC: 65N12; 65N30

Keywords: Streamline diffusion finite element method; Singularly perturbed problem; Boundary turning point

1. Introduction

In this paper, we consider a streamline diffusion finite element method for a class of the following one-dimensional singularly perturbed problem with a boundary turning point

$$-\varepsilon u'' - b(x)u' = f(x), \quad x \in (0, 1), \tag{1}$$
$$u(0) = u(1) = 0, \tag{2}$$

where the coefficient ε satisfying $0 < \varepsilon \ll 1$, and $b(0) = 0$, $b(x) > 0$, $x \in (0, 1]$. For the simplicity of presentation, we mainly discuss the case $b(x) = x^p$, $p > 0$. Our analysis can be adapted to more general case.

Although (1)–(2) is a typical elliptic partial differential equation (PDE), it is well known that the solution contains a boundary layer near the boundary point $x = 0$. The standard finite element methods (FEMs) designed for elliptic PDEs will have nonphysical oscillations [19] and the classical analysis of FEM for general elliptic equation fails because of the weak coercivity. See Section 2 for detailed explanation.

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The main result of this paper is to establish a uniform stability and optimality results for a class of one-dimensional singularly perturbed problems with a boundary turning point. We shall use a refined estimate of discrete Green functions and consistency error to prove the following stability result on a general class of grids:

$$\|u - u_h\|_\infty \leqslant C|\ln \varepsilon| \inf_{v \in V^h} \|u - v_h\|_\infty. \tag{3}$$

Since the logarithmical growth of ε is slow, we can expect almost ε-uniform second-order schemes if the grids is adapted correctly. Using the regularity result on turning point problems [13,25] and the nonlinear approximation theory [8,7,6], we can obtain almost second-order schemes for singularly perturbed problem with turning point which is not easy using traditional finite difference methods; see [14,15] for an almost first-order ε-uniform scheme.

2. Preliminaries

In this section, we shall review main ingredients in the finite element analysis and motivate our current work.

Let $\|\cdot\|$ denote the $L^2(\Omega)$-norm, $\|\cdot\|_2$ denote the norm of the standard Sobolev space $H^2(\Omega)$, and $\|u\|_a$ denote the energy norm defined by the weak form of the elliptic PDE operator. Given a grid $\mathscr{T}_h$ of (0, 1), let V^h denote the linear finite element space, u_h the finite element approximation of u and u_I the nodal interpolation of u. There are three main ingredients for the error analysis of FEM.

2.1. Stability and optimality

The elliptic and coercivity of the weak form ensures the following stability result:

$$\|u - u_h\|_a \leqslant C_1 \inf_{v_h \in V^h} \|u - v_h\|_a. \tag{4}$$

In particular, $\|u - u_h\|_a \leqslant C_1\|u - u_I\|_a$.

2.2. Approximability

Interpolation error estimates for u_I on quasi-uniform triangulations $\mathscr{T}_h$:

$$\|u - u_I\|_a \leqslant C_2 h\|u\|_2 \quad \forall u \in H^2(\Omega). \tag{5}$$

2.3. Regularity

Regularity result of elliptic operators on a smooth domain:

$$\|u\|_2 \leqslant C_3\|f\|. \tag{6}$$

Combining these three inequalities together, on quasi-uniform triangulations $\mathscr{T}_h$ of a nice domain Ω, one can obtain a first-order error estimate of the energy norm:

$$\|u - u_h\|_a \leqslant C_1\|u - u_I\|_a \leqslant C_1C_2h\|u\|_2 \leqslant C_1C_2C_3h\|f\|.$$

Furthermore, using duality arguments, one can obtain the second-order error estimate in L^2 norm or L^∞ norm with more refined analysis.

The failure of the direct application of the above analysis to singularly perturbed problem is due to the non ε-uniformality. Although all constants in the inequalities (4) and (6) are independent with the mesh size, for singularly perturbed problems, $C_i = \mathrm{O}(\varepsilon^{-\alpha})$, $i = 1, 3$ for some $\alpha > 0$. And in (5) the norm $\|u\|_2$ is proportion by inversion to ε. Therefore when ε is small, the constants in front of the convergent rate are large enough to leave a room for the oscillation.

Several efforts are made to get ε-uniform counter parts of those three ingredients. For the stability issue, to improve the degenerate coercivity of the bilinear form as $\varepsilon \to 0$, Bertoluzza et al. [2,3] considered the negative norm for the stabilized convection–diffusion operator. Sangalli [21,22] use the interpolation theory of function spaces to introduce a weaker norm. In this approach, the improved coercivity is mainly for the continuous operator. Another approach is to use mesh dependent norms. For example, Brezzi et al. [4,5] consider mesh dependent norm for residual free method. Zhang [26,27] considers the superconvergence approximation in a discrete energy norm for standard finite element method on Shishkin grid. Those L^2-type norm is a little bit weak such that the oscillation is still possible. For L^∞ norm, on quasi-uniform grid, Schatz and Wahlbin [23] obtain the error estimate for ordinary Galerkin finite element.

