Deriving the X-Z Identity from Auxiliary Space Method*

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1 Iterative Methods

In this paper we discuss iterative methods to solve the linear operator equation

$$Au = f, \tag{1}$$

posed on a finite dimensional Hilbert space $\mathcal{V}$ equipped with an inner product $(\cdot, \cdot)$. Here $A : \mathcal{V} \mapsto \mathcal{V}$ is a symmetric positive definite (SPD) operator, $f \in \mathcal{V}$ is given, and we are looking for $u \in \mathcal{V}$ such that (1) holds.

The X-Z identity for the multiplicative subspace correction method for solving (1) is introduced and proved in Xu and Zikatanov [2002]. Alternative proves can be found in Cho et al. [2008] and Vassilevski [2008]. In this paper we derive the X-Z identity from the auxiliary space method Nepomnyaschikh [1992], Xu [1996].

A basic linear iterative method for solving (1) can be written in the following form: starting from an initial guess u^0, for $k = 0, 1, 2, \cdots$

$$u^{k+1} = u^k + B(f - Au^k). \tag{2}$$

Here the non-singular operator $B \approx A^{-1}$ will be called *iterator*. Let $e^k = u - u^k$. The error equation of the basic iterative method (2) is

$$e^{k+1} = (I - BA)e^k = (I - BA)^k e^0.$$

Thus the iterative method (2) converges if and only if the spectral radius of the error operator $I - BA$ is less than one, i.e., $\rho(I - BA) < 1$.

Given an iterator B, we define the mapping $\Phi_B v = v + B(f - Av)$ and introduce its symmetrization $\Phi_{\overline{B}} = \Phi_{B^t} \circ \Phi_B$. By definition, we have the formula for the error operator $I - \overline{B}A = (I - B^t A)(I - BA)$, and thus

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$$\overline{B} = B^t(B^{-t} + B^{-1} - A)B. \tag{3}$$

Since $\overline{B}$ is symmetric, $I - \overline{B}A$ is symmetric with respect to the A-inner product $(u, v)_A := (Au, v)$. Indeed, let $(\cdot)^*$ be the adjoint in the A-inner product $(\cdot,\cdot)_A$. It is easy to show

$$I - \overline{B}A = (I - BA)^*(I - BA). \tag{4}$$

Consequently, $I - \overline{B}A$ is positive semi-definite and thus $\lambda_{\max}(\overline{B}A) \le 1$. We get

$$\|I - \overline{B}A\|_A = \max\{|1 - \lambda_{\min}(\overline{B}A)|, |1 - \lambda_{\max}(\overline{B}A)|\} = 1 - \lambda_{\min}(\overline{B}A). \tag{5}$$

From (5), we see that $I - \overline{B}A$ is a contraction if and only if $\overline{B}$ is SPD which is also equivalent to $B^{-t} + B^{-1} - A$ being SPD in view of (3).

The convergence of the scheme Φ_B and its symmetrization $\Phi_{\overline{B}}$ is connected by the following inequality:

$$\rho(I - BA)^2 \le \|I - BA\|_A^2 = \|I - \overline{B}A\|_A = \rho(I - \overline{B}A), \tag{6}$$

and the equality holds if $B = B^t$. Hence we shall focus on the analysis of the symmetric scheme in the rest of this paper.

The iterator B, when it is SPD, can be used as a preconditioner in the Preconditioned Conjugate Gradient (PCG) method, which admits the following estimate:

$$\frac{\|u - u^k\|_A}{\|u - u^0\|_A} \le 2\left(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1}\right)^k (k \ge 1), \quad \left(\kappa(BA) = \frac{\lambda_{\max}(BA)}{\lambda_{\min}(BA)}\right).$$

A good preconditioner should have the properties that the action of B is easy to compute and that the condition number $\kappa(BA)$ is significantly smaller than $\kappa(A)$. We shall also discuss construction of multilevel preconditioners in this paper.

2 Auxiliary Space Method

In this section, we present a variation of the fictitious space method Nepomnyaschikh [1992] and the auxiliary space method Xu [1996].

