INTRODUCTION TO LINEAR ELASTICITY

LONG CHEN

ABSTRACT. This notes introduces the theory of linear elasticity, which studies the deformation of elastic solid bodies under external forces. The deformation is described by the displacement vector field $\boldsymbol{u}$, and the change of shape is measured by the infinitesimal strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$. The internal forces generated by the deformation are represented by the stress tensor satisfying the constitutive equation $\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}) I + 2\mu\boldsymbol{\varepsilon}$ in the linear elasticity regime. The physical interpretation of stress and strain, the relationships between material constants, and the concepts of compressibility and incompressibility are also discussed.

1. INTRODUCTION

The theory of elasticity concerns the study of the deformation of elastic solid bodies under the influence of external forces. A body is called elastic if it returns to its original (undeformed) shape upon the removal of these external forces.

1.1. **Displacement, Strain, and Stress.** How do we describe the deformation of a solid body? To begin, let us represent a solid body by a bounded domain $\Omega \subset \mathbb{R}^3$. The deformed domain can be described as the image of a vector-valued function $\Phi : \Omega \to \mathbb{R}^3$, that is, $\Phi(\Omega)$, which we refer to as a configuration or placement of a body. For most problems of interest, it is reasonable to require that this map be one-to-one and differentiable. Given our focus on deformation, namely, the change in the domain, we define $\Phi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ or, equivalently, $\boldsymbol{u}(\boldsymbol{x}) = \Phi(\boldsymbol{x}) - \boldsymbol{x}$, and designate $\boldsymbol{u}$ as the *displacement*. Here, we engage with continuum mechanics, which assumes that systems have properties defined at every point in space, abstracting away from the details at the atomic and molecular levels.

The displacement is not the same as deformation. For instance, a translation or rotation, known as rigid body motions, of Ω results in a non-trivial displacement, i.e., $\boldsymbol{u} \neq 0$, yet the shape and volume of Ω remain unchanged.

How do we mathematically describe shape? Using vectors. For example, a cube can be defined by three orthogonal vectors. The change of shape is then described by a mapping of these vectors, which can be represented by a 3×3 matrix. For example, a rigid motion is given by

$$\Phi(\boldsymbol{x}) = \boldsymbol{x}_1 + \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0), \tag{1}$$

where $\boldsymbol{x}_0, \boldsymbol{x}_1$ are fixed material points, and $\boldsymbol{Q}$ is a rotation (unitary) matrix ($\boldsymbol{Q}^{\intercal}\boldsymbol{Q} = \boldsymbol{I}$).

Consider a point $\boldsymbol{x} \in \Omega$ and a vector $\boldsymbol{v}$ pointing from $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{v}$. Then the vector $\boldsymbol{v}$ will be deformed to $\Phi(\boldsymbol{x} + \boldsymbol{v}) - \Phi(\boldsymbol{x}) \approx D\Phi(\boldsymbol{x})\boldsymbol{v}$ provided $\|\boldsymbol{v}\|$ is small. Thus, the *deformation gradient* $\boldsymbol{F}(\boldsymbol{x}) := D\Phi(\boldsymbol{x})$ is a candidate of the mathematical quantity to describe the deformation.

To compute the change in length, consider two points $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{v}$ in Ω. The squared distance in the deformed domain will be

$$\|\Phi(\boldsymbol{x} + \boldsymbol{v}) - \Phi(\boldsymbol{x})\|^2 \approx \|D\Phi(\boldsymbol{x})\boldsymbol{v}\|^2 = \boldsymbol{v}^{\intercal}(\boldsymbol{F}^{\intercal}\boldsymbol{F})(\boldsymbol{x})\,\boldsymbol{v}.$$

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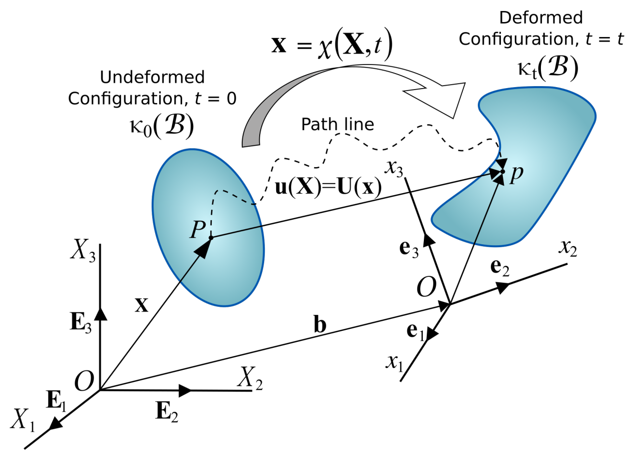


FIGURE 1. Motion of a continuum body. *From Wikipedia.*

Since Φ is a differential homeomorphism, $\det \boldsymbol{F} \neq 0$, and $(\boldsymbol{F}^{\intercal}\boldsymbol{F})(\boldsymbol{x})$ is symmetric and positive definite, defining a Riemannian metric (geometry) at $\boldsymbol{x}$.