Recently, Chen and Xu [6,7] have developed a variant of streamline diffusion finite element method (SDFEM) and proved that, when $b(x) \geqslant b_0 > 0$, the numerical solution u_h on arbitrary grid have the following ε-uniform quasi-optimal stability result:

$$\|u - u_h\|_\infty \leqslant \frac{C}{b_0} \inf_{v_h \in V_h} \|u - v_h\|_\infty, \tag{7}$$

where C independent of the mesh size and ε. Previous efforts for such uniform stability result can be found at [20,21]. The analysis is mainly based on the $(L^\infty, W^{-1,\infty})$ ε-uniform stability for the continuous problem which is firstly given by Kopteva in [12].

The separation of the stability, approximability and regularity will simplify the error analysis. Comparing to traditional approach of finite difference methods, application of this uniform stability can obtain error estimate for a large class of layer adapted grids with *a priori* or *a posteriori* information on the second derivatives [6,7]. It can be also applied to other problems, for example, the analysis of the multigrid-like solver for convection-dominated problems in the maximum norm [17,18].

The approximability of a function on arbitrary grids is well studied in the approximation theory since 1970s. In one dimension, it is called free knots approximation problem [8,9,16,10]. It turns out that the right function spaces for such problem is Besov space, in which a fractional L^p, $0 < p < 1$ metric is used. For example, de Boor [8,9] shows that if u'' is monotone and the grid equidistributes $|u''|^{1/2}$, then

$$\|u - u_I\|_\infty \leqslant CN^{-2}\|u''\|_{1/2}, \tag{8}$$

where $\|u\|_{1/2} = (\int |u''|^{1/2}\,\mathrm{d}x)^2$. To see this is indeed ε-uniform, we note that usually $u'' \approx \varepsilon^{-2}$ in an $\mathrm{O}(\varepsilon)$ region and $u'' \approx 1$ in the rest region. Thus $\|u\|_{1/2}$ is ε-uniform bounded, while the standard Sobolev norm $\|u''\| \simeq \varepsilon^{-3/2}$ is not.

The regularity result for singularly perturbed problems is also well studied in the literature. Kellogg et al. [11] gives a pointwise estimate the derivative of u for singularly perturbed problems when $b \geqslant b_0 > 0$. A special case for the second derivative is

$$|u''(x)| \leqslant C(1 + \varepsilon^{-2}\mathrm{e}^{-b_0 x/\varepsilon}). \tag{9}$$

This estimate is used in [7,6] with (7) and (8) to prove the second-order convergence of SDFEM.

It is clear that (7) cannot be applied directly to the boundary turning point case since $b_0 = 0$. We remark that the approximability of finite element spaces depends on the choice the grid but not PDEs. Pointwise regularity result for turning point problem similar to (9) is also available in the literature [13,25]. Therefore it is crucial to get an ε-uniform stability and optimality result for SDFEM.

3. Stability of SDFEM

In this section, we shall introduce streamline diffusion finite element methods and present our main result. To make the presentation more clear, proofs for several technique lemmas are left to the last section.

3.1. Problem setting

Let $\Omega = (0, 1)$. We shall use the following Hilbert spaces

$$H_0^1(\Omega) := \{v \in L^2(\Omega) : v' \in L^2(\Omega) \text{ and } v(0) = v(1) = 0\}.$$

We say that u is a weak solution to (1)–(2), if $u \in H_0^1(\Omega)$ satisfying

$$\tilde{a}(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega),$$

where the bilinear form

$$\tilde{a}(u, v) = \varepsilon(u', v') - (bu', v),$$

and $(\cdot, \cdot)$ denotes the L^2 inner produce. From the theory of elliptic partial differential equations, (1)–(2) admits a unique weak solution $u \in H_0^1(\Omega)$.

Let N be an integer and let $\mathscr{T}_N = \{x_i : 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1\}$ be a grid of [0, 1] with N interior nodes (unknowns). To be consistent with the traditional FEM analysis, we set $h = 1/N$, and also denote $\mathscr{T}_N$ by $\mathscr{T}_h$. Let $\tau_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$. Denote by φ_i the continuous piecewise linear basis function at the vertex x_i. The finite element space V^h is defined as $V^h := \{v_h : v_h = \sum_{i=1}^N a_i \varphi_i\}$. Obviously V^h is a finite dimensional subspace of $H_0^1(\Omega)$. The standard finite element discretization is to find $u_h \in V^h$ such that

$$\tilde{a}(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^h.$$

We now define a grid-dependent bilinear form by

$$a(u, v) = \tilde{a}(u, v) - \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \delta_i (-\varepsilon u'' - bu')(bv'),$$

where δ_i is a stabilization function on τ_i defined as

$$\delta_i = \begin{cases} \left(\dfrac{h_i}{\varepsilon}\right) \varphi_i \varphi_{i-1} h_i & \text{if } \varepsilon > \int_{x_{i-1}}^{x_i} b\varphi_i, \\ \left(\dfrac{\int_{x_{i-1}}^{x_i} b\varphi_i}{\int_{x_{i-1}}^{x_i} b^2 \varphi_i \varphi_{i-1}}\right) \varphi_i \varphi_{i-1} h_i & \text{if } \varepsilon \leqslant \int_{x_{i-1}}^{x_i} b\varphi_i. \end{cases} \tag{10}$$