Let $\tilde{\mathcal{V}}$ and $\mathcal{V}$ be two Hilbert spaces and let $\Pi : \tilde{\mathcal{V}} \to \mathcal{V}$ be a surjective map. Denoted by $\Pi^t : \mathcal{V} \to \tilde{\mathcal{V}}$ the adajoint of Π with respect to the default inner products

$$(\Pi^t u, \tilde{v}) := (u, \Pi\tilde{v}) \quad \text{for all } u \in \mathcal{V}, \tilde{v} \in \tilde{\mathcal{V}}.$$

Here, to save notation, we use $(\cdot,\cdot)$ for inner products in both $\mathcal{V}$ and $\tilde{\mathcal{V}}$. Since Π is surjective, its transpose Π^t is injective.

Theorem 1. *Let $\tilde{\mathcal{V}}$ and $\mathcal{V}$ be two Hilbert spaces and let $\Pi : \tilde{\mathcal{V}} \to \mathcal{V}$ be a surjective map. Let $\tilde{B} : \tilde{\mathcal{V}} \to \tilde{\mathcal{V}}$ be a symmetric and positive definite operator. Then $B := \Pi\tilde{B}\Pi^t : \mathcal{V} \to \mathcal{V}$ is also symmetric and positive definite. Furthermore*

$$(B^{-1}v, v) = \inf_{\Pi\tilde{v} = v} (\tilde{B}^{-1}\tilde{v}, \tilde{v}). \tag{7}$$

Proof. We adapt the proof given by Xu and Zikatanov [2002] (Lemma 2.4).

It is obvious that B is symmetric and positive semi-definite. Since $\tilde{B}$ is SPD and Π^t is injective, $(Bv, v) = (\tilde{B}\Pi^t v, \Pi^t v) = 0$ implies $\Pi^t v = 0$ and consequently $v = 0$. Therefore B is positive definite.

Let $\tilde{v}^* = \tilde{B}\Pi^t B^{-1} v$. Then $\Pi \tilde{v}^* = v$ by the definition of B. For any $\tilde{w} \in \tilde{\mathcal{V}}$

$$(\tilde{B}^{-1}\tilde{v}^*, \tilde{w}) = (\Pi^t B^{-1} v, \tilde{w}) = (B^{-1}v, \Pi\tilde{w}).$$

In particular $(\tilde{B}^{-1}\tilde{v}^*, \tilde{v}^*) = (B^{-1}v, \Pi\tilde{v}^*) = (B^{-1}v, v)$. For any $\tilde{v} \in \tilde{\mathcal{V}}$, denoted by $v = \Pi\tilde{v}$, we write $\tilde{v} = \tilde{v}^* + \tilde{w}$ with $\Pi\tilde{w} = 0$. Then

$$\begin{aligned}
\inf_{\Pi\tilde{v}=v} (\tilde{B}^{-1}\tilde{v}, \tilde{v}) &= \inf_{\Pi\tilde{w}=0} (\tilde{B}^{-1}(\tilde{v}^* + \tilde{w}), \tilde{v}^* + \tilde{w}) \\
&= (B^{-1}v, v) + \inf_{\Pi\tilde{w}=0} \Big(2(\tilde{B}^{-1}\tilde{v}^*, \tilde{w}) + (\tilde{B}^{-1}\tilde{w}, \tilde{w}) \Big) \\
&= (B^{-1}v, v) + \inf_{\Pi\tilde{w}=0} (\tilde{B}^{-1}\tilde{w}, \tilde{w}) \\
&= (B^{-1}v, v).
\end{aligned}$$

□

The symmetric positive definite operator B may be used as a preconditioner for solving $Au = f$ using PCG. To estimate the condition number $\kappa(BA)$, we only need to compare B^{-1} and A.

Lemma 1. *For two SPD operators A and B, if $c_0(Av, v) \le (B^{-1}v, v) \le c_1(Av, v)$ for all $v \in \mathcal{V}$, then $\kappa(BA) \le c_1/c_0$.*

Proof. Note that BA is symmetric with respect to A. Therefore

$$\lambda_{\min}^{-1}(BA) = \lambda_{\max}((BA)^{-1}) = \sup_{u \in \mathcal{V}\setminus\{0\}} \frac{((BA)^{-1}u, u)_A}{(u, u)_A} = \sup_{u \in \mathcal{V}\setminus\{0\}} \frac{(B^{-1}u, u)}{(Au, u)}.$$

Therefore $(B^{-1}v, v) \le c_1(Av, v)$ implies $\lambda_{\min}(BA) \ge c_1^{-1}$. Similarly $(B^{-1}v, v) \ge c_0(Av, v)$ implies $\lambda_{\max}(BA) \le c_0^{-1}$. The estimate of $\kappa(BA)$ then follows. □

By Lemma 1 and Theorem 1, we have the following result.