Compared with the squared distance $\|\boldsymbol{x}+\boldsymbol{v}-\boldsymbol{x}\|^2 = \boldsymbol{v}^{\intercal}\boldsymbol{I}\boldsymbol{v}$ in the original domain, we then define a symmetric matrix function

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{2}[(\boldsymbol{F}^{\intercal}\boldsymbol{F})(\boldsymbol{x}) - \boldsymbol{I}] \tag{2}$$

and call it *strain*. When Φ is a rigid body motion, $\boldsymbol{F} = \boldsymbol{Q}$ with $\boldsymbol{Q}^{\intercal}\boldsymbol{Q} = \boldsymbol{I}$, thus $\boldsymbol{E} = 0$. A nonzero strain $\boldsymbol{E}$ describes the change of geometry from the Euclidean one.

Strain is a geometrical measure of deformation that shows the relative displacement between points in a material body, essentially measuring how much a given displacement locally differs from a rigid-body displacement. It accurately describes deformation.

What is the relationship between the strain $\boldsymbol{E}$ and the displacement $\boldsymbol{u}$? By substituting $D\Phi = \boldsymbol{I} + D\boldsymbol{u}$ into (2), we obtain

$$\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{u}) = \frac{1}{2}\left[D\boldsymbol{u} + (D\boldsymbol{u})^{\intercal} + (D\boldsymbol{u})^{\intercal}D\boldsymbol{u}\right]. \tag{3}$$

We will primarily focus on the *small deformation* case, meaning $\|D\boldsymbol{u}\| \ll 1$. In this scenario, we can ignore the quadratic part and roughly use

$$\boldsymbol{E}(\boldsymbol{u}) \approx \boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla^s \boldsymbol{u} := \frac{1}{2}\left[D\boldsymbol{u} + (D\boldsymbol{u})^{\intercal}\right], \tag{4}$$

where ∇^s denotes the symmetric gradient. The strain tensor $\boldsymbol{E}$ defined in (3) is known as the *Green-St. Venant Strain Tensor* or the *Lagrangian Strain Tensor*. Its approximation $\boldsymbol{\varepsilon}$ in (4) is referred to as the *infinitesimal strain tensor*. The relation expressed in (4) is recognized as the kinematic relation. In the context of linear elasticity theory, we also refer to the kernel of $\boldsymbol{\varepsilon}(\cdot)$ as (infinitesimal) rigid body motion, and its characterization will be discussed later on.

Remark 1.1. *The approximation $\boldsymbol{E}(\boldsymbol{u}) \approx \boldsymbol{\varepsilon}(\boldsymbol{u})$ holds true only in the case of small deformation. Take, for instance, the rigid motion described in (1). Here, the deformation gradient is $\boldsymbol{F} = \boldsymbol{Q}$, which results in $\boldsymbol{E}(\boldsymbol{u}) = 0$. However, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{Q}+\boldsymbol{Q}^{\intercal})/2 - \boldsymbol{I} \neq 0$. This discrepancy shows that the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ does not accurately measure* *the correct strain in scenarios involving large rotations, even though the strain tensor* $\boldsymbol{E}$ *can effectively do so.*

1.2. **Equations.** How to find out the deformation? To cause a deformation, there should be forces applied to the solid body. In a state of static equilibrium, deformation is determined by the balance of these forces. Let us explore both external and internal forces involved.

External forces can be categorized into two types:

- Body forces are forces distributed throughout the entire volume of the body, such as gravity or a magnetic force.
- Surface forces are applied to the surface or outer boundary of the body, like the pressure from applied loads or external contacts.

Internal forces come into play due to the deformation of the body itself. When a body deforms, it generates resistance forces similar to those produced when stretching a spring or compressing rubber. To describe these internal forces, imagine conceptually slicing the body along a plane. This action creates a new internal surface, across which internal forces are exerted to maintain equilibrium between the two halves of the body. These internal forces are called *stresses*.

