Ch 5 Oscillations

22. Linearization.

$\frac{dx}{dt} = f(x)$, $x \in \mathbb{R}^n$. equilibrium $f(x_0) = 0$

$L = T - U$, $T = \frac{1}{2}(A(q)\dot{q}, \dot{q})$, $U = U(q)$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}, \quad \frac{\partial L}{\partial q} = \frac{\partial T}{\partial q} - \frac{\partial U}{\partial q}$$

$\uparrow$

$\dot{q}_0 = 0$, $\frac{\partial U}{\partial q}(q_0) = 0$.

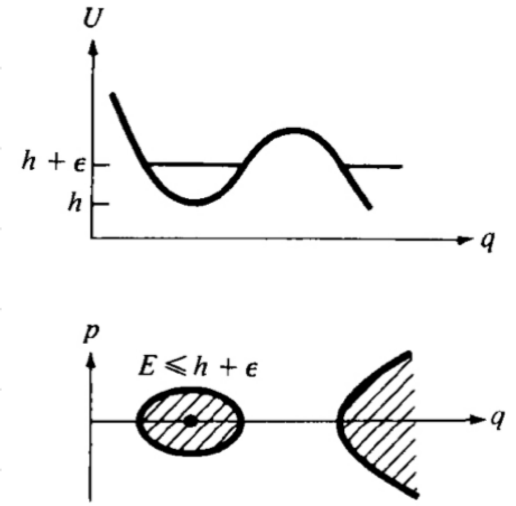


Figure 75 Stable equilibrium position

Th. Equilibrium position of E-L eqn is $\frac{\partial U}{\partial q}(q_0) = 0$, $\dot{q}_0 = 0$.

Th. If q_0 is a strict local minimum of U, then $q = q_0$ is Liapunov stable.

Recall different stability.

1. Liapunov stable. $\forall \varepsilon > 0$, $\exists \delta$, s.t. if $\|x(0) - x_0\| < \delta$, then $\|x(t) - x_0\| < \varepsilon \quad \forall t \geq 0$.

2. Asymptotically stable. Liapunov stable $+ \exists \delta$, s.t. if $\|x(0) - x_0\| < \delta$ then $\lim_{t \to +\infty} \|x(t) - x_0\| = 0$.

3. Exponentially stable. Asymptotically stable $+ \exists \alpha, \beta, \delta > 0$, s.t. if $\|x(0) - x_0\| < \delta$, then $\|x(t) - x_0\| \leq \alpha \|x(0) - x_0\| e^{-\beta t}$, $\forall t \geq 0$.

Proof of Liapunov stability is from the conservation of energy. The flow remains in the region $\{(p,q): E \leqslant U(q_0)+\varepsilon\}$.

Linearization. $\frac{dx}{dt} = f(x) \rightsquigarrow \frac{dy}{dt} = Ay$, $\quad A = \left(\frac{\partial f}{\partial x}\right)(x_0)$

$y \in T\mathbb{R}^n_{x_0}$ $\qquad$ equilibrium position

Stability is determined by the linearized system.

For Lagrange eqn, w.l.o.g. assume $q_0 = 0$ is an equilibrium point, the linearization is the E-L eqn of : A is SPD, B is symmetric

$$L_2 = T_2 - U_2, \quad T_2 = \frac{1}{2}\langle A(0)\dot{q}, \dot{q}\rangle, \quad U_2 = \frac{1}{2}\langle Bq, q\rangle, \quad \text{with } B = \nabla^2 U(0)$$

$$\frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{q}}\right) = \frac{\partial L_2}{\partial q} \rightsquigarrow A\ddot{q} = -Bq.$$

solution q is called small oscillations near an equilibrium position q_0.

23 Small Oscillations

Try to decouple the system $A\ddot{q} = -Bq$.

