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New analysis of the sphere covering problems and optimal polytope approximation of convex bodies☆

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Abstract

In this paper, we show that both sphere covering problems and optimal polytope approximation of convex bodies are related to optimal Delaunay triangulations, which are the triangulations minimizing the interpolation error between function $\|\mathbf{x}\|^2$ and its linear interpolant based on the underline triangulations. We then develop a new analysis based on the estimate of the interpolation error to get the Coxeter–Few–Rogers lower bound for the thickness in the sphere covering problem and a new estimate of the constant del_n appeared in the optimal polytope approximation of convex bodies.

Keywords: Sphere covering; Delaunay triangulation; Polytope approximation

1. Introduction and statement of results

The Delaunay triangulation (DT) of a finite point set V can be defined by the empty sphere property: no vertices in V are inside the circumsphere of any simplex in the triangulation. In [5], we characterized the DT from a function approximation point of view.

Let us denote $Q(\mathcal{T}, f, p) = \|f - f_{\mathrm{I},\mathcal{T}}\|_{L^p(\Omega)}$, where $f_{\mathrm{I},\mathcal{T}}(\mathbf{x})$ is the linear interpolation of a continuous function f based on a triangulation $\mathcal{T}$ of a domain $\Omega \subset \mathbb{R}^n$. Let $\mathcal{P}_V$ be the

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set of all triangulations that have a given set V of vertices and Ω is chosen as the convex hull of V. We have shown in [5] that

$$Q(DT, \|\mathbf{x}\|^2, p) = \min_{\mathcal{T}\in\mathcal{P}_V} Q(\mathcal{T}, \|\mathbf{x}\|^2, p), \text{ for } 1 \leqslant p \leqslant \infty. \tag{1}$$

For a more general function, a function-dependent DT is then defined to be an optimal triangulation that minimizes the interpolation error for this function and its construction can be obtained by a simple lifting and projection procedure.

The optimal Delaunay triangulation (ODT) introduced in [5] is the one that minimizes the interpolation error among all triangulations with the same number of vertices. More precisely $\mathcal{T}^*$ is an ODT in $\mathcal{P}_N$ if

$$Q(\mathcal{T}^*, f, p) = \inf_{\mathcal{T}\in\mathcal{P}_N} Q(\mathcal{T}, f, p), \text{ for some } 1 \leqslant p \leqslant \infty, \tag{2}$$

where $\mathcal{P}_N$ stands for the set of all triangulations with at most N vertices. Such a function-dependent ODT is proved to exist for any given convex continuous function and a necessary condition for an optimal triangulation is also obtained in [5].

In this paper, we will discuss two special ODTs which minimize $Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty)$ and $Q(\mathcal{T}, \|\mathbf{x}\|^2, 1)$, respectively. The first one corresponds to the sphere covering problem and the second one is related to the optimal polytope approximation of convex bodies.

Roughly speaking, sphere covering problem is to seek the most economical way to cover a domain Ω in $\mathbb{R}^n$ with overlapping balls of equal size. Let us denote $B_n(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \leqslant r\}$ and $S_n(\mathbf{x}, r)$ its boundary. If center is $\mathbf{o}$ or radius is 1, it will be omitted. For a convex domain $\Omega \subset \mathbb{R}^n$, we define the thickness θ_n as

$$\theta_n = \liminf_{r\to 0} N_r |B_n(r)|/|\Omega|,$$

where N_r is the minimum number of balls with radius r needed to cover the domain and $|\cdot|$ is the standard Lebesgue measure. In the literature, the thickness is always defined as the limit when the domain goes to $\mathbb{R}^n$ while using the unit ball [27]. However it is equivalent to let the radius go to zero by the scaling argument. The choice of the convex domain Ω in the definition above is somewhat arbitrary since we have a theorem by Hlawka (see [27, p. 4]) which says any convex domain leads to an equivalent definition. In other words, it is saying that in the asymptotic sense we can neglect the affection of the boundary of Ω.