Stress at a point $\boldsymbol{x}$ can be mathematically defined as $\lim_{A\to\{\boldsymbol{x}\}} \boldsymbol{t}_A/A$, where $\boldsymbol{t}_A$ represents the internal force applied across the cutting surface A, which encloses $\boldsymbol{x}$ and shrinks to the point $\boldsymbol{x}$. The force $\boldsymbol{t}_A$ is a vector and does not necessarily have to be orthogonal to the surface. The component of this force that is normal to the surface is known as the *normal stress* (σ_n), and the component tangential to the surface is termed the *shearing stress* (τ_n), with the subscript n indicating a normal vector to the cutting surface.

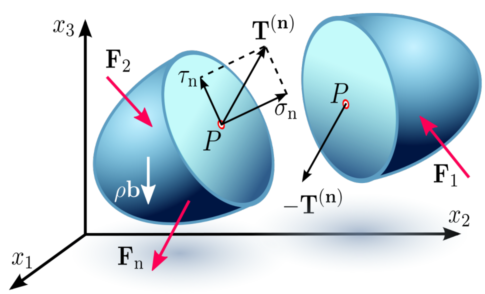


FIGURE 2. Stress vector on an internal surface S with normal vector n. The stress vector can be decomposed into two components: one component normal to the plane, called normal stress σ_n, and another component parallel to this plane, called the shearing stress τ_n. *From Wikipedia.*

At any given point $\boldsymbol{x}$ within a body, there exist infinitely many surfaces with their centroid at $\boldsymbol{x}$. Consequently, stress can be conceptualized as a function $\boldsymbol{t} : \Omega \times S^2 \to \mathbb{R}^3$, where S^2 represents the unit 2-sphere denoting direction. At a specific point $\boldsymbol{x} \in \Omega$, the stress function $\boldsymbol{t}$ maps a unit vector $\boldsymbol{n} \in S^2$ to another vector, representing the force. Cauchy's theorem facilitates the expression of this stress function by decoupling the variables, leading to

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\Sigma}(\boldsymbol{x})\, \boldsymbol{n},$$

where $\boldsymbol{\Sigma}(\boldsymbol{x})$ is referred to as the *stress tensor*.

Assuming $\boldsymbol{f}$ is the sole applied external and body-type force, the equilibrium equations can be written as:

$$\int_V \boldsymbol{f}\,\mathrm{d}\boldsymbol{x} + \int_{\partial V} \boldsymbol{\Sigma}\,\boldsymbol{n}\,\mathrm{d}S = 0 \quad \text{for all } V \subset \Omega, \tag{5}$$

$$\int_V \boldsymbol{f} \times \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} + \int_{\partial V} (\boldsymbol{\Sigma}\,\boldsymbol{n}) \times \boldsymbol{x}\,\mathrm{d}S = 0 \quad \text{for all } V \subset \Omega. \tag{6}$$

The first equation represents the balance of forces, while the second signifies the conservation of angular momentum.

From these equations, it can be deduced that the stress tensor $\boldsymbol{\Sigma}$ must be symmetric to prevent the body from undergoing unconstrained rotation. Here, we offer a simplified, albeit less rigorous, proof for the 2D case, directing readers to a more comprehensive proof using the Tensor Calculus. For this purpose, 2D vectors (x_1, x_2) are extended to 3D vectors $(x_1, x_2, 0)$ to facilitate the application of the cross product, which can then be interpreted as a scalar, namely $(a_1, a_2) \times (x_1, x_2) = a_1 x_2 - a_2 x_1$.

By considering V as a small square of size $h \ll 1$ centered at the origin and assuming $\boldsymbol{\sigma}$ as a constant function, we approximate the boundary integral using a one-point quadrature at the midpoints of the boundary edges. The volume integral $\int_V \boldsymbol{f} \times \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$ is similarly approximated using a one-point quadrature at the center, which equals zero. A straightforward calculation under these approximations reveals that $\sigma_{12} = \sigma_{21}$, thus demonstrating the symmetry of the stress tensor for a first-order approximation of the integral.

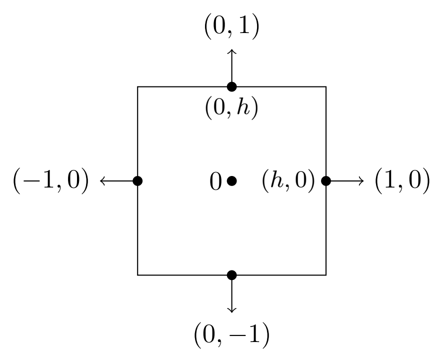


FIGURE 3. At the left and right edges, $\int_e (\boldsymbol{\sigma n}) \times \boldsymbol{x} \approx h(\sigma_{11}, \sigma_{21}) \times (h, 0) = -\sigma_{21} h^2$ and at the top and bottom edges, $\int_e (\boldsymbol{\sigma n}) \times \boldsymbol{x} \approx h(\sigma_{12}, \sigma_{22}) \times (0, h) = \sigma_{12} h^2$. The volume integral $\int_V \boldsymbol{f}\,\mathrm{d}\boldsymbol{x} \approx 0$. So (6) will imply $\sigma_{12} + \sigma_{21} = 0$ in the first order approximation.