Let (λ_i, ξ_i) be the generlized eigenvalues of B with respect to A, i.e. $B\xi_i = \lambda_i A\xi_i$. such (λ_i, ξ_i) exists as A is SPD and $\{\xi_i\}$ forms an A-orthogonal basis. Suppose $q = \sum Q_i \xi_i$. Then $A\ddot{q} = Bq$ becomes

$$\ddot{Q}_i = -\lambda_i Q_i, \quad i = 1, 2, \cdots, n.$$

When $\lambda_i = \omega_i^2 > 0$, the solution is $Q = c_1 \cos\omega t + c_2 \sin\omega t$ and
$\xi(t) = \sum_{k=1}^{n} Q_k(t)\,\xi_k = \mathrm{Re} \sum_{k=1}^{n} c_k e^{i\omega_k t}\,\xi_k$. Here c_k is a complex number.

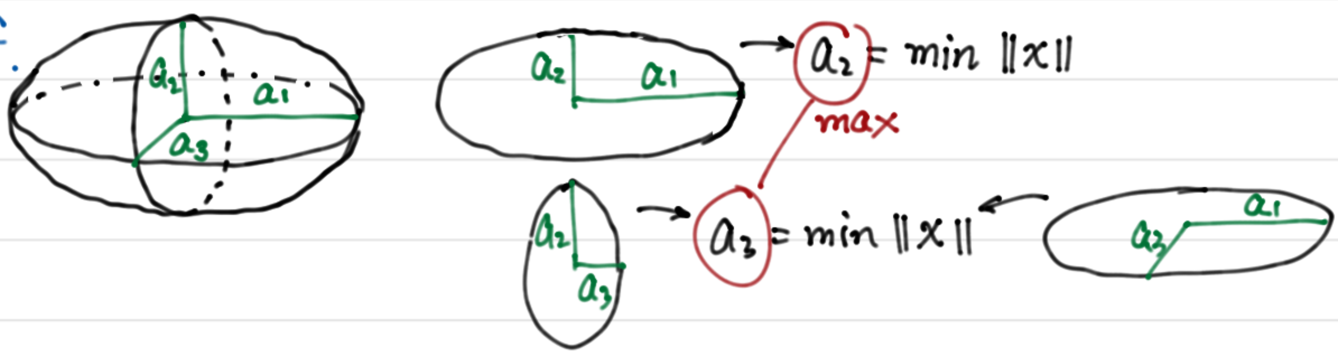
ω: characteristic frequency, $\mathrm{Re}(c_k e^{i\omega_k t})$ characteristic oscillation

24. Behavior of characteristic frequencies

Consider $A, B > 0$, w.l.o.g. assume $A = I$. The level set $(B\xi, \xi) = 1$ defines an ellipsoid in $\mathbb{R}^n$: $\sum \lambda_i Q_i^2 = 1$. The semi-axes $a_i = \frac{1}{\sqrt{\lambda_i}}$.

Theorem (Min-Max property) E: an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $a_k = \max\limits_{\{\mathbb{R}^k\}} \min\limits_{x \in \mathbb{R}^k \cap E} \|x\|$.

Pf.



$S = \mathbb{R}^{n-k+1}$ spanned by $a_k \geq a_{k+1} \cdots \geq a_n$.

For any $\mathbb{R}^k$, as $n-k+1+k = n+1$, it intersects S and let $x^* \in \mathbb{R}^k \cap S$ and x^* is on the ellipsoid. As $x^* \in S$, $\|x^*\| \leq a_k$.

$\min\limits_{x \in \mathbb{R}^k \cap E} \|x\| \leq \|x^*\| \leq a_k$. The min is acheived when $\mathbb{R}^k$ spanned by $a_1 \geq a_2 \geq \cdots \geq a_k$. #.

Based on this min-max property, we have the following propositions.

Theorem 2. $E: a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. $E' \subseteq E$ and they have the same
$E': a_1' \geqslant a_2' \geqslant \cdots \geqslant a_n'$ center.

Then $a_1 \geqslant a_1'$, $a_2 \geqslant a_2'$, $\cdots$, $a_n \geqslant a_n'$.

