Ch8 Symplectic Manifolds

37 Symplectic structure on manifolds

M^{2n}: manifold. A closed and non-dengenrate 2-form ω on M^{2n}:

$d\omega = 0$, $\forall \xi \neq 0, \exists \eta: \omega(\xi,\eta) \neq 0$

is a symplectic structure. (M^{2n}, ω): symplectic manifold

V: n-dim manifold. T^*V: cotangent bundle. 2n-dim manifold.

$$f: \underset{T_q^*V}{(p}, \underset{V}{q)} \to q \quad \text{projection}$$

(T^*V, ω) $\quad \omega = dp \wedge dq = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n = d(p\,dq)$.

$\dot{q} \in T_qV$, $p = \frac{\partial L}{\partial \dot{q}} \in T_q^*V$. Phase space (p,q) is in cotangent bundle.

Symplectic structure ω^s introduce an isomorphism : $T_xM \to T_x^*M$

$\xi \to \omega_\xi$ s.t. $\langle \omega_\xi, \eta \rangle := \omega^s(\eta, \xi)$. In matrix form, $\omega_\xi = -J\xi$

The inverse map is denoted by $I: T_x^*M \to T_xM$

$H: M^{2n} \to \mathbb{R}$. $dH \in \Lambda^1(M^{2n})$, $IdH \in \mathcal{V}$: vector field on M.

IdH is called a hamiltonian vector field.

Example. $M^{2n} = \mathbb{R}^{2n} = \{(p,q)\}$. (M^{2n}, J). $IdH = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) = -J\nabla H$

38 Hamiltonian phase flows

(M^{2n}, ω^s) : symplectic manifold, $H: M^{2n} \to \mathbb{R}$

$g^t: M^{2n} \to M^{2n}$, $\frac{d}{dt}\Big|_{t=0} g^t(x) = I\,dH(x)$

Example. $\frac{dx}{dt} = -J\nabla H(x)$ $x(t)$ is the solution of Hamiltonian system.

Theorem. A hamiltonian phase flow preserves the symplectic structure

Pf. See my lecture notes.

Integral invariants of the map $g: M \to M$ is a k-form ω s.t.

$\int_{g(c)} \omega = \int_c \omega$. In view of pull back, it is $g^* \omega = \omega$.

g : hamiltonian phase flow. Then $g^* \omega^s = \omega^s$

$(\omega^s)^k$ is also integral invariant for $k = 1, 2, \cdots, n$.

$g: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is canonical if ω^s is an integral invariant w.r.t. g.
or called symplectic mapping.

$\omega \in \Lambda^k$ is called a relative integral invariant of map g if
$\int_{g(c)} \omega = \int_c \omega$ for every closed k-chain c. By Stokes' theorem, $d\omega$
is integral invariant. **Example.** $\omega = p\,dq$, $d\omega = dp \wedge dq$.

39 The Lie algebra of vector fields

Vector space V. A product $[\cdot,\cdot]: V\times V \to V$.

Lie algebra. The product is skew-symmetric and satisfies the Jacobi identity: $[[A,B],C]+[[B,C],A]+[[C,A],B]=0$.

Example. $\mathbb{R}^3$. $[u,v]=u\times v$.

Use $(A\times B)\times C = B(A\cdot C)-A(B\cdot C)$ to verify Jacobi identity.

Example. $\mathbb{R}^{n\times n}$: squared matrix. $[A,B]=AB-BA$.

Vector fields $A: M\to TM$, $A(x)\in TM_x$

Lie derivative. In $\mathbb{R}^n$ case, $L_A\varphi = A\cdot\nabla\varphi$

For manifolds, $x+\Delta t A$ may not still on M. So we first

$$= \lim_{\Delta t\to 0}\frac{\varphi(x+\Delta t A)-\varphi(x)}{|\Delta t A|}$$

solve ODE to define a flow $\frac{d}{dt}\Big|_{t=0} A^t(x) = A(x)$.

Then $(L_A\varphi)(x) = \frac{d}{dt}\Big|_{t=0}\varphi(A^t(x))$

Poisson bracket. $C=[A,B]$ s.t. $L_C = L_B L_A - L_A L_B$

Although $L_A L_B$, $L_B L_A$ are second order diff operators, their difference is a first order diff operator.

$L_A\varphi = A\cdot\nabla\varphi$, $L_B(L_A\varphi) = (B, \nabla(A\cdot\nabla\varphi)) = (B^T\nabla A)\nabla\varphi + (B,A)_{\nabla^2\varphi}$

$$L_A(L_B\varphi) = A\cdot\nabla(B\cdot\nabla\varphi) = (A^T\nabla B)\nabla\varphi + (A,B)_{\nabla^2}\varphi$$

The second order part: $(B,A)_{\nabla^2}\varphi = (A,B)_{\nabla^2}\varphi$

As a first order diff operator, $\exists\, C$, s.t. $L_C\varphi = L_B L_A\varphi - L_A L_B\varphi$

$$C = B^T\nabla A - A^T\nabla B$$

Jacobi identity $L_{[[A,B],C]} = L_C L_{[A,B]} - L_{[A,B]} L_C$

$$= L_C L_B L_A - L_C L_A L_B - L_B L_A L_C + L_A L_B L_C$$

Commutativity of flows.