Corollary 1. *Let $\tilde{B} : \tilde{\mathcal{V}} \to \tilde{\mathcal{V}}$ be SPD and $B = \Pi\tilde{B}\Pi^t$. If*

$$c_0(Av, v) \le \inf_{\Pi\tilde{v}=v} (\tilde{B}^{-1}\tilde{v}, \tilde{v}) \le c_1(Av, v) \quad \textit{for all } v \in \mathcal{V}, \tag{8}$$

then $\kappa(BA) \le c_1/c_0$.

Remark 1. In literature, e.g. the fictitious space lemma of Nepomnyaschikh [1992], the condition (8) is usually decomposed as the following two conditions.

1. For any $v \in \mathcal{V}$, there exists a $\tilde{v} \in \tilde{V}$, such that $\Pi\tilde{v} = v$ and $\|\tilde{v}\|_{\tilde{B}^{-1}}^2 \le c_1\|v\|_A^2$.
2. For any $\tilde{v} \in \tilde{\mathcal{V}}$, $\|\Pi\tilde{v}\|_A^2 \le c_0^{-1}\|\tilde{v}\|_{\tilde{B}^{-1}}^2$.

3 Auxiliary Spaces of Product Type

Let $\mathcal{V}_i \subseteq \mathcal{V}$, $i = 0, \ldots, J$, be subspaces of $\mathcal{V}$. If $\mathcal{V} = \sum_{i=0}^{J} \mathcal{V}_i$, then $\{\mathcal{V}_i\}_{i=0}^{J}$ is called a *space decomposition* of $\mathcal{V}$. Then for any $u \in \mathcal{V}$, there exists a decomposition $u = \sum_{i=0}^{J} u_i$. Since $\sum_{i=0}^{J} \mathcal{V}_i$ is not necessarily a direct sum, decompositions of u are in general not unique.

We introduce the inclusion operator $I_i : \mathcal{V}_i \to \mathcal{V}$, the projection operator $Q_i : \mathcal{V} \to \mathcal{V}_i$ in the $(\cdot,\cdot)$ inner product, the projection operator $P_i : \mathcal{V} \to \mathcal{V}_i$ in the $(\cdot,\cdot)_A$ inner product, and $A_i = A|_{\mathcal{V}_i}$. It can be easily verified $Q_i A = A_i P_i$ and $Q_i = I_i^t$.

Given a space decomposition $\mathcal{V} = \sum_{i=0}^{J} \mathcal{V}_i$, we construct an auxiliary space of product type $\tilde{\mathcal{V}} = \mathcal{V}_0 \times \mathcal{V}_1 \times ... \times \mathcal{V}_J$, with the inner product $(\tilde{u}, \tilde{v}) := \sum_{i=0}^{J} (u_i, v_i)$. We define $\Pi : \tilde{\mathcal{V}} \to \mathcal{V}$ as $\Pi \tilde{u} = \sum_{i=0}^{J} u_i$. In operator form $\Pi = (I_0, I_1, \cdots, I_J)$. Since $\mathcal{V} = \sum_{i=0}^{J} \mathcal{V}_i$, the operator Π is surjective.

Let $R_i : \mathcal{V}_i \to \mathcal{V}_i$ be nonsingular operators, often known as smoothers, approximating A_i^{-1}. Define a diagonal matrix of operators $\tilde{R} = \operatorname{diag}(R_0, R_1, \cdots, R_J) : \tilde{\mathcal{V}} \to \tilde{\mathcal{V}}$ which is non-singular. An additive preconditioner is defined as

$$B_a = \Pi \tilde{R} \Pi^t = \sum_{i=0}^{J} I_i R_i I_i^t = \sum_{i=0}^{J} I_i R_i Q_i. \tag{9}$$

Applying Theorem 1, we obtain the following identity for preconditioner B_a.