Let

$$f(v) = (f, v) - \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \delta_i f b v'.$$

Then the SDFEM discretization is to find $u_h \in V^h$, such that

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in V^h. \tag{11}$$

Given a grid $\mathscr{T}_h$, denote by u_I the continuous piecewise linear interpolation of u. Set $e(x) = (u_I - u_h)(x) = \sum_{i=1}^N e_i \varphi_i$, where $e_i = e(x_i)$, $i = 1, 2, \ldots, N$. Noting $a(u - u_h, v_h) = 0$ for any $v_h \in V^h$, we then have the error equation

$$a(e, \varphi_i) = a(u_I - u, \varphi_i), \quad i = 1, 2, \ldots, N. \tag{12}$$

We shall analyze the left- and right-hand side of the error equation (12) to obtain our stability result.

3.2. Main results and outline of proof

We shall obtain an almost ε-uniform stability and optimality result based on refined analysis of discrete Green functions and the consistency error. We first define the discrete Green function.

Definition 3.1. $G^i \in V^h$ $(i = 1, \ldots, N)$ is called the discrete Green function at the vertex x_i, if

$$a(\varphi_j, G^i) = \delta_j^i \quad \forall \varphi_j, \quad 1 \leqslant j \leqslant N, \tag{13}$$

where δ_j^i is Kronecker symbol satisfying $\delta_j^i = 1$ when $i = j$ and 0 otherwise.

We also define the residual

$$r_i := a(u_I - u, \varphi_i).$$

For any given grid $\mathcal{T}_h$ of (0, 1), we now define a sub-grid of $\mathcal{T}_h$

$$\{x_{i_k} : 1 \leqslant k \leqslant M+1, \text{ and } 1 = i_1 < \cdots < i_M = N < i_{M+1} = N+1\},$$

and I_k, $k = 1, \ldots, M+1$ by the following algorithm.

Algorithm 1. $i_1 = 1$, $I_1 = 1$.

For $k = 1, \ldots$
  if $i_k = N$, define $M = k$; break.
  else if $2x_{i_k} \geqslant x_N$, define $M = k+1$, $i_M = N$ and $I_M = 0$; break.
  else if $2x_{i_k} \in [x_j, x_{j+1})$, define $i_{k+1} = j$.
    if $i_{k+1} = i_k$, define $I_{k+1} = 1$, $i_{k+1} = i_k + 1$.
    else define $I_{k+1} = 0$.
    end if
  end if
End for
$i_{M+1} = N+1$, $I_{M+1} = 1$.

Then we get that if $I_k = 0$, $x_{i_k}/x_{i_{k-1}} \leqslant 2$; and if $I_k = 1$, $i_k - i_{k-1} = 1$. It is also obvious that if $k < M$, $x_{i_k}/x_{i_{k-2}} \geqslant 2$, then we have $M \leqslant C|\ln x_1|$.

In [6], the global lower bound $b(x) \geqslant b_0$ is used in the estimate of the discrete Green function. We shall use this sub-grid to control the variation of the coefficient b and give estimate on the discrete Green function in each interval of this sub-grid. Roughly speaking when the interval is away from the turning point, we have a positive lower bound of b and previous analysis in [6] works. When it is close, we shall show the variation of b is bounded. See Section 5 for details.

Theorem 3.2. *The numerical approximation u_h of SDFEM with δ_i defined by* (10) *is almost ε-uniform optimal in the sense that*

$$\|u - u_h\|_\infty \leqslant CM \inf_{v \in V^h} \|u - v_h\|_\infty.$$

Furthermore, if $x_1 > \varepsilon^\alpha$ for some $\alpha > 1$, we then have

$$\|u - u_h\|_\infty \leqslant C|\ln \varepsilon| \inf_{v \in V^h} \|u - v_h\|_\infty.$$

Proof. We shall outline the proof here and prove several estimates in the last section. Let $G^i = \sum_{j=1}^N G^i_j \varphi_j$. By the definition of the discrete Green functions and the residual,

$$e(x_i) = a(e, G^i) = a\left(e, \sum_{j=1}^N G^i_j \varphi_j\right) = \sum_{j=1}^N G^i_j r_j = \sum_{k=2}^M \sum_{j=i_{k-1}}^{i_k - 1} G^i_j r_j.$$

We shall use the estimate of G^i_j and r_j in the last section (Theorem 5.5) to show that

$$\left|\sum_{j=i_{k-1}}^{i_k-1} G^i_j r_j\right| \leqslant C\|u - u_I\|_\infty. \tag{14}$$

Therefore

$$\begin{aligned}|(u_I - u_h)(x_i)| = |e(x_i)| &\leqslant \sum_{k=2}^{M+1} \left|\sum_{j=i_{k-1}}^{i_k-1} G^i_j r_j\right| \\ &\leqslant C \sum_{k=2}^{M+1} \|u - u_I\|_\infty \leqslant CM\|u - u_I\|_\infty.\end{aligned}$$

By the triangle inequality

$$\|u-u_h\|_\infty \leqslant \|u-u_I\|_\infty + \|u_I-u_h\|_\infty \leqslant CM\|u-u_I\|_\infty.$$

The result then follows by noting that the interpolation operator is stable in L^∞ norm. □

Remark 3.3. For some special cases, we can get better estimate on M. For example, $M \leqslant C \ln N$ for piecewise uniform grids (Shishkin grids).