Now we consider the problem in the other way around. Let $V = \{\mathbf{x}_i\}_{i=1}^N$ be a finite point set such that the convex hull of V is Ω. We use these points as the centers of balls and denote the minimum radius needed to cover Ω by R_V^{c}. If we let $R_N^{\mathrm{c}} = \inf_{\#V=N} R_V^{\mathrm{c}}$, by the standard $\varepsilon - N$ argument, it is easy to show that

$$\theta_n = \liminf_{N\to\infty} N|B_n(R_N^{\mathrm{c}})|/|\Omega|.$$

The sphere covering problem is then translated into finding the optimal distribution of N points which will coincide with the vertices of an ODT. More precisely, we shall prove that

$$(R_N^{\mathrm{c}})^2 = \inf_{\mathcal{T}\in\mathcal{P}_N} Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty).$$

We then derive a lower bound for the interpolation error $Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty)$ which results a new approach to obtain Coxeter–Few–Rogers lower bound of θ_n [9]. Let

$$\tau_n = \phi_n \frac{n!}{\sqrt{n+1}} \left(\frac{n}{n+1}\right)^{n/2-1},$$

where ϕ_n is the solid angle of a vertex of the n-regular simplex; See Section 3 for details.

Theorem 1.1.

$$\theta_n \geqslant \tau_n,\ 1 \leqslant n < \infty.$$

Furthermore, we can only achieve this lower bound by regular triangulation for $n = 1, 2$, *and thus*

$$\theta_1 = 1, \theta_2 = \frac{2\pi}{3\sqrt{3}}.$$

The regular triangulation in the theorem above means the triangulation with all simplices are regular, i.e. all the edge lengths in the triangulation are equal. It is well known that only for $n = 1, 2$, we can have a regular tessellation of $\mathbb{R}^n$(see, for example, [8]). This is the reason why we have so many open problems in dimensions higher than two.

Let us now discuss briefly the second problems on the optimal polytope approximation to a convex body. A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. We denote $\mathcal{C}$ the space of all convex bodies in $\mathbb{R}^n$ and $\delta^V(\cdot, \cdot)$ the volume difference metric on the space $\mathcal{C}$, i.e. $\delta^V(C, D) = |C \cup D| - |C \cap D|$. For a given convex body $C \in \mathcal{C}$, let $\mathcal{P}_N^i$ be the set of all polytopes inscribed to C with at most N vertices and $\delta^V(C, \mathcal{P}_N^i) = \inf_{P \in \mathcal{P}_N} \delta^V(C, P)$. In [18], Gruber showed that for a convex body C whose boundary is of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$ in $\mathbb{R}^{n+1}$, there exists a constant del_n depending only on n such that

$$\lim_{N \to \infty} N^{\frac{2}{n}} \delta^V(C, \mathcal{P}_N^i) = \frac{1}{2} \mathrm{del}_n \left(\int_{\partial C} \kappa_C(\mathbf{x})^{\frac{1}{n+2}} \, \mathrm{d}\sigma(\mathbf{x}) \right)^{\frac{n+2}{n}}, \tag{3}$$

where σ is the ordinary surface area measure in $\mathbb{R}^n$. Further $\mathrm{del}_1 = 1/6$, and $\mathrm{del}_2 = 1/\left(2\sqrt{3}\right)$. Again for $n \geqslant 3$, it is difficult, if it is not impossible, to get the exact value. There are some estimates about del_n [14,19,20]. We shall present a sharper estimate for the constant del_n in this paper.

For $n = 1$, (3) was indicated by Fejes Tóth [12] and proved by McClure and Vitale [21]. Proof for $n = 2$ is due to Gruber [15] and the general case was obtained by Gruber [18]. For other forms of optimal approximating polytopes with respect to other metrics, we refer to [1,13,16,17].

Remark 1.2. In view of the characterization theory of the nonlinear approximation [10], to retain the asymptotic formula for $\delta^V(C, \mathcal{P}_N^i)$, ∂C must have certain regularity in terms of Besov norms.