A, B: two vector fields. $\qquad [A,B]=0$

A^t, B^s: corresponding flows. $\qquad A^t B^s = B^s A^t$

Relation $L_{[A,B]}\varphi = \frac{\partial^2}{\partial s\partial t}\Big|_{(0,0)}\left(\varphi(A^tB^sx) - \varphi(B^sA^tx)\right) \quad \forall\varphi$

$A^tB^s = B^sA^t \Rightarrow [A,B]=0$ is based on the above identity.

If $[A,B]=0$, then $\varphi(A^tB^sx) - \varphi(B^sA^tx) = o(s^2+t^2)$.

Show $\varphi(A^tB^sx) = \varphi(B^sA^tx)$ for sufficiently small s and t.

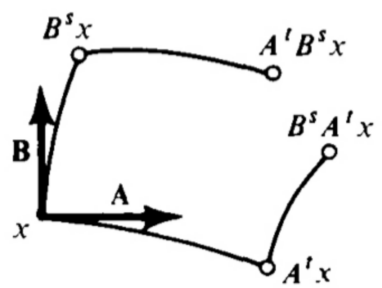

Figure 169 Non-commutative flows

40 The Lie algebra of hamiltonian functions

(M^{2n}, ω^S): symplectic manifold.

$H: M^{2n} \to \mathbb{R}$. $dH \in \Lambda^1(M^{2n})$ Through ω^S, IdH is a vector field so that $\langle IdH, \xi \rangle = \omega^S(IdH, \xi)$.

$\frac{dx}{dt} = IdH(x)$ defines a symplectic mapping: $x_0 \xrightarrow{g_H^t} x(t)$

Example. $(\mathbb{R}^{2n}, J)$. $(IdH, \eta) = (\eta, IdH)_J = (\eta, JIdH)$

On the other hand, $\langle dH, \eta \rangle = (\nabla H, \eta)$. So $JIdH = \nabla H$, $IdH = -J\nabla H$.

Poisson bracket $[F,H](x) = \frac{d}{dt}\Big|_{t=0} F(g_H^t(x))$

Example $(\mathbb{R}^{2n}, J)$ $[F,H] = (\nabla F, \nabla H)_J$

Corollary. F is a first integral of the phase flow g_H^t if and only if $[F,H] = 0$, i.e. $(\nabla F, \nabla H)_J = 0$.

Theorem. (M^{2n}, ω^S): a symplectic manifold

If H is a first integral (invariant) of the flow g_F^t, then F is a first integral of the flow g_H^t.

Proof. Equivalent to $[F,H] = 0$.

Jacobi identity $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$.

A : function, dA : 1-form, IdA : vector field, denoted by $\vec{A}$

Then $\overrightarrow{[A,B]} = [\vec{A}, \vec{B}]$

Pf. $[\vec{A}, \vec{B}] = \vec{B}^T \nabla \vec{A} - \vec{A}^T \nabla \vec{B}$, $\quad \vec{A} = J\nabla A$, $\vec{B} = J\nabla B$.

$$= (J\nabla B)^T \nabla (J\nabla A) - (J\nabla A)^T \nabla (J\nabla A) \qquad J^T J = I$$

$$= (\nabla B)^T \nabla^2 A - (\nabla A)^T \nabla^2 B.$$

$[A,B] = (\nabla A, J\nabla B)$, $\quad \overrightarrow{[A,B]} = J\nabla(\nabla A, \nabla B)_J$

$$= J(\nabla^2 A, \nabla B)_J + J(\nabla A, \nabla^2 B)_J$$

$$= (J\nabla B)^T J \nabla^2 A - J(\nabla^2 B, \nabla A)_J$$

$$= (\nabla B)^T \nabla^2 A - (\nabla A)^T \nabla^2 B.$$

Theorem (Poisson's) If F_1, F_2 are two invariant (the first integrals) of a Hamiltonian system, so is $[F_1, F_2]$.

Proof. Use Jacobi identity $[[F_1,F_2],H] = [F_1,[F_2,H]] + [F_2,[H,F_1]] = 0$.

* Hamiltonian vector fields ($J\nabla A$) on a symplectic manifold form a sub-algebra of the Lie algebra of all vector fields.

closed to $[\cdot,\cdot]$ product

* The first integrals of a hamiltonian flow form a sub-algebra.

Jacobi identity $[[A,B],C]+[[B,C],A]+[[C,A],B]=0.$

Proof. $[A,B]=(\nabla A,\nabla B)_J$,

$$[[A,B],C]=(\nabla(\nabla A,\nabla B)_J,\nabla C)_J=((\nabla^2 A,J\nabla B),J\nabla C)+((\nabla A,J\nabla^2 B),J\nabla C)$$
$$=(J\nabla B,J\nabla C)_{\nabla^2 A}-(J\nabla A,J\nabla C)_{\nabla^2 B}.$$

Similarly $[[B,C],A]=(J\nabla C,J\nabla A)_{\nabla^2 B}-(J\nabla B,J\nabla A)_{\nabla^2 C}$

and $[[C,A],B]=(J\nabla A,J\nabla B)_{\nabla^2 C}-(J\nabla C,J\nabla B)_{\nabla^2 A}$

As Hessian is symmetric, sum to get 0.