MAXWELL'S EQUATIONS

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ABSTRACT. We give a brief introduction of Maxwell's equations on electromagnetism.

1. MAXWELL'S EQUATIONS

Electromagnetism, as its name implies, is the branch of science of electricity and magnetism. Electromagnetism is described by the *electric field intensity* $\boldsymbol{E}$ and *magnetic field intensity* H which are determined by the Maxwell's equations.

We begin with the Gauss's law for *electric flux density* $\boldsymbol{D}$ and *magnetic flux density* $\boldsymbol{B}$. It is called a law since it is accepted to be true universally. In a broader sense, the flux is defined as the amount of 'flow' of a field throughout a certain amount of area. In the integral form, the Gauss's law for the electric flux density is, for any closed volume $V \subset \mathbb{R}^3$,

$$\tag{1} \int_{\partial V} \boldsymbol{D} \cdot \boldsymbol{n}\, dS = \int_V \rho(\boldsymbol{r})\, dV,$$

where ρ is the charge density. Namely the change of boundary flux is equal to the contribution of charge enclosed in this volume. The differential form

$$\tag{2} \operatorname{div} \boldsymbol{D} = \rho.$$

is obtained by using Gauss theorem.

For magnetic flux $\boldsymbol{B}$, since there is no monopole of magnet, i.e., no matter how small the volume is, it always enclose both south and north pole of a magnet. Therefore the Gauss's law is simply

$$\tag{3} \int_{\partial V} \boldsymbol{B} \cdot \boldsymbol{n}\, dS = 0.$$

or in the differential form:

$$\tag{4} \operatorname{div} \boldsymbol{B} = 0.$$

The flux $(\boldsymbol{D}, \boldsymbol{B})$ and the field $(\boldsymbol{E}, \boldsymbol{H})$ are related by the constitutive equations

$$\tag{5} \boldsymbol{D} = \epsilon \boldsymbol{E},$$

$$\tag{6} \boldsymbol{B} = \mu \boldsymbol{H},$$

where ϵ is the *permittivity* and μ is the *permeability*. The permittivity ϵ is the measure of the resistance that is encountered when forming an electric field in a medium. The permeability is the degree of magnetization of a material in response to a magnetic field. These two are in general second order tensor functions. For vacuum (free space), $\mu_0 = 4\pi 10^{-7} V \cdot s/(A \cdot m)$ and $\epsilon_0 = 1/(c_0^2 \mu_0) = 8.85 \ldots \times 10^{-12} F/m$, where c_0 is the speed of light.
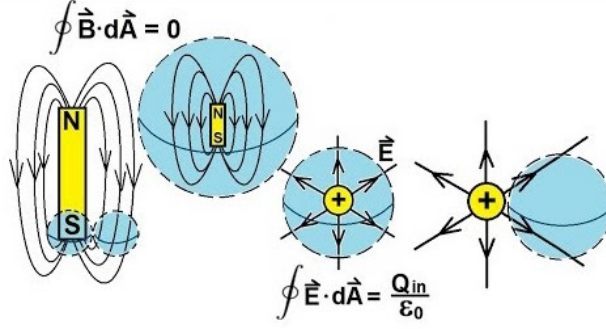

FIGURE 1. Gauss's laws of electric flux and magnetic flux.

The flux $(\boldsymbol{D}, \boldsymbol{B})$ can be thought as auxiliary fields and can be eliminated by the constitutive equations. In the terminology of differential forms, $(\boldsymbol{E}, \boldsymbol{H})$ are 1-forms associated to edges and $(\boldsymbol{D}, \boldsymbol{B})$ are 2-forms to surfaces. In terms of $(\boldsymbol{E}, \boldsymbol{H})$, the Gauss's law becomes

$$\mathrm{div}(\epsilon \boldsymbol{E}) = \rho, \tag{7}$$

$$\mathrm{div}(\mu \boldsymbol{H}) = 0. \tag{8}$$

The Gauss's law can be also thought as a constraint of the field $(\boldsymbol{E}, \boldsymbol{H})$. The 'real' equations are the interplay of electric and magnetic fields/flux. There are based on the following two facts:

- Changing magnetic field will generate current and consequently electric field.
- Changing electric field will generate magnetic field.

Faraday found that the changing magnetic field will generate electricity in 1831. The famous experiment of Faraday is that a current is detected for a wire moving in the magnetic field, which leads to the invention of all kinds of electric generators. See Fig. 2.