Since Gauss curvature only appears in the last term of (3), to estimate del_n we can choose any convenient convex body we want. By considering the paraboloid $(\mathbf{x}, \|\mathbf{x}\|^2)$, it is easy to show that (c.f. [18])

$$\mathrm{del}_n = \lim_{N\to\infty} N^{2/n} \inf_{\mathcal{T}\in\mathcal{P}_N} Q(\mathcal{T}, \|\mathbf{x}\|^2, 1)/|\Omega|^{2/n+1}$$

for any convex domain $\Omega \subset \mathbb{R}^n$. With the lower and upper bound of $Q(\mathcal{T}, \|\mathbf{x}\|^2, 1)$, we get an estimate of the constant del_n.

Theorem 1.3.

$$\frac{n+1}{n+2}\left(\frac{\tau_n}{|B_n|}\right)^{2/n} \leqslant \mathrm{del}_n \leqslant \frac{n+1}{n+2}\left(\frac{\theta_n}{|B_n|}\right)^{2/n},$$

where $|B_n|$ is the volume of the unit ball in $\mathbb{R}^n$.

Our estimate is asymptotic exact when dimension n goes to infinity.

Corollary 1.4.

$$\lim_{n\to\infty} \frac{\mathrm{del}_n}{n} = \frac{1}{2\pi e}.$$

The asymptotic exact estimate is also obtained in [20]. However our approach here is simpler and more straightforward. For $n = 1, 2$ since $\theta_n = \tau_n$, the estimate is exact, i.e. we obtain $\mathrm{del}_1 = 1/6$, and $\mathrm{del}_2 = 1/\left(2\sqrt{3}\right)$ by our estimate. Although the thickness in the upper bound are not known for $n \geqslant 3$, any reasonable upper bound of θ_n can be used to bound del_n above. For example, by choosing a special lattice sphere covering scheme (see [7, p. 36]), which is the thinnest covering known in all dimensions $n \leqslant 23$, we get a computable formula for the upper bound that is

$$\mathrm{del}_n \leqslant (n+1)^{1/n} \frac{n}{12}. \tag{4}$$

When n large, we may use the upper bound obtained by Rogers [25],

$$\theta_n < n \ln n + n \ln\ln n + 5n, \text{ for } n \geqslant 3.$$

Comparing with the result of Mankiewicz and Schütt [20],

$$\frac{n}{n+2}\frac{1}{|B_n|^{2/n}} \leqslant \mathrm{del}_n \leqslant \frac{n}{n+2}\frac{1}{|B_n|^{2/n}}\frac{\Gamma(n+2+2/n)}{(n+1)!}, n \geqslant 2$$

our lower bound is sharper and the upper bound (4) is sharper in lower dimensions ($n \leqslant 13$). The reason for the upper bound (4) becomes worse when $n \geqslant 14$ is that the sphere covering scheme we choose are away from the optimal one especially when n is large. Actually $\lim_{n\to\infty}(n+1)^{1/n}/12 = e/12 > 1/(2\pi e)$.

2. Sphere covering problem and the proof of Theorem 1.1

Let V be a finite point set such that the convex hull of V is Ω. Recall that the minimum radius needed to cover Ω is denoted by R_V^c. With the same point set V, there is a DT of Ω. The following lemma reveals the connection between sphere covering problems and DTs.