Pf. $a_k' = \max\limits_{\{\mathbb{R}^k\}} \min\limits_{x \in \mathbb{R}^k \cap E'} \|x\| \leqslant \max\limits_{\{\mathbb{R}^k\}} \min\limits_{x \in \mathbb{R}^k \cap E} \|x\| = a_k$ since $E' \subseteq E$. #

Theorem 4. $E: a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n-1} \geqslant a_n$. E' is the cross-section of E
$E': a_1' \geqslant a_2' \geqslant \cdots \geqslant a_{n-1}'$ by a hyperplane $\mathbb{R}^{n-1}$ through the
center. Then

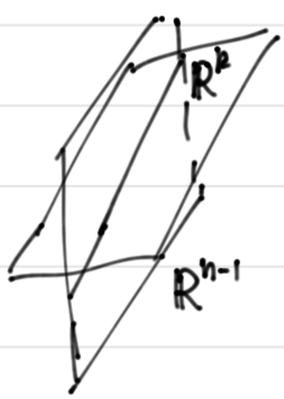
$$a_1 \geqslant a_1' \geqslant a_2 \geqslant a_2' \cdots \geqslant a_{n-1} \geqslant a_{n-1}' \geqslant a_n.$$

Pf. $a_k' \leqslant a_k$ ✓. To prove $a_k' \geqslant a_{k+1}$, any $S = \mathbb{R}^{k+1} \cap \mathbb{R}^{n-1}$.

$$a_k' = \max_{\{\mathbb{R}^k \subset \mathbb{R}^{n-1}\}} \min_{x \in \mathbb{R}^k \cap E'} \|x\| \geqslant \max_{\{\mathbb{R}^{k+1} \subset \mathbb{R}^n\}} \min_{x \in \mathbb{R}^{k+1} \cap E'} \|x\|$$

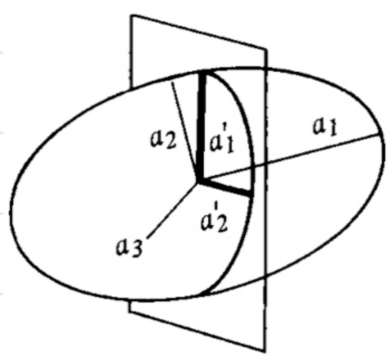
$\downarrow E' \subset E$

$$\geqslant \max_{\{\mathbb{R}^{k+1} \subset \mathbb{R}^n\}} \min_{x \in \mathbb{R}^{k+1} \cap E} \|x\| = a_{k+1}$$



$\cup$

$$\mathbb{R}^{k+1} \cap E' = \underbrace{\mathbb{R}^{k+1} \cap \mathbb{R}^{n-1}}_{\dim \geqslant k} \cap E'$$



#.

Def. A system with the same kinetic energy, and a new potential energy U' is called more rigid if $U' = \frac{1}{2}(B'q, q) \geqslant \frac{1}{2}(Bq, q) = U, \forall q.$

Theorem. Under an increase in rigidity, all the characteristic frequencies are increased.

$U: \quad \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_n \qquad$ Then $\omega_1 \leqslant \omega_1', \ \omega_2 \leqslant \omega_2', \cdots, \ \omega_n \leqslant \omega_n'.$

$U': \quad \omega_1' \leqslant \omega_2' \leqslant \cdots \leqslant \omega_n'$

Relation to the ellipsoid $E: \ (Bq, q) = 1. \qquad \omega_i = \frac{1}{a_i}$

$E': \ (B'q, q) = 1. \qquad \omega_i' = \frac{1}{a_i'}$

$$B' \geqslant B \Rightarrow E' \subseteq E \Rightarrow a_i' \leqslant a_i \Leftrightarrow \omega_i' \geqslant \omega_i.$$

Behavior of ω_i under constraint. T, U: defined on $\mathbb{R}^n$.

consider the system: T, U restrict to $\mathbb{R}^{n-1}$ (a linear constraint)

the frequency: ω_i'.

Theorem. The characteristic frequencies of the system with a constraint separate the characteristic frequencies of the original system: $\omega_1 \leqslant \omega_1' \leqslant \omega_2 \leqslant \omega_2' \leqslant \cdots \leqslant \omega_{n-1} \leqslant \omega_{n-1}' \leqslant \omega_n.$

Homework 1 & 2

Problem. 2nd cosmic velocity. a stone with v_2 can fly infinitely far from the surface of the earth.