Theorem 2. *If R_i is SPD on $\mathcal{V}_i$ for $i = 0, \ldots, J$, then B_a defined by (9) is SPD on $\mathcal{V}$. Furthermore*

$$(B_a^{-1} v, v) = \inf_{\sum_{i=0}^{J} v_i = v} \sum_{i=0}^{J} (R_i^{-1} v_i, v_i). \tag{10}$$

To define a multiplicative preconditioner, we introduce the operator $\tilde{A} = \Pi^t A \Pi$. By direct computation, the entry $\tilde{a}_{ij} = Q_i A I_j = A_i P_i I_j$. In particular $\tilde{a}_{ii} = A_i$. The symmetric operator $\tilde{A}$ may be singular with nontrivial kernel $\ker(\Pi)$, but the diagonal of $\tilde{A}$ is always non-singular. Write $\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U}$ where $\tilde{D} = \operatorname{diag}(A_0, A_1, \cdots, A_j)$, $\tilde{L}$ and $\tilde{U}$ are lower and upper triangular matrix of operators, and $\tilde{L}^t = \tilde{U}$. Note that the operator $\tilde{R}^{-1} + \tilde{L}$ is invertible. We define $\tilde{B}_m = (\tilde{R}^{-1} + \tilde{L})^{-1}$ and its symmetrization as

$$\overline{\tilde{B}}_m = \tilde{B}_m^t + \tilde{B}_m - \tilde{B}_m^t \tilde{A} \tilde{B}_m = \tilde{B}_m^t (\tilde{B}_m^{-t} + \tilde{B}_m^{-1} - \tilde{A}) \tilde{B}_m. \tag{11}$$

The symmetrizied multiplicative preconditioner is defined as

$$\overline{B}_m := \Pi \overline{\tilde{B}}_m \Pi^t. \tag{12}$$

We define the diagonal matrix of operators $\overline{\tilde{R}} = \operatorname{diag}(\overline{R}_0, \overline{R}_1, \cdots, \overline{R}_J)$, where, for each $R_i, i = 0, \cdots, J$, its symmetrization is

$$\overline{R}_i = R_i^t (R_i^{-t} + R_i^{-1} - A_i) R_i.$$

Substituting $\tilde{B}_m^{-1} = \tilde{R}^{-1} + \tilde{L}$, and $\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U}$ into (11), we have

$$\overline{\tilde{B}}_m = (\tilde{R}^{-t} + \tilde{L}^t)^{-1}(\tilde{R}^{-t} + \tilde{R}^{-1} - \tilde{D})(\tilde{R}^{-1} + \tilde{L})^{-1} \\ = (\tilde{R}^{-t} + \tilde{L}^t)^{-1}\tilde{R}^{-t}\overline{\tilde{R}}\tilde{R}^{-1}(\tilde{R}^{-1} + \tilde{L})^{-1}. \tag{13}$$

It is obvious that $\overline{\tilde{B}}_m$ is symmetric. To be positive definite, from (13), it suffices to assume $\overline{\tilde{R}}$, i.e. each $\overline{R}_i$, is symmetric and positive definite which is equivalent to the operator $I - \overline{R}_i A_i$ is a contraction and so is $I - R_i A_i$.

(A) $\|I - R_i A_i\|_{A_i} < 1$ for each $i = 0, \cdots, J$.

Theorem 3. *Suppose (A) holds. Then $\overline{B}_m$ defined by (12) is SPD, and*

$$(\overline{B}_m^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|R_i^t (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i)\|^2_{\overline{R}_i^{-1}}. \tag{14}$$

In particular, for $R_i = A_i^{-1}$, we have

$$(\overline{B}_m^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|P_i \sum_{j=i+1}^J v_j\|_A^2. \tag{15}$$

Proof. Let

$$\mathscr{M} = \tilde{R}^{-t} + \tilde{R}^{-1} - \tilde{D} = \tilde{R}^{-t}\overline{\tilde{R}}\tilde{R}^{-1}, \quad \mathscr{U} = \tilde{D} + \tilde{U} - \tilde{R}^{-1}, \quad \mathscr{L} = \mathscr{U}^t.$$

then $\tilde{R}^{-1} + \tilde{L} = \mathscr{M} + \mathscr{L}$ and $\tilde{A} = \mathscr{M} + \mathscr{L} + \mathscr{U}$. We then compute

$$\overline{\tilde{B}}_m^{-1} = (\tilde{R}^{-1} + \tilde{L})(\tilde{R}^{-t} + \tilde{R}^{-1} - \tilde{D})^{-1}(\tilde{R}^{-t} + \tilde{L}^t) \\ = (\mathscr{M} + \mathscr{L})\mathscr{M}^{-1}(\mathscr{M} + \mathscr{U}), \\ = \tilde{A} + \mathscr{L}\mathscr{M}^{-1}\mathscr{U} \\ = \tilde{A} + \Big[\tilde{R}^t(\tilde{D} + \tilde{U} - \tilde{R}^{-1})\Big]^t \overline{\tilde{R}}^{-1} \Big[\tilde{R}^t(\tilde{D} + \tilde{U} - \tilde{R}^{-1})\Big].$$

