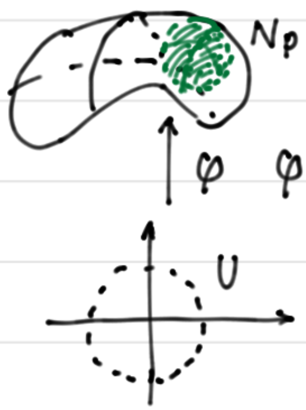
Ch4 Lagrangian Mechanics on Manifolds

18. Differentiable manifolds

A. Definition. M: Hausdorff space (can separate points).

If $\forall p \in M$, $\exists$ neighborhood N of p, s.t. $\phi: U \to N$ is isomorphism, where U is an open set of $\mathbb{R}^n$, then we call M is an n-dimensional topological manifold.

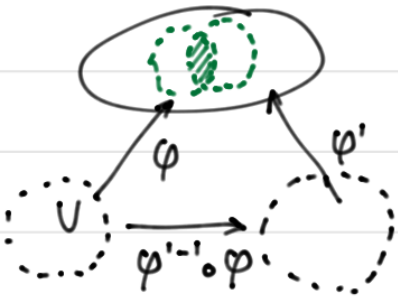
Local chart / local coordinate:



φ is a local parametrization

$\varphi^{-1}(p) = (x^1(p), x^2(p), \cdots, x^n(p))$

local coordinate



compatible local charts: $\varphi'^{-1}\varphi: V \to V'$ is differentible

atlas: $\bigcup_\alpha (U_\alpha, \varphi_\alpha)$, $M = \bigcup \varphi_\alpha(U_\alpha)$, $\{(U_\alpha, \phi_\alpha)\}$ are compatible.

A differentible manifold is a class of equivalent atlases.

B Examples number of d.o.f. = dimension

Sphere, configuration spaces, rigid body motion $SO(3)$.

Embedded manifold. $\forall x \in M$, $\exists$ a neighborhood $U \subset \mathbb{R}^n$, and there are $n-k$ functions $f_i : U \to \mathbb{R}$, $i = 1, \cdots, n-k$ s.t. $U \cap M = \{x \in \mathbb{R}^n : f_i(x) = 0$ for $i = 1, \cdots, n-k\}$ and $\nabla f_i(x)$ are linearly independent.

Whitney proves that a separable n-manifold can be always embedded into $\mathbb{R}^{2n+1}$.

C Tangent space.

For embedded manifold, $TM_x = \text{span}\{\nabla f_i(x)\}^{\perp}$

In general, consider a curve $\varphi : \mathbb{R} \to M$, $\varphi(0) = x$. Two curves are equivalent if $\varphi(0) = \psi(0) = x$ and $\lim_{t \to 0} \frac{\varphi(t) - \psi(t)}{t} = 0$ in some chart

Def. A tangent vector $= \{\varphi\}$ equivalent class of curves.

All tangent vectors form a linear/vector space TM_x.

In particular, consider $\varphi(q_i)$, the corresponding tangent vector is ξ_i.

∂x_i or ∂q_i can be interepted as a tangent vector.

D The tangent bundle

$$TM := \bigcup_{x \in M} TM_x$$

Local charts: $(q_1, q_2, \cdots, q_n) \times (\xi_1, \xi_2, \cdots, \xi_n) = (q_1, q_2, \cdots, q_n, \xi_1, \xi_2, \cdots, \xi_n)$

$$\begin{array}{ccc} (q_1, q_2, \cdots, q_n) & & = T_x M \cong \mathbb{R}^n \\ \downarrow \varphi & & \\ M & & \end{array}$$

