INF-SUP CONDITIONS

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CONTENTS

1. INF-SUP CONDITIONS

In this section, we shall study the well posedness of the weak formulation of the steady-state Stokes equations

$$-\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1}$$

$$-\operatorname{div} \boldsymbol{u} = 0, \tag{2}$$

where u can be interpreted as the velocity field of an incompressible fluid motion, and p is then the associated pressure, the constant μ is the viscosity coefficient of the fluid. For simplicity, we consider homogenous Dirichlet boundary condition for the velocity, i.e. $\boldsymbol{u}|_{\partial\Omega} = 0$ and $\mu = 1$. The conditions for the well posedness is known as inf-sup condition or Ladyzhenskaya-Babuška-Breezi (LBB) condition.

Multiplying test function $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ to the momentum equation (1) and $q \in L^2(\Omega)$ to the mass equation (2), and applying integration by part for the momentum equation, we obtain the weak formulation of the Stokes equations: Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and a pressure $p \in L^2(\Omega)$ such that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$$

$$-(\operatorname{div} \boldsymbol{u}, q) = 0 \qquad \text{for all } q \in L^2(\Omega).$$

1.1. **Variational problem in the mixed form.** We shall consider an abstract mixed variational problem first. Let $\mathbb{V}$ and $\mathbb{P}$ be two Hilbert spaces. For given $(f, g) \in \mathbb{V}' \times \mathbb{P}'$, find $(u, p) \in \mathbb{V} \times \mathbb{P}$ such that:

$$a(u, v) + b(v, p) = \langle f, v\rangle, \qquad \text{for all } v \in \mathbb{V},$$

$$b(u, q) = \langle g, q\rangle, \qquad \text{for all } q \in \mathbb{P}.$$

Let us introduce linear operators

$$A : \mathbb{V} \mapsto \mathbb{V}', \text{ as } \langle Au, v\rangle = a(u, v)$$

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and

$$B : \mathbb{V} \mapsto \mathbb{P}', B' : \mathbb{P} \mapsto \mathbb{V}', \text{ as } \langle Bv, q\rangle = \langle v, B'q\rangle = b(v,q).$$

Written in the operator form, the problem becomes

$$Au + B'p = f, \tag{3}$$

$$B\,u = g, \tag{4}$$

or

$$\begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{5}$$

We shall study the well posedness of this abstract mixed problem.

1.2. **Babuška theory I.** Let

$$a(\cdot,\cdot) : \mathbb{U} \times \mathbb{V} \mapsto \mathbb{R}$$

be a bilinear form on two Hilbert spaces $\mathbb{U}$ and $\mathbb{V}$. It will introduce two linear operators

$$A : \mathbb{U} \mapsto \mathbb{V}', \text{ and } A' : \mathbb{V} \mapsto \mathbb{U}'$$

$$\text{by} \quad \langle Au, v\rangle = \langle u, A'v\rangle = a(u,v).$$

We consider the operator equation: Given a $f \in \mathbb{V}'$, find $u \in \mathbb{U}$ such that

$$Au = f, \qquad \text{in } \mathbb{V}', \tag{6}$$

or equivalently

$$a(u,v) = \langle f, v\rangle \quad \text{for all } v \in \mathbb{V}.$$

To begin with, we have to assume A is continuous. We skip the subscript of the norm for different spaces. It should be clear from the context.

(C) The bilinear form $a(\cdot,\cdot)$ is continuous in the sense that

$$a(u,v) \leq C\|u\|\|v\|, \quad \text{for all } u \in \mathbb{U}, v \in \mathbb{V}.$$

The minimal constant satisfies the above inequality will be denoted by $\|a\|$. With this condition, it is easy to check that A and A' are bounded operators and $\|A\| = \|A'\| = \|a\|$. The following conditions discuss when A^{-1} is well defined and the norm of $\|A^{-1}\|$.

Existence of a solution to (6) $\iff$ A is onto $\iff$ A' is into $\iff$

$$(\mathbf{E}) \qquad \inf_{v\in\mathbb{V}} \sup_{u\in\mathbb{U}} \frac{a(u,v)}{\|u\|\|v\|} = \alpha_E > 0.$$

Uniqueness of the solution to (6) $\iff$ A is into $\iff$ A' is onto $\iff$

$$(\mathbf{U}) \qquad \inf_{u\in\mathbb{U}} \sup_{v\in\mathbb{V}} \frac{a(u,v)}{\|u\|\|v\|} = \alpha_U > 0.$$

The equivalence: A is onto $\iff$ A' is into, can be easily verified using the definition of the dual operator. The difficulty is to characterize the into by the inf-sup condition.