Lemma 2.1. $$(R_V^c)^2 = Q(DT, \|\mathbf{x}\|^2, \infty) = \min_{\mathcal{T}\in\mathcal{P}_V} Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty).$$

Proof. Let us look at a simplex τ with vertices $\{\mathbf{x}_i\}_{i=1}^{n+1}$. For $f(\mathbf{x}) = \|\mathbf{x}\|^2$, by the multiple points Taylor expansion ([6, p. 128]), we know

$$f_I(\mathbf{x}) - f(\mathbf{x}) = \sum_{j=1}^{n+1} \lambda_j(\mathbf{x})\|\mathbf{x} - \mathbf{x}_j\|^2 \leqslant E_{\max}, \tag{5}$$

where $\lambda_i(\mathbf{x})$ is the barycenter coordinate of $\mathbf{x}$ in τ and $E_{\max}$ denotes $Q(\tau, \|\mathbf{x}\|^2, \infty)$. Since $\sum \lambda_i(\mathbf{x}) = 1$, (5) implies that, first, there exists a vertex $\mathbf{x}_i$ such that $\|\mathbf{x} - \mathbf{x}_i\|^2 \leqslant E_{\max}$ which means $\tau \subset \bigcup_{i=1}^{n+1} B(\mathbf{x}_i, E_{\max}^{1/2})$, and, secondly for $\mathbf{x}^*$ at which the error attains the maximum value, $\sum_{j=1}^{n+1} \|\mathbf{x}^* - \mathbf{x}_j\|^2 = E_{\max}$, which means to cover τ with balls of equal size centered at its vertices, the minimum radius is $E_{\max}^{1/2}$.

We thus proved that for any triangulation $\mathcal{T} \in \mathcal{P}_V$, if we use vertices as centers of balls, the square of the minimum radius needed to cover the domain is $Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty)$. By the optimality of DTs (see (1)) we finish the proof. □

As a direct consequence, the optimal distribution of the centers of the covering balls coincides with the vertices of an optimal DT.

Corollary 2.2.

$$(R_N^c)^2 = \inf_{\mathcal{T}\in\mathcal{P}_N} Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty).$$

We then derive a lower bound for the interpolation error $Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty)$.

Lemma 2.3.

$$Q(\tau, \|\mathbf{x}\|^2, \infty) \geqslant \frac{n}{n+1}\, \frac{n!^{2/n}}{(n+1)^{1/n}} |\tau|^{2/n}, \tag{6}$$

where the equality holds if and only if τ is regular.

Proof. By (5), $E(\mathbf{x}) := f_I(\mathbf{x}) - f(\mathbf{x})$ only depends on the quadratic part of the function. Hence we may consider $g(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_o\|^2$, where $\mathbf{x}_o$ is the circum center of τ. By looking at this way, we get

$$E(\mathbf{x}) = f_I(\mathbf{x}) - f(\mathbf{x}) = g_I(\mathbf{x}) - g(\mathbf{x}) = R_\tau^2 - \|\mathbf{x} - \mathbf{x}_o\|^2, \tag{7}$$

where R_τ is the radius of the circum sphere of τ. If $\mathbf{x}_o \in \tau$, then $E_{\max} = R_\tau^2$. (6) is a well known geometric inequality for a simplex, for example see [24, p. 515] and the equality holds if and only if τ is regular.

Otherwise $E(\mathbf{x})$ attains its maximum at $\mathbf{x}^*$, the projection of $\mathbf{x}_o$ to τ, i.e. $E_{\max} = R_\tau^2 - \|\mathbf{x}_o - \mathbf{x}^*\|^2$. In this case $\mathbf{x}^*$ is on some facet σ of τ, which is an $(n-1)$-simplex. By the definition of the projection, for $\mathbf{x} \in \sigma$

$$\|\mathbf{x} - \mathbf{x}^*\|^2 + \|\mathbf{x}^* - \mathbf{x}_o\|^2 = \|\mathbf{x} - \mathbf{x}_o\|^2. \tag{8}$$