$p : TM \to M$ natural projection, $p^{-1}(x) = TM_x$: fiber

E. Riemannian manifolds

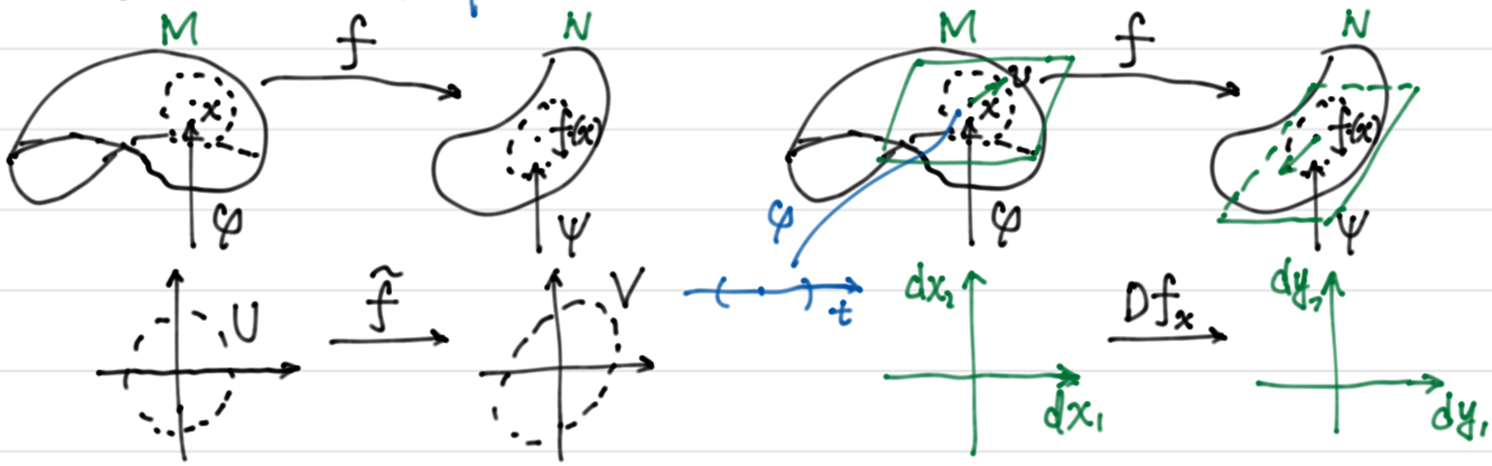
Def. Riemannian metric on TM_x : SPD matrix A

Then $\forall \xi \in TM_x$, $(A\xi, \xi)$ defines a positive definite quadratic form.

$ds^2 = (A\,d\vec{q}, d\vec{q})$ or $(A\,d\vec{x}, d\vec{x})$

$A = A(q)$ is a differentiable function of q.

F. The derivative map



$f : M \to N$

$f_{*x} : TM_x \to TN_{f(x)}$

$f_* : TM \to TN$

$$Df_x\, v = \frac{d}{dt} f(\varphi(t))\Big|_{t=0}, \quad v = \frac{d\varphi}{dt}\Big|_{t=0}$$

$$v \to f_* v$$

$$v = \sum \xi_i \partial x_i, \quad f_* v = \sum \eta_i \partial y_i$$

$$\vec{\eta} = \left(\frac{\partial y}{\partial x}\right)\vec{\xi}, \quad \eta_i = \sum_j \frac{\partial y_i}{\partial x_j}\xi_j .$$