Let us introduce the notation

- $N(A) = \ker(A) = \{u \in \mathbb{U} : Au = 0\}$ which forms a linear subspace of $\mathbb{U}$.
- For a subset $Z \subseteq \mathbb{U}$, $Z^\circ := \{f \in \mathbb{U}', \langle f, u\rangle = 0, \text{for all } u \in Z\}$.
- For a subset $Z \subseteq \mathbb{U}$, $Z^\perp := \{v \in \mathbb{U}, (v,u) = 0, \text{for all } u \in Z\}$.

Roughly speaking, both Z° and $Z^\perp$ are "orthogonal" to Z. But they are in different spaces.

Exercise 1.1. For a linear and continuous operator B defined on a Hilbert space $\mathbb{U}$, write the projection operator $P : \mathbb{U} \to \ker(B)$ and $P^\perp : \mathbb{U} \to \ker(B)^\perp$ in terms of B.

Theorem 1.2. *For a continuous bilinear form $a(\cdot,\cdot)$, the problem (6) is well-posed if and only if (E) and (U) hold. Furthermore if (E) and (U) hold, then*

$$\|A^{-1}\| = \|(A')^{-1}\| = \alpha_U^{-1} = \alpha_E^{-1} = \alpha^{-1},$$

and thus

$$\|u\| \le \frac{1}{\alpha}\|f\|_{\mathbb{V}'}.$$

Proof. We will prove the following conditions are equivalent.

(1) **(E)**
(2) $\|A'v\|_{\mathbb{U}'} \ge \alpha_E\|v\|, \quad$ for all $v \in \mathbb{V}$.
(3) $A' : \mathbb{V} \mapsto N(A)^\circ$ is an isomorphism.
(4) $A : N(A)^\perp \mapsto \mathbb{V}'$ is an isomorphism.

(1) $\iff$ (2). It can be proved by the definition of the dual norm

$$\|A'v\|_{U'} = \sup_{u\in\mathbb{U}} \frac{\langle u, A'v\rangle}{\|u\|} = \sup_{u\in\mathbb{U}} \frac{a(u,v)}{\|u\|}.$$

(2) $\Longrightarrow$ (3). An obvious consequence of (2) is A' is an injection. We now prove that (2) also implies that the range $R(A')$ is closed and thus form a linear subspace of $\mathbb{U}'$. Choosing a convergent sequence $\{A'v_k\}$, by (2), we know $\{v_k\}$ is also a Cauchy sequence and thus converges to some $v \in \mathbb{V}$. The continuity of A' shows that $A'v_k$ converges to $A'v$ and thus $R(A')$ is closed.

We can then conclude that $A' : \mathbb{V} \mapsto R(A')$ is an isomorphism. Next we prove $R(A') = N(A)^\circ$. For a subset $Z \subseteq \mathbb{U}$, let us recall the definition $Z^\circ := \{f \in \mathbb{U}', \langle f, u\rangle = 0, \text{for all } u \in Z\}$. Using the definition of A'

$$\langle u, A'v\rangle = \langle Au, v\rangle,$$

we see that $R(A') \subseteq N(A)^\circ$. If $R(A') \subset N(A)^\circ$, i.e. there exists $f \in N(A)^\circ \backslash R(A')$. Since $R(A')$ is closed, by Hahn-Banch theorem and Risez representation theorem, there exists $u \in \mathbb{U}$ such that $\langle u, A'v\rangle = 0, \text{ for all } v \in \mathbb{V}$ and $\langle u, f\rangle = 1$. But $\langle u, A'v\rangle = \langle Au, v\rangle = 0, \text{ for all } v \in \mathbb{V}$ implies that $Au = 0$, i.e. $u \in N(A)$ and thus $\langle u, f\rangle = 0$ for $f \in N(A)^\circ$. Contradiction.