Without lose of generality, we may assume σ is opposite to vertex $\mathbf{x}_{n+1}$, namely it is made up by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. By (8), all the distances between $\mathbf{x}_i$ $(1 \leqslant i \leqslant n)$ and $\mathbf{x}^*$ are equal. Thus $\mathbf{x}^*$ is the circum center of σ and $E_{\max}$ is the square of the radius of the circum sphere of σ. By the characterization of the projection $(\mathbf{x}_o - \mathbf{x}^*) \cdot (\mathbf{x}_{n+1} - \mathbf{x}^*) \leqslant 0$, we get

$$\begin{aligned}\|\mathbf{x}_{n+1} - \mathbf{x}^*\|^2 &= \|\mathbf{x}_{n+1} - \mathbf{x}_o\|^2 + \|\mathbf{x}_o - \mathbf{x}^*\|^2 + 2(\mathbf{x}_{n+1} - \mathbf{x}_o) \cdot (\mathbf{x}_o - \mathbf{x}^*) \\ &= R_\tau^2 - \|\mathbf{x}_o - \mathbf{x}^*\|^2 + 2(\mathbf{x}_{n+1} - \mathbf{x}^*) \cdot (\mathbf{x}_o - \mathbf{x}^*) \\ &\leqslant R_\tau^2 - \|\mathbf{x}_o - \mathbf{x}^*\|^2.\end{aligned}$$

Thus $\tau \subset B_n(\mathbf{x}^*, E_{\max}^{1/2})$.

We then construct a simplex τ' with $|\tau'| \geqslant |\tau|$ which is inscribed to $B(\mathbf{x}^*, E_{\max}^{1/2})$. Let us choose a coordinate such that $\mathbf{x}^*$ is the origin and σ is on $x^{n+1} = 0$. Suppose the coordinate of the vertex which opposites to σ is $\mathbf{v} = (v^1, v^2, \ldots, v^{n+1})$. We change it to $\mathbf{v}' = (v^1, v^2, \ldots, (E_{\max} - \sum_{i=1}^n (v^i)^2)^{1/2})$. Then $\mathbf{v}'$ and σ gives us an inscribed simplex τ' and obviously $|\tau'| \geqslant |\tau|$. Applying the first case to τ', we finish the proof. $\square$

Theorem 2.4. *Let N_{T} be the number of simplices in the triangulation, we have*

$$Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty) \geqslant LC_{n,\infty} N_{\mathrm{T}}^{-\frac{2}{n}} |\Omega|^{\frac{2}{n}},$$

where

$$LC_{n,\infty} = \frac{n}{n+1} \frac{n!^{2/n}}{(n+1)^{1/n}}.$$

The equality holds if and only if $\mathcal{T}$ is a regular triangulation, namely all edges of $\mathcal{T}$ are equal.

Proof. By Lemma 6 and the Cauchy inequality,

$$\begin{aligned}Q(\mathcal{T}, \|\mathbf{x}\|^2, \infty) &= \max_{\tau \in \mathcal{T}} Q(\tau, \|\mathbf{x}\|^2, \infty) \geqslant \sum_{\tau \in \mathcal{T}} Q(\tau, \|\mathbf{x}\|^2, \infty)/N_{\mathrm{T}} \\ &\geqslant LC_{n,\infty} \sum_{\tau \in \mathcal{T}} |\tau|^{2/n}/N_{\mathrm{T}} \geqslant LC_{n,\infty} N_{\mathrm{T}}^{-\frac{2}{n}} \Omega^{\frac{2}{n}}.\end{aligned}$$

The equality holds if and only if $Q(\tau, \|\mathbf{x}\|^2, \infty) = LC_{n,\infty}|\tau|^{2/n} = \text{constant}$, $\forall \tau \in \mathcal{T}$, i.e. $\mathcal{T}$ is a regular triangulation. $\square$

Now we are going to connect the number of simplices N_{T} and number of vertices N. For $n = 1$, it is trivial to show $\lim_{N \to \infty} N/N_{\mathrm{T}} = 1$. Let us consider triangulations in two dimension. We sum angles $\phi_{\tau,k}$ of triangles in two different ways, namely elementwise and pointwise. We can easily show

$$2\pi N > \sum_{i=1}^{N_{\mathrm{T}}} \left(\sum_{k=1}^{3} \phi_{\tau_i,k} \right) = \pi N_{\mathrm{T}},$$

and

$$\lim_{N\to\infty} \frac{N}{N_{\mathrm{T}}} = \frac{1}{2}.$$

Thus with Theorem 2.4 we proved that $\theta_1 = 1$, $\theta_2 = 2\pi/3\sqrt{3}$.