For any $\tilde{v} \in \overline{\mathcal{V}}$, denoted by $v = \Pi\tilde{v}$, we have

$$(\tilde{A}\tilde{v}, \tilde{v}) = (\Pi^t A \Pi \tilde{v}, \tilde{v}) = (A\Pi\tilde{v}, \Pi\tilde{v}) = \|v\|_A^2.$$

Using component-wise formula of $\tilde{R}^t(\tilde{D} + \tilde{U} - \tilde{R}^{-1})\tilde{v}$, e.g. $((\tilde{D} + \tilde{U})\tilde{v})_i = \sum_{j=i}^J \tilde{a}_{ij} v_j = \sum_{j=i}^J A_i P_i v_j$, we get

$$(\mathscr{M}^{-1}\mathscr{U}\tilde{v}, \mathscr{U}\tilde{v}) = \sum_{i=0}^J \|R_i^t (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i)\|^2_{\overline{R}_i^{-1}}.$$

The identity (14) then follows. □

If we further introduce the operator $T_i = R_i A_i P_i : \mathcal{V} \to \mathcal{V}_i$, then $T_i^* = R_i^t A_i P_i$, $\overline{T}_i := T_i + T_i^* - T_i^* T_i = \overline{R}_i A_i P_i$, and $(\overline{R}_i^{-1} w_i, w_i) = (A_i \overline{T}_i^{-1} w_i, w_i) = (\overline{T}_i^{-1} w_i, w_i)_A$. Here $\overline{T}_i^{-1} := (\overline{T}_i|_{\mathcal{V}_i})^{-1} : \mathcal{V}_i \to \mathcal{V}_i$ is well defined due to the assumption (A). We then recovery the original formulation in Xu and Zikatanov [2002]

$$(\overline{B}_m^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J (\overline{T}_i^{-1} T_i^* w_i, T_i^* w_i)_A,$$

with $w_i = \sum_{j=i}^J v_j - T_i^{-1} v_i$. With these notation and $w_i = \sum_{k>i} v_k$, we can also use (13) to recovery the formula in Cho et al. [2008]

$$(\overline{B}_m^{-1} v, v) = \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J (\overline{T}_i^{-1} (v_i + T_i^* w_i), v_i + T_i^* w_i)_A.$$

4 Method of Subspace Correction

In this section, we view the method of subspace correction Xu [1992] as an auxiliary space method and provide identities for the convergence analysis.

Let $\mathcal{V} = \sum_{i=0}^J \mathcal{V}_i$ be a space decomposition of $\mathcal{V}$. For a given residual $r \in \mathcal{V}$, we let $r_i = Q_i r$ be the restriction of the residual to the subspace $\mathcal{V}_i$ and solve the residual equation in the subspaces

$$A_i e_i = r_i \quad \text{approximately by} \quad \hat{e}_i = R_i r_i.$$

Subspace corrections $\hat{e}_i$ are assembled together to give a correction in the space $\mathcal{V}$. There are two basic ways to assemble subspace corrections.

Parallel Subspace Correction (PSC)

This method is to do the correction on each subspace in parallel. In operator form, it reads

$$u^{k+1} = u^k + B_a(f - Au^k), \tag{16}$$

where

$$B_a = \sum_{i=0}^J I_i R_i I_i^t = \sum_{i=0}^J I_i R_i Q_i. \tag{17}$$

Thus PSC is also called additive methods. Note that the formula (17) and (9) are identical and thus identity (10) is useful to estimate $\kappa(B_a A)$.

Successive Subspace Correction (SSC)

This method is to do the correction in a successive way. In operator form, it reads

$$v^0 = u^k, \quad v^{i+1} = v^i + R_i Q_i (f - Av^i), i = 0, \ldots, N, \quad u^{k+1} = v^{J+1}. \tag{18}$$

For each subspace problem, we have the operator form $v^{i+1} = v^i + R_i(f - Av^i)$, but it is not easy to write out the iterator for the space $\mathcal{V}$. We define B_m such that the error operator

$$I - B_m A = (I - R_J Q_J A)(I - R_{J-1} Q_{J-1} A)...(I - R_0 Q_0 A).$$

Therefore SSC is also called multiplicative method. We now derive a formulation of B_m from the auxiliary space method.