Sol. Conservation of energy. $E = T + U$, $T = \frac{1}{2} m v^2$, $U = -\frac{GMm}{r}$

earth

$$E_0 = T_0 + U_0 = \frac{1}{2} m v_0^2 - \frac{GMm}{r_0}$$

$$E_\infty = T_\infty + U_\infty = 0 \qquad \text{as } r \to +\infty,\ U_\infty = 0$$

$$v \to 0,\quad T_\infty = 0$$

$$E_0 = E_\infty = 0$$

$$\Rightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$$, G: gravation constant, M: mass of earth, r_0: radius of earth.

$$\approx 11.2 \text{ km/s}$$

Problem. E_0 is a local minimum of U at point ξ. Find the period $T_0 = \lim_{E \to E_0} T(E)$ of small osicallations near ξ.

Sol. $\ddot{x} = -\nabla U(x)$. $\nabla U(x) = \nabla U(\xi) + \left(\nabla^2 U(\xi)(x-\xi), (x-\xi)\right) + o(\|x-\xi\|^2)$

Consider the scalar case. The linearized eqn is

$$\ddot{x} + U''(\xi)(x-\xi) = 0, \qquad x - \xi = c_1 \cos\sqrt{U''(\xi)}\, t + c_2 \sin\sqrt{U''(\xi)}\, t.$$

So the period is $2\pi/\sqrt{U''(\xi)}$.

Problem. Consider the ODE system $\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{\partial U}{\partial x} \end{cases}$.

Assume $\exists\, t_*$, either ① $\lim\limits_{t \to t^*} |x(t)| = +\infty$ or ② $\lim\limits_{t \to t^*} |y(t)| = +\infty$.

By the conservation of energy $E = \frac{1}{2} y^2 + U(x) = \text{const}$. Then $y^2 = 2(E - U) \leqslant 2E$ as $U \geqslant 0$. Namely $|y|$ is bounded, ② impossible.

if $x(t_*) = \infty$, by $t_* - t_1 = \int_{x_1}^{x(t_*)} \frac{dx}{\sqrt{E-U}} \geqslant \int_{x_1}^{+\infty} \frac{dx}{\sqrt{E}} = +\infty$.

So ① is possible only if $t^* = +\infty$, i.e. the solution exists for all t.

Problem. $\begin{cases} \ddot{x}_1 = -x_1 \\ \ddot{x}_2 = -x_2 \end{cases}$ Let $y_i = \dot{x}_i$. Then the energy is

$$E = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

$$\begin{cases} x_1 = C_1 \cos t + C_2 \sin t \\ y_1 = -C_1 \sin t + C_2 \cos t \end{cases} \qquad \begin{cases} x_2 = C_3 \cos t + C_4 \sin t \\ y_2 = -C_3 \sin t + C_4 \cos t \end{cases}$$

$$C_1 x_1 + C_2 y_1 = (C_1^2 + C_2^2) \cos t \qquad C_3 x_2 + C_4 y_2 = (C_3^2 + C_4^2) \cos t$$

So
$$\frac{C_1}{C_1^2+C_2^2} x_1 + \frac{C_2}{C_1^2+C_2^2} y_1 = \frac{C_3}{C_3^2+C_4^2} x_2 + \frac{C_4}{C_3^2+C_4^2} y_2. \qquad (1)$$

Similarly we can use $\sin t$ to get another relation:

$$\frac{C_2}{C_1^2+C_2^2} x_1 - \frac{C_1}{C_1^2+C_2^2} y_1 = \frac{C_4}{C_3^2+C_4^2} x_2 - \frac{C_3}{C_3^2+C_4^2} y_2. \qquad (2)$$

The intersection of two hyper-planes (1) & (2) is a 2-dimensional plane passing 0.
The intersection of the sphere $E = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$ is a great circle.