In the differential form the Faraday's law is:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{9}$$

and its integral form

$$\int_{\partial \Sigma} \boldsymbol{E} \cdot \boldsymbol{t} \, \mathrm{d}l = -\int_{\Sigma} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{10}$$

where Σ is a surface bounded by the closed contour $\partial \Sigma$. So $\boldsymbol{B}$ is also called *magnetic induction*. In (10), the orientation of Σ and $\partial \Sigma$ is chosen according to the right hand rule. When the flux is changing, a voltage is induced in the wire loop in an attempt by the system to 'fight' the change. Therefore a negative sign is in the equation.

To complete the system, we need one more equation which describes the change of electricity will generate magnetism. In 1820, Oersted found a compass was deflected when put on the top of a wire with current and the the needle was perpendicular to the wire; see Fig. 3. That means there exists a magnetic field surrounding an electric wire. This is the first time people found that the electricity can generate magnetism.

If two such wires next to each other, then both of them will generate magnetic field and therefore exert force between them. Just a few months after Oersted's experiment, Ampère gave the mathematical description of the force between two electric wires and obtain the
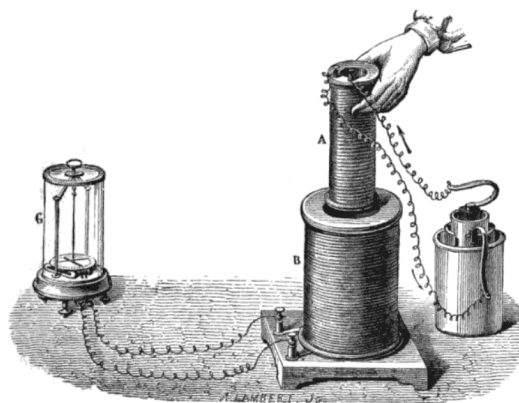
FIGURE 2. One of Faraday's 1831 experiments demonstrating induction. The liquid battery (right) sends an electric current through the small coil (A). When it is moved in or out of the large coil (B), its magnetic field induces a momentary voltage in the coil, which is detected by the galvanometer (G). *The picture and caption is copied from Wiki.*
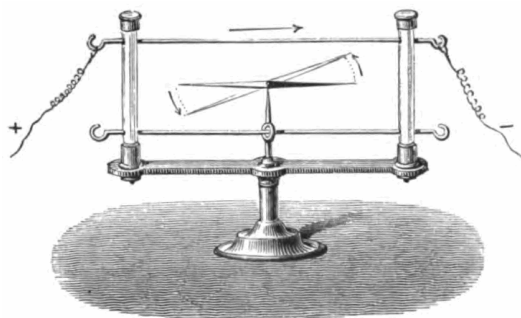

FIGURE 3. Oersted's experiment (1820) that when an electric current is passed through a conducting wire, a magnetic field is produced around it. The presence of magnetic field at a point around a current carrying wire can be detected with the help of a compass needle. *The picture and caption is copied from Wiki.*

Ampère's law to characterize the generated magnetic field $\boldsymbol{H}$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J},$$

and its integral form

$$(11) \qquad \int_{\partial \Sigma} \boldsymbol{H} \cdot \boldsymbol{t} \, \mathrm{d}l = \int_{\Sigma} J \cdot \boldsymbol{n} \, \mathrm{d}S,$$

where J is the *current density* and again Σ is a surface bounded by the closed contour $\partial \Sigma$.

Maxwell found a flaw in the Ampère's law. In (11), Σ can be an arbitrary surface. Let us consider a circuit with a capacitor which consists of two conducting plates with distance d. While the capacitor is charging, positive charge accumulated on one plate and

(a) $\sum B_{//} \Delta l = \mu_0 I$ (b) Force between two wires with currents
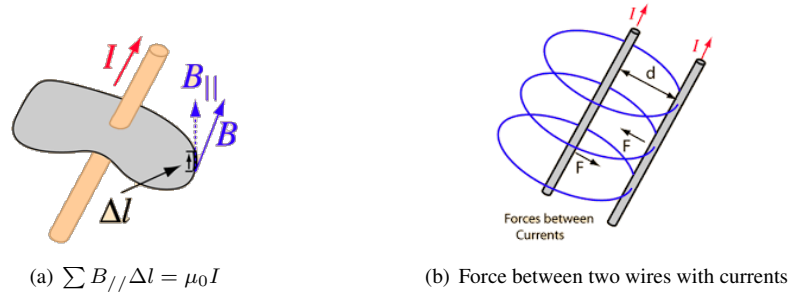
FIGURE 4. Ampère's circuital law (1823) relates the integrated magnetic field around a closed loop to the electric current passing through the loop.