(3) $\Longrightarrow$ (2). By the assumption, $(A')^{-1} : N(A)^\circ \mapsto \mathbb{V}$ is a well defined and bounded linear operator. Thus

$$\|v\| = \|(A')^{-1}A'v\| \le C\|A'v\|_{U'}.$$

(3) $\iff$ (4). Obviously (4) $\iff$ $A' : \mathbb{V} \mapsto (N(A)^\perp)'$ is an isomorphism. Thus we only need to show the isomorphism $(N(A)^\perp)' \cong N(A)^\circ$. For any $f \in (N(A)^\perp)'$, we define $\bar{f}$ such that $\langle \bar{f}, v\rangle := \langle f, P^\perp v\rangle$ for all $v \in \mathbb{V}$, where $P^\perp : \mathbb{U} \to N(A)^\perp$ is the projection. Then $\bar{f} \in N(A)^\circ$. One can easily prove $f \to \bar{f}$ defines an isometric isomorphism.

The uniqueness is obtained by the dual argument. If both (**E**) and (**U**) hold, then

$$\|A^{-1}\| = \|(A')^{-1}\| = \alpha_U^{-1} = \alpha_E^{-1} = \alpha^{-1}.$$

□

Let us take the inf-sup condition (E) as an example to show how to verify it. To verify (E), one way is

$$\text{for all } v \in \mathbb{V}, \text{ find } u \in \mathbb{U}, \ s.t. \ a(u,v) \geq \alpha \|u\| \|v\|. \tag{7}$$

We shall present a slightly different characterization of (E). With this characterization, the verification is then transformed to a construction of a suitable function.

Theorem 1.3. *The inf-sup condition (E) is equivalent to that for any $v \in \mathbb{V}$, there exists $u \in \mathbb{U}$, such that*

$$a(u,v) \geq C_1 \|v\|^2, \quad \text{and} \ \ \|u\| \leq C_2 \|v\|. \tag{8}$$

Proof. Obviously (8) will imply (7) with $\alpha = C_1/C_2$. We now prove (E) implies (8). Recall that (E) is equivalent to $A : N(A)^{\perp} \mapsto \mathbb{V}'$ is an isomorphism. We identify $\mathbb{V}$ as $\mathbb{V}'$ by the Riesz map $J : \mathbb{V} \mapsto \mathbb{V}'$ such that $\langle Jv, v\rangle = (v,v) = \|v\|^2$. Then for a given $v \in \mathbb{V}$, we can find $u \in \mathbb{U}$ such that $Au = Jv$ and thus $a(u,v) = \langle Au, v\rangle = \langle Jv, v\rangle = \|v\|^2$. Since $u \in N(A)^{\perp}$, we also have A^{-1} is bounded and thus $\|u\| = \|A^{-1}v\| \leq C\|v\|$. ☐

In (8) u could dependent on v in a subtle way. A special case is $u = v$ when $\mathbb{U} = \mathbb{V}$. It is known as the corcevity

$$a(u,u) \geq \alpha \|u\|^2.$$

The corresponding result is known as Lax-Milgram Theorem.

Corollary 1.4 (Lax-Milgram)**.** *For a bilinear form $a(\cdot,\cdot)$ on $\mathbb{V} \times \mathbb{V}$, if it satisfies*

(1) Continuity: $a(u,v) \leq C_1 \|u\| \|v\|$;
(2) Corcevity: $a(u,u) \geq C_2 \|u\|^2$,

then for any $f \in \mathbb{V}'$, there exists a unique $u \in \mathbb{V}$ such that

$$a(u,v) = \langle f, v\rangle,$$

and

$$\|u\| \leq C_1/C_2 \|f\|_{\mathbb{V}'}.$$

The most simplest case is the bilinear form $a(\cdot,\cdot)$ is symmetric and positive definite. Then $a(\cdot,\cdot)$ defines a new inner product. Lax-Milgram theorem is simply the Riesz representation theorem.

1.3. **Brezzi theory I.** We consider the mixed problem

$$Au + B'p \ = \ f, \tag{9}$$
$$B\,u \ = \ g, \tag{10}$$

First we assume all bilinear forms are continuous.