Using the Gauss theorem and letting $V \to \{\boldsymbol{x}\}$, the equation (5) for the balance of forces becomes

$$\boldsymbol{f} + \operatorname{div} \boldsymbol{\Sigma} = 0. \tag{7}$$

The 3×3 matrix function $\boldsymbol{\Sigma}$ has 9 components while (5)-(6) contains only 6 equations. Or the symmetric matrix function $\boldsymbol{\Sigma}$ has 6 components while (7) contains only 3 equations. We need 3 more equations to determine $\boldsymbol{\Sigma}$.

The internal force within a material results directly from deformation. Therefore, it is logical to presume that stress is a function of strain, and subsequently, a function of displacement, expressed as:

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{E}) = \boldsymbol{\Sigma}(\boldsymbol{u}), \tag{8}$$

This relationship is known as the *constitutive equation*. Substituting this equation into the balance of forces enables solving for the 3 unknown components of the displacement vector function $\boldsymbol{u}$ by 3 equations in (5).

To delineate the relation $\boldsymbol{\Sigma}(\boldsymbol{E})$ within the constitutive equation, two primary assumptions are made: firstly, that the stress tensor is intrinsic, meaning it is independent of the coordinate system chosen. Secondly, the material is considered isotropic, implying that the stress remain unchanged under any rotation of the non-deformed body. Under these assumptions, the stress tensor can be represented as:

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{F}^{\intercal}\boldsymbol{F}) = \boldsymbol{\Sigma}(I + 2\boldsymbol{E}).$$

Following the Rivlin-Ericksen theorem, this expression can be expanded to:

$$\boldsymbol{\Sigma} = \lambda \operatorname{trace}(\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} + o(\boldsymbol{E}),$$

where λ and μ are two positive constants referred to as the Lamé constants. In the realm of *linear elasticity*, the higher-order terms are neglected, leading to the approximation:

$$\boldsymbol{\Sigma} \approx \boldsymbol{\sigma} = \lambda \operatorname{trace}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} = \lambda \operatorname{div} \boldsymbol{u}\boldsymbol{I} + 2\mu\nabla^s\boldsymbol{u}. \tag{9}$$

We now summarize the unknowns and equations for *linear elasticity* as the follows.

- $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{u})$: (stress, strain, displacement).
  $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ are 3×3 symmetric matrix functions and $\boldsymbol{u}$ is a 3×1 vector function.
- Kinematic equation
$$\boldsymbol{\varepsilon} = \nabla^s\boldsymbol{u} = \frac{1}{2}(D\boldsymbol{u} + (D\boldsymbol{u})^{\intercal}).$$
- Constitutive equation
$$\boldsymbol{\sigma} = \lambda \operatorname{trace}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon} = \lambda \operatorname{div} \boldsymbol{u} + 2\mu\nabla^s\boldsymbol{u}.$$
- Balance equation
$$\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma} = 0.$$

Given certain boundary conditions, the combination of the kinematic equation, constitutive equation, and balance equation uniquely defines the deformation characterized by the trio $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{u})$.

Remark 1.2. *Across various mechanics disciplines, the kinematic and balance equations maintain their form consistently. The distinction among these disciplines arises in the constitutive equations – essentially, how stress is related to strain varies.*

1.3. **Boundary conditions.** In linear elasticity, there are several types of boundary conditions that can be prescribed on the boundary of the domain Ω. The boundary $\partial\Omega$ is typically divided into two disjoint parts: the Dirichlet boundary Γ_D and the Neumann boundary Γ_N, such that $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

1. **Dirichlet boundary conditions** (also known as displacement boundary conditions). These conditions specify the displacement vector $\boldsymbol{u}$ on the Dirichlet boundary Γ_D. Mathematically: $\boldsymbol{u} = \boldsymbol{g}$ on Γ_D, where $\boldsymbol{g}$ is a given function. Physically, this means that the boundary Γ_D is fixed ($\boldsymbol{g} = 0$) or has a prescribed displacement.