To deal with higher dimensions, we shall introduce the concept of the solid angle. The following definition and lemma are adopted from Zong's book [27].

Definition 2.5. Let P be a polytope in $\mathbb{R}^n$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and write

$$V_i = \{\mathbf{v}_i + \lambda(\mathbf{x} - \mathbf{v}_i) : \mathbf{x} \in P, \lambda \geqslant 0\}.$$

Then we call

$$\phi(\mathbf{v}_i) = |S_n(\mathbf{v}_i, 1) \cap V_i|_s$$

the *solid angle* of P at $\mathbf{v}_i$, where $|\cdot|_s$ means the surface area.

For a regular simplex, all the solid angles are equal. We denote it by ϕ_n. Let $\kappa_n := |S_n|_s/\phi_n$ be the number of equilateral simplices surrounding a vertex. The integer κ_n is corresponding to a regular triangulation which is only possible for $n = 1, 2$; See for example [8].

The following important lemma was introduced by Coxeter et al. [9]. The proof can be found in [27].

Lemma 2.6. *Let τ be an n-dimensional simplex, with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$. Then*

$$\sum_{i=1}^{n+1} \phi(\mathbf{v}_i) \geqslant (n+1)\phi_n,$$

where the equality holds when τ is a regular simplex.

With this lemma, we apply the same argument as that in the two dimensions to get an inequality between N and N_{T}.

Corollary 2.7. *For a triangulation $\mathcal{T}$,*

$$\frac{N}{N_{\mathrm{T}}} \geqslant \frac{n+1}{\kappa_n}.$$

Now we are in the position to prove Theorem 1.1

Proof of Theorem 1.1. We will use abbreviation $Q_p(\mathcal{T}) = Q(\mathcal{T}, \|\mathbf{x}\|^2, p),\ 1 \leqslant p \leqslant \infty$. Since

$$\theta_n = \liminf_{N\to\infty} \frac{N|B_n(Q_\infty^{1/2}(T))|}{|\Omega|} = \liminf_{N\to\infty} |B_n| \frac{N}{N_{\mathrm{T}}} \frac{N_{\mathrm{T}} Q_\infty^{n/2}(T)}{|\Omega|},$$

the result follows from Theorem 2.4 and Corollary 2.7. □

However it is not easy to get a computable formula for τ_n. Here we list an asymptotic formula obtained by Rogers [26].

$$\tau_n \sim \frac{n}{e\sqrt{e}} \text{ when } n \to \infty. \tag{9}$$

The proof can be found at [27].

3. Optimal polytope approximation of convex bodies and the proof of Theorem 1.3

In this section, we follow the same line in Section 2 to estimate the constant del_n. Recall that

$$\mathrm{del}_n = \lim_{N\to\infty} N^{2/n} \inf_{\mathcal{T}\in\mathcal{P}_N} Q(\mathcal{T}, \|\mathbf{x}\|^2, 1)/|\Omega|^{2/n+1}.$$

We first present an explicit formula for the interpolation error $Q(\mathcal{T}, \|\mathbf{x}\|^2, 1)$.

Lemma 3.1.

$$Q(\mathcal{T}, \|\mathbf{x}\|^2, 1) = \frac{1}{(n+2)(n+1)} \sum_{\tau\in\mathcal{T}} |\tau| \sum_{k=1}^{n(n+1)/2} d_{\tau,k}^2,$$

where $d_{\tau,k}$ is the kth edge length of τ.