4. Convergence analysis

In this section, we shall prove the convergence of the SDFEM on two types of layer adapted grids, Shishkin-type grid [24] and Bakhvalov-type grid [1]. Let us consider the following particular singularly perturbed problem with a boundary turning point:

$$-\varepsilon u'' - x^p u' = x^p g(x), \tag{15}$$
$$u(0) = u(1) = 0, \tag{16}$$

where $g(x) \in C^1([0,1])$. We have the following estimate of the second derivatives of u [14],

$$|u''(x)| \leqslant C\left(1+\varepsilon^{-2/(p+1)}\exp\left(-\frac{x^{p+1}}{\varepsilon(p+1)}\right)\right). \tag{17}$$

Let us consider layer-adapted grids to obtain almost second-order schemes. First we construct a Shishkin-type grid [24]. Let $N+1$ be an even integer and the transition point

$$\theta = \min\{\tfrac{1}{2}, (2\varepsilon(p+1)\ln N)^{1/(p+1)}\}.$$

In practice, ε is so small that $\theta = (2\varepsilon(p+1)\ln N)^{1/(p+1)}$. Then $[0,\theta]$ and $[\theta,1]$ are divided into $(N+1)/2$ subintervals. Let

$$h_i = \begin{cases} \dfrac{2\theta}{N+1}, & 1 \leqslant i \leqslant (N+1)/2, \\ \dfrac{2(1-\theta)}{N+1}, & (N+1)/2 < i \leqslant N+1. \end{cases} \tag{18}$$

Lemma 4.1. *Let u be the solution to* (15)–(16). *For Shishkin grid*,

$$\|u-u_I\|_{L^\infty(I_i)} \leqslant \begin{cases} CN^{-2}(\ln N)^{2/(p+1)}, & 1 \leqslant i \leqslant (N+1)/2, \\ CN^{-2}, & (N+1)/2+1 < i \leqslant N+1. \end{cases}$$

Proof. Since $1+\varepsilon^{-2/(p+1)}\exp(-x^{p+1}/\varepsilon(p+1))$ is monotone in τ_i, we get that [10,7],

$$\|u-u_I\|_{L^\infty(I_i)} \leqslant C\left(\int_{x_{i-1}}^{x_i}\left(1+\varepsilon^{-2/(p+1)}\exp\left(-\frac{x^{p+1}}{\varepsilon(p+1)}\right)\right)^{1/2}\right)^2.$$

When $i \leqslant (N+1)/2$,

$$\|u-u_I\|_{L^\infty(I_i)} \leqslant Ch_i^2\varepsilon^{-2/(p+1)} \leqslant CN^{-2}(\ln N)^{2/(p+1)}.$$

When $i > (N+1)/2$,

$$\begin{aligned}\|u-u_I\|_{L^\infty(I_i)} &\leqslant CN^{-2} + C\left(\int_\theta^{\theta+h_{(N+3)/2}} \left(\varepsilon^{-2/(p+1)} \exp\left(-\frac{x^{p+1}}{\varepsilon(p+1)}\right)\right)^{1/2}\right)^2 \\ &\leqslant CN^{-2} + C\left(\varepsilon^{-1/(p+1)} \int_0^{h_{(N+3)/2}} \exp\left(-\frac{\theta^{p+1}+(p+1)\theta^p x}{2\varepsilon(p+1)}\right)\right)^2 \\ &\leqslant CN^{-2} + C\left(\varepsilon^{-1/(p+1)} N^{-1} \int_0^{h_{(N+3)/2}} \exp\left(-\frac{\theta^p x}{2\varepsilon}\right)\right)^2 \\ &\leqslant CN^{-2}\left(1+\varepsilon^{-1/(p+1)}\frac{2\varepsilon}{\theta^p}\right)^2 \\ &\leqslant CN^{-2}.\end{aligned}$$

Then we complete the proof. □

Lemma 4.2. *Let u be the solution to* (15)–(16), *u_h is the SDFEM approximation on Shishkin grid* (18) *with δ_i defined by* (10). *Then*

$$\|u-u_h\|_\infty \leqslant CN^{-2}(\ln N)^{(p+3)/(p+1)}.$$

Proof. Since Shishkin grids are piecewise uniform, we get that $M \leqslant C \ln N$. Then the conclusion follows directly by Lemma 4.1 and Theorem 3.2. □

Now we present the numerical experiment to support our theoretic result. Let us consider the following boundary value problem,

$$-\varepsilon u'' - xu' = -2\varepsilon x - x^3, \quad x \in (0,1), \tag{19}$$

$$u(0) = u(1) = 0. \tag{20}$$

The true solution to (19)–(20) is

$$u(x) = \frac{1}{3}\left[x^3 - \frac{\int_0^x \exp(-t^2/2\varepsilon)\,\mathrm{d}t}{\int_0^1 \exp(-t^2/2\varepsilon)\,\mathrm{d}t}\right],$$

which has a boundary layer near the boundary turning point $x = 0$. Now we consider the SDFEM approximation on the Shishkin grid defined by (18).