2. **Neumann boundary conditions** (also known as traction boundary conditions). These conditions specify the traction vector $\boldsymbol{t}$ (force per unit area) on the Neumann boundary Γ_N. Mathematically: $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}$ on Γ_N, where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{n}$ is the unit outward normal vector to the boundary, and $\boldsymbol{t}$ is a given function. Physically, this means that the boundaryΓ_N is subjected to a specified force or traction.

3. **Mixed boundary conditions**. These conditions involve a combination of Dirichlet and Neumann boundary conditions on different parts of the boundary: $\boldsymbol{u} = \boldsymbol{g}$ on Γ_D and $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}$ on Γ_N.

4. **Robin boundary conditions** (also known as spring boundary conditions or elastic support boundary conditions). These conditions relate the displacement and the traction on the boundary through a spring-like relation. Mathematically: $\boldsymbol{\sigma} \cdot \boldsymbol{n} + k\boldsymbol{u} = \boldsymbol{t}$ on Γ_R, where k is a positive constant representing the stiffness of the elastic support and Γ_R is the portion of the boundary where Robin conditions are prescribed. Physically, this means that the boundary is supported by an elastic medium or springs.

5. **Periodic boundary conditions**. These conditions are used when the domain has a periodic structure, and the displacements and tractions are required to be periodic across opposite boundaries. Mathematically: $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{L})$ and $\boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n} = -\boldsymbol{\sigma}(\boldsymbol{x} + \boldsymbol{L}) \cdot \boldsymbol{n}$ on periodic boundaries, where $\boldsymbol{L}$ is the periodic length vector.

These boundary conditions are essential for well-posedness of the linear elasticity problem and to model various physical situations. The choice of boundary conditions depends on the specific problem and the physical characteristics of the system being modeled.

2. Physical Interpretation

In this section, we explain the physical meaning of stress and strain, offering insights into kinematic and constitutive equations from a physical standpoint.

2.1. Strain.

In the one-dimensional case, the equation $\varepsilon = u'$ implies that strain is essentially the relative displacement per unit distance between two points within a body. In a multi-dimensional setting, deformation encompasses not only changes in length but also alterations in angles. We can thus distinguish between two main types of strain:

- *Normal strain*, which quantifies the degree of elongation along a specific direction.
- *Shear strain*, which measures the amount of distortion or, more simply, the change in angles.

Mathematically, considering three points $\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{v}$, and $\boldsymbol{x} + \boldsymbol{w}$ in the undeformed body, the geometry (lengths and angles) of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is characterized by the inner product $(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}^{\intercal}\boldsymbol{v}$. After deformation, $\Phi(\boldsymbol{x}+\boldsymbol{v}) - \Phi(\boldsymbol{x}) \approx D\Phi(\boldsymbol{x})\boldsymbol{v} = \boldsymbol{F}\boldsymbol{v}$ and similarly for $\Phi(\boldsymbol{x} + \boldsymbol{w}) - \Phi(\boldsymbol{x}) \approx \boldsymbol{F}\boldsymbol{w}$, the inner product becomes $(\boldsymbol{F}\boldsymbol{w}, \boldsymbol{F}\boldsymbol{v}) = \boldsymbol{v}^{\intercal}\boldsymbol{F}^{\intercal}\boldsymbol{F}\boldsymbol{w}$.

Taking $\boldsymbol{w} = \boldsymbol{v}$, we obtain the change in squared distance, particularly

$$\|\delta \boldsymbol{e}_1\|^2 = \|\boldsymbol{F}\boldsymbol{e}_1\|^2 - \|\boldsymbol{e}_1\|^2 = 2(\boldsymbol{E}\boldsymbol{e}_1, \boldsymbol{e}_1) = 2\boldsymbol{\varepsilon}_{11}.$$

Thus, $\boldsymbol{\varepsilon}_{ii}$ signifies half of the change in squared length in the i-th coordinate direction for $i = 1, 2, 3$.

The alteration in angle is similarly reflected in the strain tensor $\boldsymbol{E} = (\boldsymbol{F}^{\intercal}\boldsymbol{F} - \boldsymbol{I})/2$. Selecting $\boldsymbol{w} = \boldsymbol{e}_1$ and $\boldsymbol{v} = \boldsymbol{e}_2$ as the coordinate axes vectors. The original angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is $\pi/2$ and thus $(\boldsymbol{e}_1, \boldsymbol{e}_2) = 0$. Denoting the angle of the deformed vectors by θ and the change in angle by $\delta\theta$, such that $\theta = \pi/2 - \delta\theta$, we find

$$\begin{aligned} 2\boldsymbol{\varepsilon}_{12} &\approx 2(\boldsymbol{E}\boldsymbol{e}_1, \boldsymbol{e}_2) = (\boldsymbol{F}\boldsymbol{e}_1, \boldsymbol{F}\boldsymbol{e}_2) = \|\boldsymbol{F}\boldsymbol{e}_1\|\|\boldsymbol{F}\boldsymbol{e}_2\| \cos\theta \\ &= (1 + \|\delta\boldsymbol{e}_1\|)(1 + \|\delta\boldsymbol{e}_2\|) \sin\delta\theta \approx \delta\theta. \end{aligned}$$