(C) The bilinear form $a(\cdot,\cdot)$, and $b(\cdot,\cdot)$ are continuous

$$\begin{aligned} a(u,v) &\leq C\|u\|\|v\|, \quad \text{for all } u, v \in \mathbb{V}, \\ b(v,q) &\leq C\|v\|\|q\|, \quad \text{for all } v \in \mathbb{V}, q \in \mathbb{P}. \end{aligned}$$

We use the decomposition $\mathbb{V} = N(B) \oplus N(B)^{\perp}$ to write $u = u_0 + u_1, \ u_0 \in N(B)$ and $u_1 \in N(B)^{\perp}$. Then (10) becomes $Bu_1 = g$. Since $u_1 \in N(B)^{\perp}$, the existence and uniqueness of u_1 is equivalent to B is onto or B' is into, i.e. the following inf-sup condition

$$(\mathbf{B}) \qquad \inf_{q\in\mathbb{P}}\sup_{v\in\mathbb{V}}\frac{b(v,q)}{\|v\|\|q\|}=\beta>0$$

After we get a unique u_1, to determine a unique u_0, we restrict the test function space of (9) to $N(B)$. Since $\langle v, B'q\rangle = \langle Bv, q\rangle = 0$ for $v \in N(B)$, we get the following variational form: find $u_0 \in N(B)$ such that

$$a(u_0, v) = \langle f, v\rangle - a(u_1, v), \quad \text{for all } v \in N(B). \tag{11}$$

The existence and uniqueness of u_0 is then equivalent to the two inf-sup conditions for $a(u, v)$ on space $\mathbb{Z} = N(B)$.

(A)

$$\inf_{u\in\mathbb{Z}}\sup_{v\in\mathbb{Z}}\frac{a(u,v)}{\|u\|\|v\|}=\inf_{v\in\mathbb{Z}}\sup_{u\in\mathbb{Z}}\frac{a(u,v)}{\|u\|\|v\|}=\alpha>0.$$

After we determine a unique u in this way, we solve

$$B'p = f - Au \tag{12}$$

to get p. Since u_0 is the solution to (11), the right hand side $f - Au \in N(B)^\circ$. Thus we require $B' : \mathbb{V} \mapsto N(B)^\circ$ is an isomorphism which is also equivalent to the condition (B).

Theorem 1.5. *The continuous variational problem (5) is well-posed if and only if (A) and (B) hold. When (A) and (B) hold, we have the stability result*

$$\|u\|_{\mathbb{V}} + \|p\|_{\mathbb{P}} \lesssim \|f\|_{\mathbb{V}'} + \|g\|_{\mathbb{P}'}.$$

The following characterization of the inf-sup condition for the operator B is useful. The verification is again transfered to a construction of a suitable function. The proof is similar to that in Theorem 1.3 and thus skipped here.

Theorem 1.6. *The inf-sup condition (B) is equivalent to that: for any $q \in \mathbb{P}$, there exists $v \in \mathbb{V}$, such that*

$$b(v, q) \geq C_1\|q\|^2, \quad \textit{and} \;\; \|v\| \leq C_2\|q\|. \tag{13}$$

Note that $v = v(q)$ and the construction of v may not be straightforward for some problems.

1.4. **Application to Stokes equations.** Let us return to the Stokes equations. The setting for the Stokes equations:

- Spaces:

$$\mathbb{V} = \boldsymbol{H}_0^1(\Omega),\ \mathbb{P} = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0.\}.$$

- Bilinear form:

$$a(\boldsymbol{u},\boldsymbol{v}) = \mu\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}, \qquad b(\boldsymbol{v}, q) = -\int_\Omega (\mathrm{div}\boldsymbol{v})\, q.$$

- Operator:

$$\begin{aligned} &A = -\Delta : \boldsymbol{H}_0^1(\Omega) \mapsto \boldsymbol{H}^{-1}(\Omega), && \langle Au, v\rangle = a(u, v) = \mu(\nabla u, \nabla v),\\ &B = -\mathrm{div} : \boldsymbol{H}_0^1(\Omega) \mapsto L_0^2(\Omega), && \langle Bv, q\rangle = b(v, q) = -(\mathrm{div} v, q),\\ &B' = \nabla : L_0^2(\Omega) \mapsto \boldsymbol{H}^{-1}(\Omega), && \langle v, \nabla q\rangle = -(\mathrm{div} v, q).\end{aligned}$$

Remark 1.7. A natural choice of the pressure space is $L^2(\Omega)$. Note that

$$\int_\Omega \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n}\, dS = 0$$

due to the boundary condition. Thus div operator will map $\boldsymbol{H}_0^1(\Omega)$ into the subspace $L_0^2(\Omega)$. In $L_0^2(\Omega)$ the pressure of the Stokes equations is unique. But in $L^2(\Omega)$, it is unique up to a constant.