Table 1 is on the maximum error of the true solution and SDFEM solution on Shishkin grid to the problem (19)–(20), where the values of rate denote the error on N-node grid divided by the error on $2N+1$-node grid. The numerical experiment supports our theoretic result, and indeed indicates a better convergence speed than the error estimate in Lemma 4.2.

Table 1
The maximum error on Shishkin grid with different grid and ε

N	$\varepsilon = 1\mathrm{E}-6$	Rate	$\varepsilon = 1\mathrm{E}-8$	Rate	$\varepsilon = 1\mathrm{E}-10$	Rate	$\frac{(2N+1)^2\ln^2 N}{N^2\ln^2(2N+1)}$
31	5.0671E − 4	3.06	4.9955E − 4	3.05	4.9871E − 4	3.05	2.84
63	1.5512E − 4	3.55	1.5250E − 4	3.56	1.5214E − 4	3.57	2.97
127	4.3734E − 5	3.69	4.2793E − 5	3.70	4.2626E − 5	3.71	3.08
255	1.1845E − 5	3.79	1.1558E − 5	3.77	1.1479E − 5	3.79	3.17
511	3.1284E − 6	3.86	3.0633E − 6	3.81	3.0288E − 6	3.83	3.25
1023	8.1124E − 7	3.92	8.0434E − 7	3.83	7.9064E − 7	3.85	3.31
2047	2.0707E − 7	3.99	2.0995E − 7	3.85	2.0527E − 7	3.86	3.36
4095	5.1909E − 8		5.4494E − 8		5.3155E − 8		

Let us take Bakhvalov-type [1] grid as another example. Define the transition point $\theta = (2\varepsilon|\ln\varepsilon|)^{1/(p+1)}$. In $[0,\theta]$, we put $(N+1)/2$ elements such that

$$\int_{x_{i-1}}^{x_i} \varepsilon^{-1/(p+1)} \exp\left(-\frac{x^{p+1}}{2\varepsilon(p+1)}\right) \leqslant \frac{2C_0}{N+1}\int_0^\theta \varepsilon^{-1/(p+1)} \exp\left(-\frac{x^{p+1}}{2\varepsilon(p+1)}\right), \tag{21}$$

where $C_0 \geqslant 1$ is a given positive constant. In $[\theta, 1]$, we put $(N+1)/2$ uniform grid.

Lemma 4.3. *Let u be the solution to* (15)–(16). *For Bakhvalov grid*,

$$\|u - u_I\|_\infty \leqslant CN^{-2}.$$

Proof. Similar as the proof of Lemma 4.1, we have

$$\|u-u_I\|_{L^\infty(I_i)} \leqslant C\left(\int_{x_{i-1}}^{x_i}\left(1+\varepsilon^{-2/(p+1)}\exp\left(-\frac{x^{p+1}}{\varepsilon(p+1)}\right)\right)^{1/2}\right)^2.$$

Note that $\varepsilon^{-2/(p+1)}\exp(-x^{p+1}/\varepsilon(p+1)) \geqslant 1$ on $[0,\theta]$, and $\int_0^{+\infty} \varepsilon^{-1/(p+1)}\exp(-x^{(p+1)}/2\varepsilon(p+1)) \leqslant C$, we get by (21),

$$\|u-u_I\|_{L^\infty(I_i)} \leqslant CN^{-2}, \quad 1 \leqslant i \leqslant (N+1)/2.$$

Since $\varepsilon^{-2/(p+1)}\exp(-x^{p+1}/\varepsilon(p+1)) \leqslant 1$ on $[\theta, 1]$, we get that

$$\|u-u_I\|_{L^\infty(I_i)} \leqslant CN^{-2}, \quad (N+1)/2 < i \leqslant N+1. \qquad \square$$

Lemma 4.4. *Let u be the solution to* (15)–(16), *u_h be the SDFEM approximation on Bakbvalov grid with δ_i defined by* (10). *We have*

$$\|u-u_h\|_{L^\infty} \leqslant C(|\ln\varepsilon| + \ln N)N^{-2}.$$

Proof. By (21), we get that $x_1 \geqslant C\varepsilon^{1/(p+1)}\theta/N$. The number of sub-grid nodes in $[0,\theta]$ (denoted by M_1) satisfies

$$2^{M_1} \leqslant C\frac{\theta}{x_1} \leqslant C(|\ln\varepsilon| + \ln N).$$

Noting that the uniform grid in $[\theta, 1]$ with $(N+1)/2$ nodes, we have $M \leqslant C(\ln\varepsilon + \ln N)$. $\square$

Table 2 is on the maximum error of the true solution and SDFEM solution on Bakhvalov grid to the problem (19)–(20), where the values of rate denote the error on N-node grid divided by the error on $2N+1$-node grid. The numerical experiment supports our theoretic result.