In the sequel, we consider the SSC applied to the space decomposition $\mathcal{V} = \sum_{i=0}^{J} \mathcal{V}_i$ with smoothers $R_i, i = 0, \cdots, J$. Recall that $\tilde{\mathcal{V}} = \mathcal{V}_0 \times \mathcal{V}_1 \times ... \times \mathcal{V}_J$ and $\tilde{A} = \Pi^t A \Pi$. Let $\tilde{f} = \Pi^t f$. Following Griebel and Oswald [1995], we view SSC for solving $Au = f$ as a Gauss-Seidel type method for solving $\tilde{A}\tilde{u} = \tilde{f}$.

Lemma 2. *Let $\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U}$ and $\tilde{B} = (\tilde{R}^{-1} + \tilde{L})^{-1}$. Then SSC for $Au = f$ with smothers R_i is equivalent to the Gauss-Seidel type method for solving $\tilde{A}\tilde{u} = \tilde{f}$:*

$$\tilde{u}^{k+1} = \tilde{u}^k + \tilde{B}(\tilde{f} - \tilde{A}\tilde{u}^k). \tag{19}$$

Proof. By multiplying $\tilde{R}^{-1} + \tilde{L}$ to (19) and rearranging the term, we have

$$\tilde{R}^{-1}\tilde{u}^{k+1} = \tilde{R}^{-1}\tilde{u}^k + \tilde{f} - \tilde{L}\tilde{u}^{k+1} - (\tilde{D} + \tilde{U})\tilde{u}^k.$$

Multiplying $\tilde{R}$, we obtain

$$\tilde{u}^{k+1} = \tilde{u}^k + \tilde{R}(\tilde{f} - \tilde{L}\tilde{u}^{k+1} - (\tilde{D} + \tilde{U})\tilde{u}^k),$$

and its component-wise formula, for $i = 0, \cdots, J$

$$\begin{aligned} u_i^{k+1} &= u_i^k + R_i(f_i - \sum_{j=0}^{i-1} \tilde{a}_{ij} u_j^{k+1} - \sum_{j=i}^{J} \tilde{a}_{ij} u_j^k) \\ &= u_i^k + R_i Q_i (f - A\sum_{j=0}^{i-1} u_j^{k+1} - A\sum_{j=i}^{J} u_j^k). \end{aligned}$$

Let

$$v^{i-1} = \sum_{j=0}^{i-1} u_j^{k+1} + \sum_{j=i}^{J} u_j^k.$$

Noting that $v^i - v^{i-1} = u_i^{k+1} - u_i^k$, we then get, for $i = 1, \cdots, J+1$

$$v^i = v^{i-1} + R_i Q_i (f - Av^{i-1}),$$

which is exactly the correction on $\mathcal{V}_i$; see (18). $\square$

Lemma 3. *For SSC, we have*

$$B_m = \Pi \tilde{B}_m \Pi^t \quad \textit{and} \quad \overline{B}_m = \Pi \overline{\tilde{B}}_m \Pi^t.$$

Proof. Let $u^k = \Pi \tilde{u}^k$. Applying Π to (19) and noting that

$$\tilde{f} = \Pi^t f, \quad \text{and} \quad \tilde{A}\tilde{u}^k = \Pi^t A u^k,$$

we then get

$$u^{k+1} = u^k + \Pi \tilde{B} \Pi^t (f - Au^k).$$

Therefore $B_m = \Pi \tilde{B}_m \Pi^t$. The formulae for $\overline{B}_m$ is obtained similarly. □

Combining Lemma 3, (5), (6), and Theorem 3, we obtain the X-Z identity.

Theorem 4 (X-Z identity). *Suppose assumption (A) holds. Then*

$$\|(I - R_J Q_J A)(I - R_{J-1} Q_{J-1} A)...(I - R_0 Q_0 A)\|_A^2 = 1 - \frac{1}{1 + c_0}, \tag{20}$$

where

$$c_0 = \sup_{\|v\|_A = 1} \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|R_i^t (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i)\|_{\overline{R}_i^{-1}}^2.$$

In particular, for $R_i = A_i^{-1}$,

$$\|(I - P_J)(I - P_{J-1}) \cdots (I - P_0)\|_A^2 = 1 - \frac{1}{1 + c_0}, \tag{21}$$

where

$$c_0 = \sup_{\|v\|_A = 1} \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|P_i \sum_{j=i+1}^J v_j\|_A^2.$$

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