Remark 1.8. By the same reason, for Stokes equations with non-homogenous Dirichlet boundary condition $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$, the data $\boldsymbol{g}$ should satisfy the compatible condition

$$\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\, dS = \int_{\partial\Omega} \operatorname{div} \boldsymbol{u}\, d\boldsymbol{x} = 0.$$

The continuity of $a(\cdot,\cdot)$ is trivial. The continuity of $b(\cdot,\cdot)$ can be proved using the identity in the following exercise.

Exercise 1.9. Prove

$$-\Delta = -\text{grad div} + \text{curl curl}$$

holds as an operator from $\boldsymbol{H}_0^1 \to \boldsymbol{H}^{-1}$. Namely for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1$

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) + (\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v}).$$

We need to verify two inf-sup conditions. (A) is easy by the Poincáre inequality.

Lemma 1.10. *Inf-sup conditions (A) is satisfied since the following inequality*

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \geq C\|\boldsymbol{u}\|_1, \quad \textit{for all } \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega).$$

The key is the inf-sup condition (B) which is equivalent to either

- div : $\boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ is onto, or
- grad : $L_0^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is into.

Exercise 1.11. Define the Sobolev space

$$\boldsymbol{H}_0(\text{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{div} \boldsymbol{v} \in L^2(\Omega), \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}.$$

Prove div : $\boldsymbol{H}_0(\text{div};\Omega) \to L_0^2(\Omega)$ is onto, i.e., the inf-sup condition holds for a weaker norm $\|\boldsymbol{v}\|_{\text{div}} = (\|\boldsymbol{v}\|^2 + \|\operatorname{div} \boldsymbol{v}\|^2)^{1/2}$

$$\inf_{q\in L_0^2(\Omega)} \sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\text{div};\Omega)} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\text{div}}\|q\|} = \tilde{\beta} > 0.$$

The inf-sup condition holds for a weaker norm $\|\boldsymbol{v}\|_{\text{div}} = (\|\boldsymbol{v}\|^2 + \|\operatorname{div} \boldsymbol{v}\|^2)^{1/2}$. The difficulty is to control the tangential trace. In view of Theorem 1.6, we shall construct a suitable function to verify the inf-sup condition.

Lemma 1.12. *For any $q \in L_0^2(\Omega)$, there exists a $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that*

$$\operatorname{div} \boldsymbol{v} = q, \quad \textit{and} \quad \|\boldsymbol{v}\|_1 \lesssim \|q\|_0.$$

Consequently the inf-sup condition (B) holds.

Proof. We first consider the case when Ω is smooth or convex. We can solve the Poisson equation

$$\Delta\psi = q \ \text{ in } \ \Omega$$

$$\frac{\partial\psi}{\partial n} = 0 \ \text{ on } \partial\Omega.$$

The equation is well posed since $q \in L_0^2(\Omega)$. If we set $\boldsymbol{v} = \nabla\psi$, then $\operatorname{div} \boldsymbol{v} = \Delta\psi = q$ and $\|\boldsymbol{v}\|_1 = \|\psi\|_2 \lesssim \|p\|_0$ by the regularity result.

The remaining part is to verify the boundary condition. First $\boldsymbol{v} \cdot \boldsymbol{n} = \nabla\psi \cdot \boldsymbol{n} = 0$ by the construction. To take care of the tangential component $\boldsymbol{v} \cdot \boldsymbol{t}$, we invoke the trace theorem for $H^2(\Omega)$ to conclude that: there exist $\phi \in H^2(\Omega)$ such that $\phi|_{\partial\Omega} = 0$ and $\nabla\phi \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{t}$ and $\|\phi\|_2 \lesssim \|\boldsymbol{v}\|_1$. Let $\tilde{\boldsymbol{v}} = \operatorname{curl}\phi$. We have

$$\begin{aligned} &\operatorname{div} \tilde{\boldsymbol{v}} = 0, \\ &\tilde{\boldsymbol{v}} \cdot \boldsymbol{n} = \operatorname{curl}\phi \cdot n = \operatorname{grad}\phi \cdot \boldsymbol{t} = 0, \\ \text{and } \ &\tilde{\boldsymbol{v}} \cdot \boldsymbol{t} = -\operatorname{grad}\psi \cdot \boldsymbol{n} = -\boldsymbol{v} \cdot \boldsymbol{t}. \end{aligned}$$

Then we set $\boldsymbol{v}_q = \boldsymbol{v} + \tilde{\boldsymbol{v}}$ to obtain the desired result.