Problem. $\begin{cases} x_1(t) = A_1 \sin(t + \varphi_1) \\ x_2(t) = A_2 \sin(\omega t + \varphi_2) \end{cases}$

- **Case 1** $\omega = \frac{m}{n}$. Let $T = 2\pi n$. Then $x_1(t+T) = x_1(t)$ and

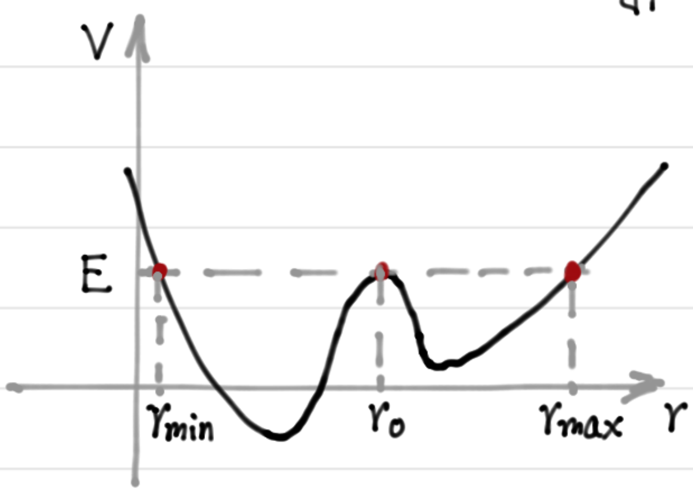
$$x_2(t+T) = A_2 \sin\left(\frac{m}{n}(t + 2\pi n) + \varphi_2\right) = A_2 \sin\left(\frac{m}{n} t + \varphi_2 + 2\pi m\right) = x_2(t)$$

i.e. $\begin{cases} x_1(t+T) = x_1(t) \\ x_2(t+T) = x_2(t) \end{cases}$ The curve $(x_1(t), x_2(t))$ is closed.

Case 2. ω is irrational number. If $\exists\, T$, s.t. $x_i(t+T) = x_i(t)$, $i=1,2$.

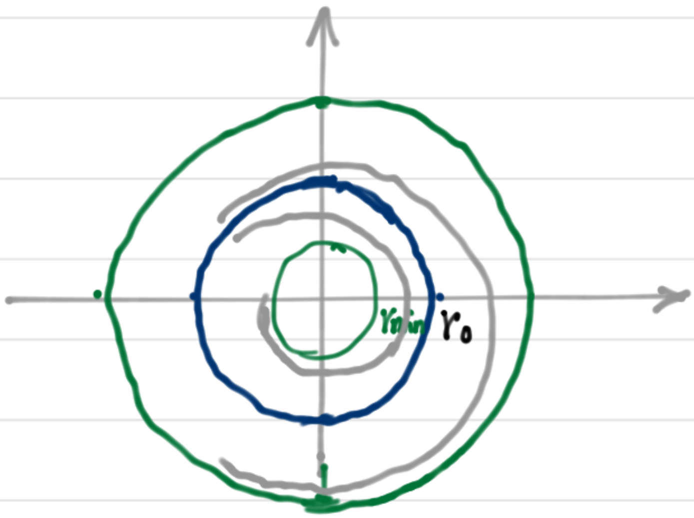
Then $T = 2\pi k$ from $x_1(t+T) = x_1(t)$ and $\omega T = 2\pi l$ from $x_2(t+T) = x_2(t)$. So $\omega = \frac{l}{k}$ should be a rational number.

Problem (p36). Consider a typical energy near a local maximum

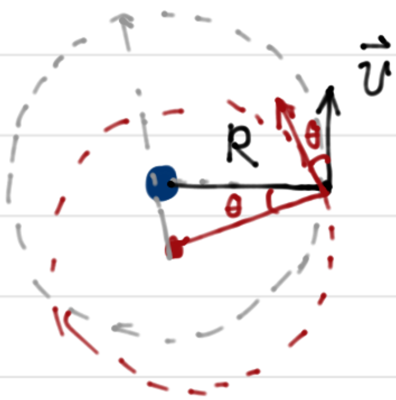


consider initial condition $r(0) = r_0$ and $\dot{r}(0) = 0$. Then the trajectory will be the circle $r(t) \equiv r_0$.

But with a slight perturbation of $r(0)$, the trajectory will remain in one of the valley, i.e. $r_{min} \leq r(t) \leq r_0$ or $r_0 \leq r(t) \leq r_{max}$.



Problem P50



It is almost a circle for a trajectory near the equilibrium (circle).
second order in $e \ll 1$.

So the shift of perigee = shift of the center of circles $= R \sin 1^\circ \approx 110$