In the approximation, we utilize the linear approximation $\sin \delta\theta \approx \delta\theta, 1+|\delta \boldsymbol{e}_i| \approx 1$, and omit quadratic and higher order terms, given the smallness of length and angle changes. Consequently,

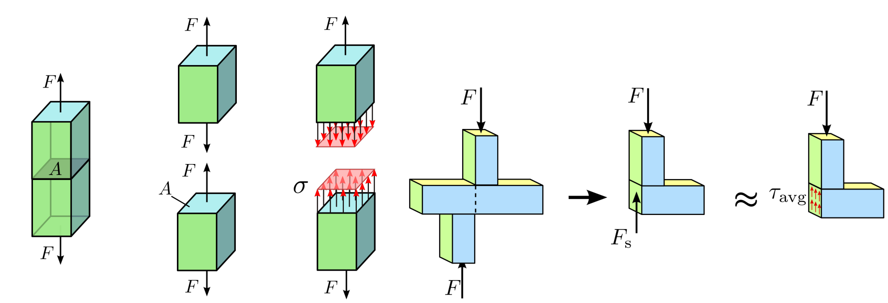
$$\varepsilon_{12} = \delta\theta/2,$$

implying that the shear strain ε_{ij} represents half of the angle change.

2.2. **Stress.** With an understanding of strain, let us explore the concept of stress. Consider the deformation occurring within a bar-like structure. When we envision a cut through this body, essentially introducing an imaginary cutting plane, the deformation gives rise to two primary types of stress:

- *Normal stress*, which acts perpendicular to the plane.
- *Shear stress* , which acts parallel to the plane.

Normal stress is deemed positive when it results in tension within the material, stretching it apart. Conversely, it is considered negative when it leads to compression, pressing the material together. The distinction between normal and shear stress is context-dependent. For example, in a bar that is being stretched horizontally, there would be no shear stress if the cutting plane is vertical, as the stress is purely normal to the plane. Similarly, if the cutting plane were horizontal, we would observe no normal stress but shear stress, depending on the direction and nature of the applied forces. See Fig. 3.



(a) Normal stress

(b) Shear stress

FIGURE 4. Normal and shear stress.

For a symmetric matrix, employing eigen-vectors to establish an orthonormal basis allows for the stress to be represented in diagonal form within a new coordinate system. These eigen-vectors, known as *principal directions*, vary with location. When analyzing stress along these principal directions, only normal stress is observed, demonstrating a direct relationship between the orientation of the stress tensor and the nature of the stress experienced.

Similarly, the strain tensor also has principal directions. Note that the principal directions for strain and stress tensors do not always coincide.

We use a Cartesian coordinate system to describe deformation. Changing the coordinate system will alter the description, but the deformation itself is intrinsic, meaning it is independent of the coordinate system chosen. Both stress and strain are described by symmetric matrix functions, underscoring their role as intrinsic material properties. Importantly, these properties are coordinate-independent, meaning their mathematical representation must remain consistent across coordinate transformations. For instance, consider a coordinate transformation where $\hat{\boldsymbol{x}} = \boldsymbol{Q}\boldsymbol{x}$, with $\boldsymbol{Q}$ representing a rotation matrix. The transformed strain and stress tensors, $\hat{\varepsilon}$ and $\hat{\boldsymbol{\sigma}}$, respectively, are related to their original counterparts by:

$$\hat{\varepsilon} = \boldsymbol{Q}\varepsilon\boldsymbol{Q}^{\intercal}, \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^{\intercal}, \tag{10}$$

demonstrating the rotational invariance of these quantities. So mathematically speaking, while a matrix can be considered a 2nd order tensor, a 2nd order tensor is essentially an equivalent class of matrices.

2.3. **Constitutive Equation.** We first derive a simple relation between the normal stress and the normal strain. Consider the case of a spring that is stretched in the x-direction. According to Hooke's Law, the stress resulting from this elongation is directly proportional to the strain experienced by the spring, mathematically expressed as:

$$\boldsymbol{\sigma}_{11} = \mathbb{E}\,\varepsilon_{11}, \quad \text{or} \quad \varepsilon_{11} = \frac{1}{\mathbb{E}}\boldsymbol{\sigma}_{11}.$$

The positive parameter $\mathbb{E}$ is called the *modulus of elasticity in tension* or *Young's modulus* which is a property of the material. Usually $\mathbb{E}$ is large (tens for solid and hundreds for metals), which means a small strain will lead to a large stress.