Table 2
The maximum error on Bakhvalov grid with different grid and ε

N	$\varepsilon = 1E-8$	Rate	$\varepsilon = 1E-10$	Rate	$\varepsilon = 1E-12$	Rate	$\frac{(2N+1)^2\ln N}{N^2\ln(2N+1)}$
31	9.2668E − 4	3.67	1.0321E − 3	3.65	1.1216E − 3	3.64	3.42
63	2.5280E − 4	3.69	2.8245E − 4	3.68	3.0797E − 4	3.67	3.48
127	6.8523E − 5	3.73	7.6792E − 5	3.71	8.3853E − 5	3.71	3.52
255	1.8384E − 5	3.78	2.0687E − 5	3.74	2.2601E − 5	3.75	3.57
511	4.8601E − 6	3.85	5.5246E − 6	3.78	6.0330E − 6	3.78	3.61
1023	1.2602E − 6	3.94	1.4630E − 6	3.81	1.5972E − 6	3.80	3.64
2047	3.2018E − 7	3.99	3.8387E − 7	3.86	4.2007E − 7	3.82	3.67
4095	8.0220E − 8		9.9566E − 8		1.0987E − 7		

5. Estimate of discrete Green functions and the residual

Let us denote $u_i = u_h(x_i)$, then $u_h(x) = \sum_{i=1}^{N} u_i \varphi_i(x)$. Let $v_h = \varphi_i$, $i = 1, 2, \ldots, N$ in (11), we get the system of equations

$$Au_h = f_h,$$

where $A = (a_{ij})$ is a tri-diagonal matrix with $a_{ij} = a(\varphi_j, \varphi_i)$, $u_h = (u_1, u_2, \ldots, u_N)^{\mathrm{T}}$ and $f_h = (f(\varphi_1), f(\varphi_1), \ldots, f(\varphi_N))^{\mathrm{T}}$. Here with a slightly abuse of the notation, we identify a function in the finite element space with a vector in $\mathbb{R}^N$.

Direct calculation gives us that

$$\begin{aligned}
a_{i,i-1} &= -\frac{\varepsilon}{h_i} + \frac{\int_{x_{i-1}}^{x_i} b\varphi_i}{h_i} - \frac{\int_{x_{i-1}}^{x_i} \delta_i b^2}{h_i^2}, \\
a_{i,i} &= \frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}} - \frac{\int_{x_{i-1}}^{x_i} b\varphi_i}{h_i} + \frac{\int_{x_i}^{x_{i+1}} b\varphi_i}{h_{i+1}} + \frac{\int_{x_{i-1}}^{x_i} \delta_i b^2}{h_i^2} + \frac{\int_{x_i}^{x_{i+1}} \delta_{i+1} b^2}{h_{i+1}^2}, \\
a_{i,i+1} &= -\frac{\varepsilon}{h_{i+1}} - \frac{\int_{x_i}^{x_{i+1}} b\varphi_i}{h_{i+1}} - \frac{\int_{x_i}^{x_{i+1}} \delta_{i+1} b^2}{h_{i+1}^2},
\end{aligned}$$

with standard modifications for $i = 1$ and N. It is easy to see that if δ_i is determined by (10), A is an M-matrix.

The following lemma shows the basic properties of the discrete Green function for the SDFEM.

Lemma 5.1. *Let $G^i = \sum_{j=1}^{N} G_j^i \varphi_j$. Then G_j^i satisfy*

(1) $0 \leqslant G_1^i < \cdots < G_i^i > G_{i+1}^i > \cdots > G_N^i \geqslant 0$, *and*
(2) $G_j^i \leqslant \frac{1}{\bar{b}_{j+1}} \leqslant C x_{j+1}^{-p}$, *where* $\bar{b}_{j+1} = \frac{\int_{x_j}^{x_{j+1}} b(x)}{h_{j+1}}$.

Proof. Since A is M-matrix, we immediately know $G_j^i \geqslant 0$.

We first prove $G_{j-1}^i < G_j^i$ for $j < i$ by induction. First because $a_{1,1} G_1^i + a_{2,1} G_2^i = 0$, we get $G_1^i < G_2^i$. Suppose $G_{k-1}^i < G_k^i$ holds. Noting that $a_{k-1,k} + a_{k,k} + a_{k+1,k} = \int_{x_k}^{x_{k+1}} b/h_{k+1} - \int_{x_{k-1}}^{x_k} b/h_k \geqslant 0$, we have

$$\begin{aligned}
0 &= a_{k-1,k} G_{k-1}^i + a_{k,k} G_k^i + a_{k+1,k} G_{k+1}^i \\
&> a_{k-1,k} G_k^i + a_{k,k} G_k^i + a_{k+1,k} G_{k+1}^i \\
&\geqslant a_{k+1,k} (G_{k+1}^i - G_k^i).
\end{aligned}$$

Since $a_{k+1,k} < 0$, we conclude $G_k^i < G_{k+1}^i$. Similarly, we can prove that $G_j^i < G_{j-1}^i$ when $j > i$.