negative accumulated on the other. But there is no conducting current inside the capacitor. The magnetic field between the plates should be the same as that outside the plates. But if choosing Σ inside the capacitor in (11), will indicate that there was no magnetic field. Choosing the Σ containing the the wires connected the capacitor will imply there is a magnetic field.
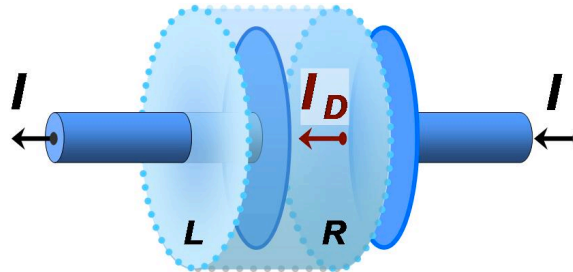

FIGURE 5. An electrically charging capacitor with an imaginary cylindrical surface surrounding the left-hand plate. Right-hand surface R lies in the space between the plates and left-hand surface L lies to the left of the left plate. No conduction current enters cylinder surface R, while current I leaves through surface L. Consistency of Ampère's law requires a displacement current $I_D = I$ to flow across surface R. *The picture and caption is copied from Wiki.*

To fix the gap, Maxwell introduced the concept 'displacement current', which is not a real current but the change of electric field. The displacement current extends the notion of current beyond a mere transport of charge and is of form $\boldsymbol{J}_d = \partial_t \boldsymbol{D}$. So $\boldsymbol{D}$ is also called the *electric displacement*. The form $\partial_t \boldsymbol{D}$ is motivated by the symmetry of Faraday's law (9): compare $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$ and $\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}$. The sign difference can be easily figured out by the physical meaning.

Adding it into Ampère's law, we obtain the Maxwell-Ampère's equation

$$\int_{\partial \Sigma} \boldsymbol{H} \cdot \boldsymbol{t} \, \mathrm{d}l = \int_{\Sigma} \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}S,$$

and its differential form

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}.$$

Now we summarize the four Maxwell's equations in terms of $(\boldsymbol{E}, \boldsymbol{H})$

$$\nabla \times \boldsymbol{E} = -\frac{\partial(\mu \boldsymbol{H})}{\partial t}, \qquad \text{(Faraday law)}$$

$$\nabla \times \boldsymbol{H} = J + \frac{\partial(\epsilon \boldsymbol{E})}{\partial t}, \qquad \text{(Ampère-Maxwell's law)}$$

$$\nabla \cdot (\epsilon \boldsymbol{E}) = \rho, \qquad \text{(Gauss's law for electricity)}$$

$$\nabla \cdot (\mu \boldsymbol{H}) = 0, \qquad \text{(Gauss's law for magnetism)}.$$

It is remarkable that using only these four equations it is possible to explain all known electromagnetic phenomena.

The volume charge density ρ and the electric current density J in the Maxwell's equations are refer to external charges. They are related. Let us take a tube with ΔS cut surface and the charge flow with velocity $\boldsymbol{v}$ inside the tube. Then the flux is $J = \Delta Q/(\Delta S \Delta t) = \Delta I/\Delta S$ by definition. On the other hand, the total charge $\Delta Q = \rho \Delta V = \rho \Delta S v \Delta t$. Therefore $\boldsymbol{J} = \rho \boldsymbol{v}$, which leads to another important law of electricity: Ohm's law

$$(12) \qquad\qquad \boldsymbol{J} = \sigma \boldsymbol{E},$$

where σ is called the *conductivity*:

- $\sigma > 0$: conductor;
- $\sigma = 0$: dielectric;
- $\sigma = \infty$: perfect conductor.