If the domain is not smooth, we can still construct such ψ; see [1, 4, 3]. □

Remark 1.13. Since

$$(\operatorname{div} \boldsymbol{v}, q) \le \|\operatorname{div} \boldsymbol{v}\|\|q\| \le \|\nabla\boldsymbol{v}\|\|q\|,$$

we have a upper bound on the inf-sup constant

$$\beta = \inf_{q \in \mathbb{P}} \sup_{\boldsymbol{v} \in \mathbb{V}} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\nabla\boldsymbol{v}\|\|q\|} \le 1.$$

We shall also sketch the other approach to prove grad is into which can be derived from the generalized Poincare inequality

$$\|\operatorname{grad} p\|_{-1} \ge \beta\|p\| \quad \text{for any } p \in L_0^2(\Omega). \tag{14}$$

The natural domain of the gradient operator is $H^1(\Omega)$, i.e. $\nabla : H^1(\Omega) \to L^2(\Omega)$. We can continuously extend the gradient operator from $H^1(\Omega)$ to $L^2(\Omega)$ and prove the range $\operatorname{grad}(L^2)$ is a closed subspace of $\boldsymbol{H}^{-1}$. The most difficult part is the following norm equivalence.

Theorem 1.14. *Let $X(\Omega) = \{v \,|\, v \in H^{-1}(\Omega), \operatorname{grad} v \in (H^{-1}(\Omega))^n\}$ endowed with the norm $\|v\|_X^2 = \|v\|_{-1}^2 + \|\operatorname{grad} v\|_{-1}^2$. Then for Lipschitz domains, $X(\Omega) = L^2(\Omega)$.*

Proof. The proof for $\|v\|_X \lesssim \|v\|$, consequently $L^2(\Omega) \subseteq X(\Omega)$, is trivial (using the definition of the dual norm). The non-trival part is to prove the inequality

$$\|v\|^2 \lesssim \|v\|_{-1}^2 + \|\operatorname{grad} v\|_{-1}^2 = \|v\|_{-1}^2 + \sum_{i=1}^{d} \|\frac{\partial v}{\partial x_i}\|_{-1}^2. \tag{15}$$

The difficulty is associated to the non-computable dual norm. We only present a special case $\Omega = \mathbb{R}^n$ and refer to [5, 2] for general cases.

We use the characterization of H^{-1} norm using Fourier transform. Let $\hat{u}(\xi) = \mathscr{F}(u)$ be the Fourier transform of u. Then

$$\|u\|_{\mathbb{R}^n}^2 = \|\hat{u}\|_{\mathbb{R}^d}^2 = \|1/(\sqrt{1+|\xi|^2})\hat{u}\|_{\mathbb{R}^n}^2 + \sum_{i=1}^{d} \|\xi_i/(\sqrt{1+|\xi|^2})\hat{u}\|_{\mathbb{R}^n}^2 = \|u\|_X^2.$$

□

Exercise 1.15. Use the fact L^2 is compactly embedded into H^{-1} and the inequality (15) to prove the Poincaré inequality (14).

Exercise 1.16. For Stokes equations, we can solve $\boldsymbol{u} = A^{-1}(f - B'p)$ and substitute into the second equation to get the Schur complement equation

$$BA^{-1}B'p = BA^{-1}f - g. \tag{16}$$

Define a bilinear form on $\mathbb{P} \times \mathbb{P}$ as

$$s(p, q) = \langle A^{-1}B'p, B'q\rangle.$$

Prove the well-posedness of (16) by showing:

- the continuity of $s(\cdot, \cdot)$ on $L_0^2 \times L_0^2$;
- the coercivity $s(p, p) \geq c\|p\|^2$ for any $p \in L_0^2$.
- relate the constants in the continuity and coercivity of $s(\cdot, \cdot)$ to the inf-sup condition of A and B.

In summary, we have established the well-posedness of Stokes equations.

Theorem 1.17. *There exists a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ *to the weak formulation of the Stokes equations and*

$$\|\boldsymbol{u}\|_1 + \|p\| \lesssim \|f\|_{-1}.$$

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