It is left to prove (2). By the definition

$$a(v_h, G^i) = v_h(x_i), \quad \forall v_h \in V^h. \tag{22}$$

We shall choose a special $v_h \in V^h$ to prove (2). When $j \geqslant i$, setting $v_h = \sum_{k=1}^{j} \varphi_k$ in (22), we get

$$\sum_{l=1}^{N} \sum_{k=1}^{j} a_{l,k} G_l^i = 1.$$

By the formula $a_{j,j-1} + a_{j,j} + a_{j,j+1} = 0$, we get

$$(a_{1,1} + a_{1,2}) G_1^i + a_{j+1,j} (G_{j+1}^i - G_j^i) + (a_{j,j-1} + a_{j,j} + a_{j+1,j}) G_j^i = 1. \tag{23}$$

The first two terms of (23) are positive. Note that

$$a_{j,j-1}+a_{j,j}+a_{j+1,j}=\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} b(x)=\bar{b}_{j+1}.$$

Then we can conclude that $G^i_j\leqslant \bar{b}^{-1}_{j+1}$ when $j\geqslant i$. (2) is also valid for $j<i$ by (1) and $G^i_i\leqslant 1/\bar{b}_{i+1}$. $\square$

We obtain the following formula for the residual $r_i=a(u_I-u,\varphi_i)$ by direct computation.

Lemma 5.2.

$$r_i=t_i-t_{i+1},\quad i=1,\ldots,N.$$

where

$$t_i=\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b(u_I-u)+\frac{\varepsilon}{h_i}\int_{x_{i-1}}^{x_i} b\delta_i(u_I-u)''+\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b^2\delta_i(u_I-u)'+\int_0^{x_i} b'(u_I-u)\sum_{k=1}^{i}\varphi_i. \tag{24}$$

The following technique lemma estimates the residual. It is the counterpart of the consistency error in the finite difference analysis.

Lemma 5.3.

$$|t_i|\leqslant Cx_i^p\|u-u_I\|_\infty.$$

Proof. Let us estimate the terms of t_i in (24) on by one. The estimate on the first and last term is given by the following two inequalities.

$$\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b(u_I-u)\right|\leqslant x_i^p\|u-u_I\|_\infty,$$

$$\left|\int_0^{x_i} b'(u-u_I)\sum_{k=1}^{i}\varphi_i\right|\leqslant\|u-u_I\|_\infty\int_0^{x_i} b'\leqslant x_i^p\|u-u_I\|_\infty.$$

Let us consider the second term of t_i in (24).

$$\left|\frac{\varepsilon}{h_i}\int_{x_{i-1}}^{x_i} b\delta_i(u_I-u)''\right|\leqslant\frac{\varepsilon}{h_i}\|u-u_I\|_\infty\left|\int_{x_{i-1}}^{x_i}(b\delta_i)''\right|. \tag{25}$$

Assume that $\delta_i=\kappa_i\varphi_i\varphi_{i-1}h_i$, where

$$\kappa_i=\begin{cases}\dfrac{h_i}{\varepsilon} & \text{if } \varepsilon>\int_{x_{i-1}}^{x_i} b\varphi_i,\\[2ex] \dfrac{\int_{x_{i-1}}^{x_i} b\varphi_i}{\int_{x_{i-1}}^{x_i} b^2\varphi_i\varphi_{i-1}} & \text{if } \varepsilon\leqslant\int_{x_{i-1}}^{x_i} b\varphi_i,\end{cases} \tag{26}$$

when $\varepsilon>\int_{x_{i-1}}^{x_i} b\varphi_i$, $\varepsilon\kappa_i=h_i$; when $\varepsilon\leqslant\int_{x_{i-1}}^{x_i} b\varphi_i$,

$$\varepsilon\kappa_i\leqslant\frac{(\int_{x_{i-1}}^{x_i} b\varphi_i)^2}{\int_{x_{i-1}}^{x_i} b^2\varphi_i\varphi_{i-1}}\leqslant\frac{(h_ib(x_i))^2}{Ch_ib^2(x_i)}\leqslant Ch_i.$$

Then we get that $\varepsilon\kappa_i\leqslant Ch_i$ for all $1\leqslant i\leqslant(N+1)$.

$$\begin{aligned}\frac{\varepsilon}{h_i}\left|\int_{x_{i-1}}^{x_i}(b\delta_i)''\right|&=\frac{\varepsilon\kappa_ih_i}{h_i}\left|\int_{x_{i-1}}^{x_i}(b\varphi_i\varphi_{i-1})''\right|\\&\leqslant Ch_i\int_{x_{i-1}}^{x_i} b''\varphi_i\varphi_{i-1}-b(\varphi_i\varphi_{i-1})''+2b'|(\varphi_i\varphi_{i-1})'|.\end{aligned}$$

Note that $(\varphi_i\varphi_{i-1})'' = -2/h_i^2$, $h_i b''\phi_i\phi_{i-1} \leqslant xb'' \leqslant Cb'$ on τ_i and $h_i b'|(\varphi_i\varphi_{i-1})'| \leqslant b'$, we get that

$$\frac{\varepsilon}{h_i}\left|\int_{x_{i-1}}^{x_i}(b\delta_i)''\right| \leqslant C\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b + \int_{x_{i-1}}^{x_i} b'\right) \leqslant Cx_i^p.$$

By the above inequality and (25), we finish the estimate of the second term of t_i in (24).