Proof. Recall that in a simplex τ, $f_{\mathrm{I}}(\mathbf{x}) - f(\mathbf{x}) = \sum_j \lambda_j(\mathbf{x})\|\mathbf{x} - \mathbf{x}_j\|^2$. Let us write $\mathbf{x} - \mathbf{x}_j = \sum_i \lambda_i(\mathbf{x}_i - \mathbf{x}_j)$, we then have

$$f_{\mathrm{I}}(\mathbf{x}) - f(\mathbf{x}) = \sum_{i,j=1}^{n+1} \lambda_i\lambda_j(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x} - \mathbf{x}_j). \tag{10}$$

Using the symmetry of index i, j, we can write it as

$$f_{\mathrm{I}}(\mathbf{x}) - f(\mathbf{x}) = \sum_{i,j=1}^{n+1} \lambda_i\lambda_j(\mathbf{x}_j - \mathbf{x}_i)\cdot(\mathbf{x} - \mathbf{x}_i). \tag{11}$$

Combining (10) and (11) gives us

$$f_{\mathrm{I}}(\mathbf{x}) - f(\mathbf{x}) = \sum_{i,j=1,i<j}^{n+1} \lambda_i(\mathbf{x})\lambda_j(\mathbf{x})(\mathbf{x}_i - \mathbf{x}_j)^2.$$

Taking the integration and using the fact

$$\int_\tau \lambda_i(\mathbf{x})\lambda_j(\mathbf{x})\,d\mathbf{x} = \frac{|\tau|}{(n+2)(n+1)},$$

we get the key formula

$$\int_\tau |f_{\mathrm{I}}(\mathbf{x}) - f(\mathbf{x})|\,d\mathbf{x} = \frac{|\tau|}{(n+2)(n+1)} \sum_{k=1}^{n(n+1)/2} d_{\tau,k}^2. \tag{12}$$

The desired result follows by summing them up. □

The following geometric inequality can be found at [24, p. 517]. For two dimensions, it is a direct consequence of the well-known Heron's formula of a triangle.

Lemma 3.2. *For an n-simplex τ, we have*

$$\sum_{k=1}^{n(n+1)/2} d_k^2 \geqslant \frac{n(n+1)n!^{2/n}}{(n+1)^{1/n}} |\tau|^{2/n}$$

and the equality holds if and only if τ is regular.

Theorem 3.3. *For a triangulation $\mathcal{T}$ of a bounded domain Ω with N_{T} simplices, we have*

$$Q(\mathcal{T}, \|\mathbf{x}\|^2, 1) \geqslant LC_{n,1} N_{\mathrm{T}}^{-\frac{2}{n}} |\Omega|^{\frac{n+2}{n}},$$

where

$$LC_{n,1} = \frac{n}{n+2} \frac{n!^{2/n}}{(n+1)^{1/n}}.$$

The equality holds if and only if $\mathcal{T}$ is an regular triangulation, namely all edges of $\mathcal{T}$ are equal.

Proof. By Lemmas 3.1 and 3.2, we have

$$Q(\mathcal{T}, \|\mathbf{x}\|^2, 1) \geqslant LC_{n,1} \sum_{i=1}^{N_{\mathrm{T}}} |\tau_i|^{2/n+1} \geqslant LC_{n,1} N_{\mathrm{T}}^{-2/n} |\Omega|^{\frac{n+2}{n}}.$$

First equality holds if and only if τ_i's are regular and the second one holds if and only if $|\tau_i|$'s are equal. Thus the equality holds if and only if all edge lengths are equal. □

Remark 3.4. In general, we have

$$Q(\mathcal{T}, \|\mathbf{x}\|^2, p) \geqslant LC_{n,p} N_{\mathrm{T}}^{-\frac{2}{n}} |\Omega|^{\frac{n+2}{n}}, 1 \leqslant p \leqslant \infty.$$

The expression of $LC_{n,p}$ is implicitly contained in [4].