When the electric field $\boldsymbol{E}$ is uniform and oriented along the length of the conductor, we can compute the voltage by $V = EL$. The current density $J = I/\Delta S$. Substituting into (12), we obtain the most familiar form of Ohm's law

$$I = \frac{V}{R}, \quad \text{with } R = \frac{L}{\sigma \Delta S}.$$

If the conductor is moving at velocity $\boldsymbol{v}$ in a magnetic field, an extra term must be added to account for the current induced by the Lorentz force

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

The Ohm's law is less fundamental than Maxwell's equations and will break down when the electric field is too strong. The Ampère-Maxwell's law becomes

$$(13) \qquad\qquad \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \frac{\partial(\epsilon \boldsymbol{E})}{\partial t} + J_a.$$

The source J_a is for another type of current density independent of $\boldsymbol{E}$.

From the Maxwell's equations, we can also derive the conservation of charges. Take divergence of Ampère-Maxwell's equation and substitute the Gauss's law for electricity. We get the charge conservation law

$$\partial_t \rho + \nabla \cdot J = 0.$$

Bibliography remark. Here is a list of important work by Maxwell.

- 1855. On Faraday's Lines of Force.
- 1862. On Physical Lines of Force.
- 1865. A Dynamical Theory of the Electromagnetic Field.
- 1873. Treatise on Electricity and Magnetism.

The book *A Dynamical Theory of the Electromagnetic Field* is a great book comparable to Newton's *Philosophy of Nature* and Darwin's *Origin of Species*. The original equations formulated by Maxwell contains 20 equations for 20 variables. The current form is simplified by Heaviside using vectors notation. It can be further simplified in terms of differential forms:

$$d\boldsymbol{F} = 0, \quad \star d \star \boldsymbol{F} = \boldsymbol{J}.$$

A comprehensive but yet concise introduction of Maxwell's equations can be found at the website www.maxwells-equations.com.

2. Variant Formulations of Maxwell's Equations

When further properties are assumed, e.g. time periodic, we obtain different formulations of Maxwell's equations.

2.1. **Wave Equations.** For wave propagation problems, ρ and J are localized, e.g., restricted inside the cellular tower. The generated electric and magnetic fields will then radiated away from the source. The self-perpetuating waves of oscillating electric and magnetic fields and each field will drive the other. Mathematically in source-free space, Maxwell's equations are simplified to

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \quad \epsilon_0 \nabla \cdot E = 0, \quad \nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \partial_t E, \quad \nabla \cdot \boldsymbol{B} = 0.$$

We can eliminate $\boldsymbol{E}$ in the following way

$$\partial_{tt}\boldsymbol{B} = -\nabla \times \boldsymbol{\partial}_t E = -\frac{1}{\mu_0 \epsilon_0}\nabla \times \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{\mu_0 \epsilon_0}\Delta \boldsymbol{B}.$$

In the last step, we use the identity for vector Laplacian

$$\Delta = \nabla\nabla \cdot - \nabla \times \boldsymbol{\nabla} \times,$$

and the constrain $\nabla \cdot \boldsymbol{B} = 0$. We could obtain the same equation for $\boldsymbol{E}$.

Let $c_0 = 1/\sqrt{\mu_0 \epsilon_0}$. Then both $\boldsymbol{E}$ and $\boldsymbol{B}$ will satisfy the wave equation

$$\partial_{tt}u - c_0^2 \Delta u = 0.$$

The speed c_0 happens to be the speed of light. From the derivation the displacement current in Maxwell-Ampère's equation is essential in predicting the existence of propagating electromagnetic waves.

'The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to the electromagnetic laws.' – Maxwell 1865.

2.2. **Plane Wave Solutions.** We begin with a simple cosine function in 1-D to explain terminology of waves. Consider the function

$$A(x, t) = A_0 \cos(kx - \omega t + \varphi)$$

where x and t are space and time variables and others are:
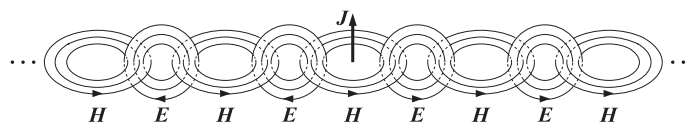
- A_0 is the amplitude of the wave.