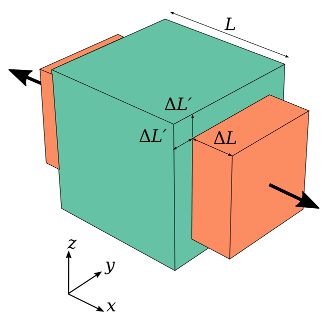


FIGURE 5. A cube with sides of length L of an isotropic linearly elastic material subject to tension along the x axis, with a Poisson's ratio of ν. The green cube is unstrained, the orange is expanded in the x direction by ΔL, and contracted in the y and z directions by $\Delta L'$. The ration $-\Delta L'/\Delta L = \nu$. *From Wikipedia.*

Stretching a body in one direction will usually have the effect of changing its shape in others – typically decreasing it. The strain in y- and z- directions caused by the stress $\boldsymbol{\sigma}_{11}$ can be described as

$$\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} = -\frac{\nu}{\mathbb{E}}\boldsymbol{\sigma}_{11},$$

where ν is a property of the material and called *Poisson's ratio*. It can be mathematically shown $\nu \in (0, 0.5)$ and usually takes values in the range $0.25 - 0.3$. See Fig. 6.

By the superposition, which holds for the linear elasticity, we have the relation

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{\mathbb{E}}\left(\quad \boldsymbol{\sigma}_{11} - \nu\boldsymbol{\sigma}_{22} - \nu\boldsymbol{\sigma}_{33}\right), \\
\varepsilon_{22} &= \frac{1}{\mathbb{E}}\left(-\nu\boldsymbol{\sigma}_{11} + \boldsymbol{\sigma}_{22} - \nu\boldsymbol{\sigma}_{33}\right), \\
\varepsilon_{33} &= \frac{1}{\mathbb{E}}\left(-\nu\boldsymbol{\sigma}_{11} - \nu\boldsymbol{\sigma}_{22} + \boldsymbol{\sigma}_{33}\right).
\end{aligned} \tag{11}$$

Material	Young's modulus (GPa)	Poisson ratio
rubber	0.01–0.1	0.4999
titanium	110.3	0.265 - 0.34
copper	117	0.33
steel	200	0.27 - 0.30
glass	17.2	0.18 - 0.3
concrete	30	0.1 - 0.2

FIGURE 6. Young's modulus $\mathbb{E}$ and Poisson ratio ν

For shear stress and shear strain, we now show:

$$\varepsilon_{ij} = \frac{1+\nu}{\mathbb{E}}\boldsymbol{\sigma}_{ij} \quad \text{for } i \neq j. \tag{12}$$

This equation demonstrates that, for off-diagonal elements representing shear components, the strain is proportional to the stress, modulated by material properties expressed by $G = \frac{\mathbb{E}}{2(1+\nu)}$ which is called *shear modulus* or *modulus of rigidity*, representing the material's resistance to shear deformation.

To illustrate this, consider a strain consisting only of shear strain with $\boldsymbol{\sigma}_{12} = \boldsymbol{\sigma}_{21} = 1$. We can identify a unitary matrix $\boldsymbol{Q}$ such that:

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boldsymbol{Q}^\intercal \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{Q} =: \boldsymbol{Q}^\intercal \hat{\boldsymbol{\sigma}} \boldsymbol{Q}.$$

In the coordinate system defined by the principal directions (the columns of $\boldsymbol{Q}$), the stress is diagonal, $\hat{\boldsymbol{\sigma}}_{11} = 1, \hat{\boldsymbol{\sigma}}_{22} = -1$). Applying the relationship (11) between normal stress and normal strain, we obtain

$$\begin{aligned} \hat{\varepsilon}_{11} =& \frac{1}{\mathbb{E}}(\hat{\boldsymbol{\sigma}}_{11} - \nu\hat{\boldsymbol{\sigma}}_{22}) = \frac{1+\nu}{\mathbb{E}}, \\ \hat{\varepsilon}_{22} =& \frac{1}{\mathbb{E}}(\hat{\boldsymbol{\sigma}}_{22} - \nu\hat{\boldsymbol{\sigma}}_{11}) = -\frac{1+\nu}{\mathbb{E}} \end{aligned}$$

in the rotated coordinate system. Transforming back to the original coordinate system, we find:

$$\boldsymbol{\varepsilon} = \frac{1}{\mathbb{E}} \begin{pmatrix} 0 & 1+\nu \\ 1+\nu & 0 \end{pmatrix}, \text{ for } \boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

thus establishing the stated relationship between shear stress and shear strain.