Proof of Theorem 1.3. Combing Theorem 3.3 with the lower bound of N/N_{T} (see Corollary 2.7), we will prove the lower bound for del_n, i.e.

$$\mathrm{del}_n \geqslant \frac{n+1}{n+2}\left(\frac{\tau_n}{|B_n|}\right)^{2/n}.$$

Without loss of generality, we may choose Ω such that $|\Omega| = 1$. For any triangulation $\mathcal{T}$ of Ω, we have

$$N^{2/n} Q_1(\mathcal{T}) = \left(\frac{N}{N_{\mathrm{T}}}\right)^{2/n} N_{\mathrm{T}}^{2/n} Q_1(\mathcal{T}) \geqslant \left(\frac{n+1}{\kappa_n}\right)^{2/n} LC_{n,1} = \frac{n+1}{n+2}\left(\frac{\tau_n}{|B_n|}\right)^{2/n}.$$

The desired result is obtained by sending N to ∞.

To prove the upper bound

$$\mathrm{del}_n \leqslant \frac{n+1}{n+2}\left(\frac{\theta_n}{|B_n|}\right)^{\frac{2}{n}},$$

we use a geometric inequality for a simplex τ (see [24, p. 515])

$$\sum_{i=1}^{n+1} d_{\tau,i}^2 \leqslant (n+1)^2 R_\tau^2.$$

R_τ^2 in the right side can be modified to $Q(\tau, \|\mathbf{x}\|^2, \infty)$ by the same argument as that in Lemma 6. Combining with Lemma 3.1, we know

$$Q_1(\mathcal{T}) = \frac{1}{(n+2)(n+1)} \sum_{\tau\in\mathcal{T}} \left(|\tau| \sum_{i=1}^{n(n+1)/2} d_{\tau,i}^2\right) \leqslant \frac{n+1}{n+2} Q_\infty(T)|\Omega|.$$

For simplicity, we choose $|\Omega| = 1$. For any $\mathcal{T}$ with N vertices we have

$$\mathrm{del}_n \leqslant N^{2/n} Q_1(\mathcal{T}) \leqslant \frac{n+1}{n+2}(N Q_\infty^{\frac{n}{2}}(\mathcal{T}))^{2/n}.$$

The desired result then follows. □

Proof of Corollary 1.4. By the asymptotic formula of τ_n (9), we know $\lim_{n\to\infty} \tau_n^{2/n} = 1$. On the other hand, Rogers [25] gives an upper bound for θ_n,

$$\theta_n < n\ln n + n\ln\ln n + 5n, \text{ for } n \geqslant 3.$$

Thus $\lim_{n\to\infty} \theta_n^{2/n} = 1$.

It is well known that

$$|B_n| = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

With Stirling's formula

$$\Gamma(n/2+1) \sim \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{(n+1)/2},$$

we get

$$\lim_{n\to\infty} \frac{\mathrm{del}_n}{n} = \lim_{n\to\infty} \frac{1}{n|B_n|^{2/n}} = \frac{1}{2\pi e}. \qquad \square$$

4. Concluding remarks

In this paper, we have shown the new connection between sphere covering problems, optimal polytope approximation of convex bodies and linear approximation of function $\|\mathbf{x}\|^2$. Based on this approach, we give a new analysis of those problems and get a new proof of Coxeter–Few–Rogers lower bound for the thickness in the sphere covering problem. More importantly, we get a new estimate of the constant del_n in the optimal polytopes approximation to the convex bodies.

Note that the Hessian matrix of function $\|\mathbf{x}\|^2$ corresponds to Euclidean metric. By changing the approximate function, we may study the sphere covering problems and polytopes approximation of convex bodies in a more general metric. For example using our approach, it is easy to derive some classic results [11] in the spherical space.

We have developed some mesh generation techniques by minimizing the interpolation error $Q(\mathcal{T}, f, p)$ in [2,3]. The connection of the ODT with the sphere covering and the duality of sphere covering and sphere packing suggest that sphere packing or covering can be used in the mesh generation; See [23,22]. On the other hand, the techniques used in the mesh improvements like mesh smoothing [2] can be used as a numerical algorithm to compute a better sphere covering scheme.

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