FIGURE 6. Electromagnetic waves propagating from a localized source. Extracted from [1].
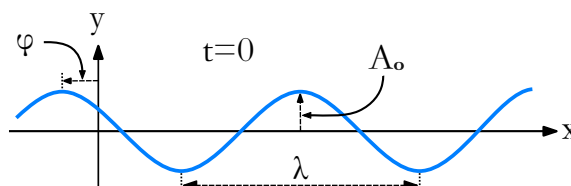

FIGURE 7. A wave function

- k is the wave number. One wave is one period of the cos function. The number k will be equal to how many waves in $(0, 2\pi)$.
- $\lambda = 2\pi/k$ is the wavelength.
- ω is the angular frequency (thinking of a rotated plate). Let T be the period of the wave. Then $\omega = 2\pi/T$.
- φ is the phase shift. A positive phase shift will shifts the wave in the negative x-axis direction.

So $A(x, t)$ is the magnitude of the wave at a given point in space and time. To satisfy the wave equation, a dispersion relation $\omega = \omega(k)$ is needed. For speed c, the dispersion relation is $\omega = ck$ and with such relation the plane wave becomes

$$A(x, t) = A_0 \cos(k(x - ct) + \varphi).$$

The graph of this function will be a wave propagate from the left to right (the direction of x-axis) with speed c.

Now we describe a wave propagate in an arbitrary direction in space $\mathbb{R}^3$:

$$A(\boldsymbol{x}, t) = A_0 \cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t + \varphi).$$

The only difference is the wave vector $\boldsymbol{k}$. The magnitude $|\boldsymbol{k}| = k$ is still called the wave number. The direction of $\boldsymbol{k}$ is the wave direction. One can easily rotate the coordinate such that $\boldsymbol{k}$ is along the x-axis and thus we return to the scalar case. Again to satisfy the wave equation $u_{tt} = c\Delta u$, $\omega = ck$.

We can separate the magnitude and direction by denoting $\boldsymbol{k} = k\boldsymbol{d}$ with $\boldsymbol{d}$ being a unit vector and thus rewrite the plane wave

$$A(\boldsymbol{x}, t) = A_0 \cos(k(\boldsymbol{d} \cdot \boldsymbol{x} - ct) + \varphi).$$

It is called plane wave because the wave is propagate along the normal direction $\boldsymbol{d}$ of the plane and all wave front in the plane orthogonal to $\boldsymbol{d}$ are the same.

These waves are scalar functions. For electromagnetic waves, one can simply replace the scalar amplitude A_0 by a constant vector $\boldsymbol{A}_0$. A transverse wave is one in which $\boldsymbol{A}_0 \cdot \boldsymbol{d} = 0$ and a longitudinal wave is one in which $\boldsymbol{A}_0 // \boldsymbol{d}$. The electromagnetic waves is

a transverse wave while acoustic waves in gas or fluid is longitudinal. Therefore

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0 \cos(k(\boldsymbol{d} \cdot \boldsymbol{x} - c_0 t) + \varphi)$$

will be a solution of the electromagnetic waves equation in the free space $\partial_{tt} \boldsymbol{E} = c_0^2 \Delta \boldsymbol{E}$ provided $\boldsymbol{E}_0 \cdot \boldsymbol{d} = 0$. The orthogonality is from the constrain $\nabla \cdot \boldsymbol{E} = 0$. Physically it means the oscillations of the waves is perpendicular to the propagate direction.

The complex form of plane waves is

$$U(\boldsymbol{x}, t) = U_0 e^{i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t} = e^{-i\omega t} U_0 e^{i\frac{\omega}{c} \boldsymbol{d} \cdot \boldsymbol{x}}.$$

Here $U_0 = A_0 e^{i\varphi}$ could be a complex valued amplitude. Then $Re(U(\boldsymbol{x}, t)) = A(\boldsymbol{x}, t)$. A plane wave solution of the time-harmonic Maxwell's equations will be

$$\boldsymbol{E} = \boldsymbol{P} e^{-i\omega \boldsymbol{x} \cdot \boldsymbol{d}}, \quad \boldsymbol{H} = -\boldsymbol{d} \times \boldsymbol{P} e^{-i\omega \boldsymbol{x} \cdot \boldsymbol{d}},$$

where $d \in \mathbb{S}^2, P \in \mathbb{C}^3$ and $\boldsymbol{d} \cdot \boldsymbol{P} = 0$. It can be easily verify that (E, H) is the solution to

$$\nabla \times \boldsymbol{E} = i\omega H, \quad \nabla \times \boldsymbol{H} = -i\omega E, \quad \nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{H} = 0.$$

the time-harmonic Maxwell's equations in free space (after normalization) $\mu = \epsilon = 1$ and $\rho = 0, \boldsymbol{J} = \boldsymbol{0}$.