At last, we consider the third term of t_i in (24).

$$\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b^2\delta_i(u_I - u)'\right| \leqslant \frac{\|u-u_I\|_\infty}{h_i}\left|\int_{x_{i-1}}^{x_i}(b^2\delta_i)'\right|.$$

From (26), we get that $\kappa_i \leqslant Cx_i^{-p}$ for all $1 \leqslant i \leqslant N+1$. Then

$$\begin{aligned}\frac{1}{h_i}\left|\int_{x_{i-1}}^{x_i}(b^2\delta_i)'\right| &\leqslant \kappa_i\left(\int_{x_{i-1}}^{x_i} 2bb'\varphi_i\varphi_{i-1} + \int_{x_{i-1}}^{x_i} b^2|(\varphi_i\varphi_{i-1})'|\right)\\ &\leqslant \frac{C}{x_i^p h_i}\int_{x_{i-1}}^{x_i} b^2\\ &\leqslant Cx_i^p. \qquad \square\end{aligned}$$

The following summation by part formula is a discrete version of the integration by part. The proof is straightforward and thus skipped.

Lemma 5.4. *Let $\{c_j, j=1,\ldots,k+1\}$ and $\{d_j, j=1,\ldots,k+1\}$ be two sequences. Then we have*

$$\sum_{j=1}^{k} c_j(d_j - d_{j+1}) = c_1d_1 - c_kd_{k+1} + \sum_{j=1}^{k-1}(c_{j+1}-c_j)d_{j+1}.$$

Now we are in the position to present our main estimate.

Theorem 5.5.

$$\left|\sum_{j=i_{k-1}}^{i_k-1} G_j^i(t_j - t_{j+1})\right| \leqslant C\|u-u_I\|_\infty.$$

Proof. If $I_k = 1$, $i_k - 1 = i_{k-1}$; we get the conclusion directly by Lemmas 5.3 and 5.1.

If $I_k = 0$, then we apply the summation by part to obtain

$$\sum_{j=i_{k-1}}^{i_k-1} G_j^i(t_j - t_{j+1}) = G_{i_{k-1}}^i t_{i_{k-1}} - G_{i_k-1}^i t_{i_k} + \sum_{j=i_{k-1}}^{i_k-2}(G_{j+1}^i - G_j^i)t_{j+1}. \tag{27}$$

By the estimate of Green function and residual, we have

$$G_{i_{k-1}}^i t_{i_{k-1}} \leqslant C\left(\frac{x_{i_{k-1}}}{x_{i_{k-1}}}\right)^p\|u-u_I\|_\infty \leqslant C\|u-u_I\|_\infty.$$

Similarly we can prove

$$G_{i_k-1}^i t_{i_k} \leqslant C\left(\frac{x_{i_k}}{x_{i_k-1}}\right)^p\|u-u_I\|_\infty \leqslant C\left(\frac{x_{i_k}}{x_{i_{k-1}}}\right)^p\|u-u_I\|_\infty \leqslant C\|u-u_I\|_\infty,$$

where in the last step, we have used that if $I_k = 0$, $x_{i_k}/x_{i_{k-1}} \leqslant 2$. For the last term in the right-hand side of (27), let us first deal with the case $i \geqslant i_k$. We use the monoticity of the Green function to obtain

$$\begin{aligned}\sum_{j=i_{k-1}}^{i_k-2} (G^i_{j+1} - G^i_j)t_{j+1} &\leqslant \sum_{j=i_{k-1}}^{i_k-2} |G^i_{j+1} - G^i_j| \max_{i_{k-1}+1 \leqslant j \leqslant i_k} t_j \\ &= (G^i_{i_k-1} - G^i_{i_{k-1}}) \max_{i_{k-1}+1 \leqslant j \leqslant i_k} t_j \\ &\leqslant CG^i_{i_k} x^p_{i_k} \|u - u_I\|_\infty \\ &\leqslant C\|u - u_I\|_\infty.\end{aligned}$$

When $i \leqslant i_{k-1}$, using similar argument, we will end with

$$\begin{aligned}\sum_{j=i_{k-1}}^{i_k-2} (G^i_{j+1} - G^i_j)t_{j+1} &\leqslant CG^i_{i_{k-1}} x^p_{i_k} \|u - u_I\|_\infty \\ &\leqslant C\left(\frac{x_{i_k}}{x_{i_{k-1}}}\right)^p \|u - u_I\|_\infty \\ &\leqslant C\|u - u_I\|_\infty,\end{aligned}$$

where in the last step we have used that if $I_k = 0$, $x_{i_k}/x_{i_{k-1}} \leqslant 2$. The case $i_{k-1} < i < i_k$ can be done in a similar way. □

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