19 Lagrangian Dynamical Systems

A Definition

M : differentiable manifold

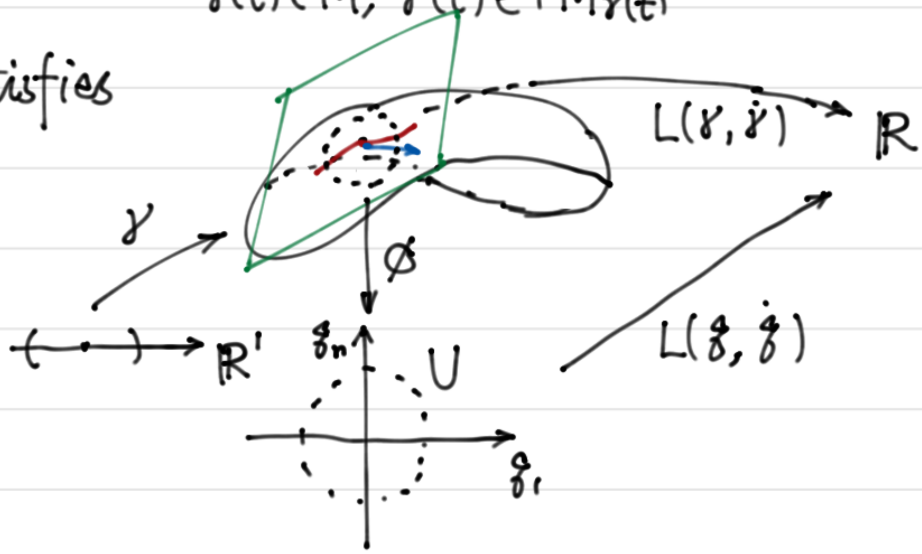
TM : tangent bundle

$L : TM \to \mathbb{R}$ differentiable

$\gamma : \mathbb{R} \to M$ a curve

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))\, dt$$

$\gamma(t) \in M$, $\dot{\gamma}(t) \in TM_{\gamma(t)}$



Theorem Extremal curve satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}.$$

where $L(q, \dot q)$ is the expression of L in the local coordinate $(q, \dot q)$.

B Natural Systems

$L = T - U$, where $T = \frac{1}{2}\langle v, v\rangle$, $v \in TM_x$, $U : M \to \mathbb{R}$

Kinetic energy　　potential energy

C Systems with holonomic constraints is natural

D Procedure for solving problems with constraints

1. Determine the configuration manifold and introduce local coordinates
2. $T = \frac{1}{2}\langle A\dot q, \dot q\rangle$
3. $L = T - U(q)$ and solve Lagrange's equations.

20 Noether's Theorem

Conservation law is from a group translation invariance.

A Formulation

M: a smooth manifold

$L: TM \to \mathbb{R}$ a smooth function

$h: M \to M$ a smooth map

$h^s: M \to M$, $s \in \mathbb{R}$ one-parameter group of diffeomorphism.

Def. (M, L) admits the mapping h if $\forall\, v \in TM$, $L(h_* v) = L(v)$

$v = (q, \xi)$ two components

Noether's Theorem. If (M, L) admits h^s, then $\dfrac{dI(\gamma(t), \dot\gamma(t))}{dt} = 0$, i.e.
$I: TM \to \mathbb{R}$ is a first integral of E-L eqn.

In local coordinate, $I(q, \dot q) = \dfrac{\partial L}{\partial \dot q}\dfrac{dh^s(q)}{ds}\Big|_{s=0}$.

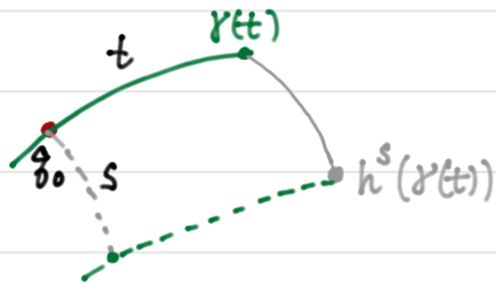
B. Proof.

$h^s: M \to M$

$$q \to h^s(q) = q + s\frac{dh^s(q)}{ds}\Big|_{s=0} + O(s^2)$$

E-L eqn's solution: $\gamma: t \to q(t)$

$q: \mathbb{R} \times \mathbb{R} \to M \qquad q(s,t) = h^s(\gamma(t))$



$L(q(s,t), \dot q(s,t))$ is independent of s.

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial q}\cdot q' + \frac{\partial L}{\partial \dot q}\cdot \dot q' = 0.$$

And if $\gamma(t)$ satisfies E-L, so is $h^s(\gamma(t))$

why?

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot q}(q(s,t), \dot q(s,t))\right) = \frac{\partial L}{\partial q}(q(s,t), \dot q(s,t))$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot q}\right)\cdot q' + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt} q'\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\cdot q'\right) = 0.$$