In summary, we can write the relation between the strain $\boldsymbol{\varepsilon}$ and the stress $\boldsymbol{\sigma}$ in the matrix form

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \frac{1}{\mathbb{E}} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{11} \\ \boldsymbol{\sigma}_{22} \\ \boldsymbol{\sigma}_{33} \\ \boldsymbol{\sigma}_{12} \\ \boldsymbol{\sigma}_{13} \\ \boldsymbol{\sigma}_{23} \end{pmatrix}, \tag{13}$$

and abbreviate (13) in a simple form

$$\boldsymbol{\varepsilon} = \mathcal{A}\boldsymbol{\sigma}, \tag{14}$$

where $\mathcal{A}$ is called *compliance tensor* of fourth order. Solving $\boldsymbol{\varepsilon}$ from (18), we obtain

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\sigma})I\right). \tag{15}$$

Inverting the matrix in (13), we have the relation

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon} \tag{16}$$

where

$$\mathcal{C} = \frac{\mathbb{E}}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{pmatrix}. \tag{17}$$

Comparing (16) with the constitutive equation

$$\boldsymbol{\sigma} = \lambda\,\operatorname{trace}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon}, \tag{18}$$

we obtain the relation of the parameters

$$\mathbb{E} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)},$$

and

$$\lambda = \frac{\mathbb{E}\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{\mathbb{E}}{2(1+\nu)},$$

which implies $\nu \in (0, 0.5)$ and $\lim_{\nu\to 0.5}\lambda = +\infty$. When ν is close to 0.5 or equivalently $\lambda \gg 1$, it will cause difficulties in the numerical approximation.

We have derived the constitutive equation assuming the material is isotropic and the deformation it undergoes is linear. In a broader context, the constitutive equation can still be formulated using a fourth-order tensor, $\mathcal{C}$, with $\boldsymbol{\sigma}_{ij} = \mathcal{C}_{ijkl}\varepsilon_{kl}$. Mathematically, the matrix $\mathcal{C}$, which could be visualized as a 9×9 matrix, implies the possibility of 81 parameters. However, the symmetry inherent in both the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$ effectively reduces the number of independent parameters to 36.

For materials that exhibit isotropy, meaning their properties are uniform in all directions, the complexity of $\mathcal{C}$ is further diminished. Isotropy implies that $\mathcal{C} = \mathcal{C}'$, where $\mathcal{C}'$ represents the tensor in a rotated coordinate system. By examining specific rotations, it becomes evident that $\mathcal{C}$ for isotropic materials relies only on two parameters. These parameters can be represented as either the pair $(\mathbb{E}, \nu)$, consisting of Young's modulus and Poisson's ratio, or the Lamé constants (λ, μ).

2.4. **Compressibility and Incompressibility.** When considering a small volume V, the change in volume due to deformation is given by

$$\delta V = \int_{\partial V} \boldsymbol{u}\cdot\boldsymbol{n}\;\mathrm{d}S = \int_V \operatorname{div}\boldsymbol{u}\;\mathrm{d}\boldsymbol{x} = \int_V \operatorname{tr}(\boldsymbol{\varepsilon})\,\mathrm{d}\boldsymbol{x}.$$

This relationship provides insight into the material's behavior under stress in terms of volume change.

Compressible materials have the capacity to undergo significant volume changes when subjected to pressure. Gases are the quintessential example of compressible materials because their volume can be easily reduced or expanded based on the applied pressure.

Incompressible materials, theoretically, experience no volume change when pressure is applied. Many liquids fall into this category as their volume remains largely constant under pressure variations.

Nearly incompressible materials demonstrate very slight volume changes in response to pressure. While not perfectly incompressible, these materials exhibit a strong resistance to volume alteration. Examples include certain rubbers, biological tissues, and metals under specific conditions, all of which maintain nearly constant volume when subjected to external forces.

The concept of compressibility is closely tied to the Lamé constant λ, which is related to the material's response to changes in density. As indicated in the constitutive equation involving $\lambda \operatorname{tr}(\boldsymbol{\varepsilon})$, a very large value of λ suggests that $\operatorname{tr}(\boldsymbol{\varepsilon}) = \operatorname{div} \boldsymbol{u}$ is very small, signifying minimal volume change under stress.