Nice plot of plane wave solutions can be found at stack exchange website: How do I plot a plane EM wave?

2.3. **Time-harmonic Maxwell's Equations.** Assume the vector fields are periodic in time such that we can apply Fourier transform in time. The partial derivative to t will become a multiplication. More precisely, let $X(x, t) = e^{-i\omega t} \hat{X}(x)$. Then $\partial_t X = -i\omega X$. Here ω is a positive constant called the *frequency*.

Remark 2.1. In the engineering literature, $e^{j\omega t}$ is used while $e^{-i\omega t}$ is more often in the physics literature. This could be a source of confusion on the signs. Using $e^{-i\omega t}$, the sign of time-harmonic Maxwell's equations will be different with original Maxwell's equations.

With such change, the Maxwell-Faraday and Maxwell-Ampère equations is changed to

$$(14) \qquad \nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H}$$

$$(15) \qquad \nabla \times \boldsymbol{H} = -i\omega\epsilon\boldsymbol{E} + \sigma E + J_a = -i\omega(\epsilon + i\sigma/\omega)E + J_a.$$

We denote by $\tilde{\epsilon} = (\epsilon + i\sigma/\omega)$ and call it *effective permittivity*. The Gauss's laws remains the same since they do not involve time derivatives.

Next we do some normalization. Let

$$\epsilon_r = \frac{1}{\epsilon_0}(\epsilon + i\sigma/\omega), \mu_r = \frac{\mu}{\mu_0}, \ k = \omega\sqrt{\epsilon_0\mu_0},$$

and

$$E_r = \sqrt{\epsilon_0}E, \ H_r = \sqrt{\mu_0}H, \ F = ik\sqrt{\mu_0}J_a$$

The rescaled k is called the *wave number* and ϵ_r, μ_r are *relative permittivity* and *relative permeability*, respectively.

Then we rewrite Maxwell-Faraday and Maxwell-Ampère equations as

$$(16) \qquad \nabla \times \boldsymbol{E}_r = ik\mu_r H_r,$$

$$(17) \qquad \nabla \times \boldsymbol{H}_r = -ik\epsilon_r E + \frac{1}{ik}F.$$

After normalization, for vacuum, the relative permittivity and permeability is 1. Note that (16)-(17) are just change of notation of (14)-(15). So we shall still use the original formulation (14)-(15) to keep the physical meaning of parameters.

We can further eliminate one variable to get just one equation. Divide μ in (14) and take $\nabla\times$ to get

$$\nabla \times \left(\mu^{-1}\nabla \times \boldsymbol{E}\right) = i\omega\nabla \times \boldsymbol{H} = i\omega(-i\omega(\epsilon + i\sigma/\omega)E + J_a) = \omega^2\tilde{\epsilon}\boldsymbol{E} + \tilde{J}.$$

The source term $\tilde{J}$ in the normalized equation is F, a complex function. Similarly divide $\tilde{\epsilon}$ and take curl to get the equation of H:

$$\nabla \times \left(\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}\right) = -i\omega\nabla \times \boldsymbol{E} + \nabla \times \left(\frac{1}{\tilde{\epsilon}}J_a\right) = \omega^2\mu\boldsymbol{H} + \nabla \times \tilde{\boldsymbol{J}}.$$

In the normalized equation $\tilde{J} = \sqrt{\mu_0}/\epsilon_r J_a$.

In summary the time-harmonic Maxwell equation for $\boldsymbol{E}$ is

$$\nabla \times \left(\mu^{-1}\nabla \times \boldsymbol{E}\right) - \omega^2\tilde{\epsilon}\boldsymbol{E} = \tilde{J}$$
$$\nabla \cdot (\epsilon\boldsymbol{E}) = \rho.$$

The time-harmonic Maxwell equation for H is

$$\nabla \times \left(\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}\right) - \omega^2\mu\boldsymbol{H} = \nabla \times \tilde{\boldsymbol{J}}$$
$$\nabla \cdot (\mu\boldsymbol{H}) = 0.$$

The divergence constrain is usually skipped in the discretization and imposed weakly.

3. Physical Properties of Maxwell's Equations

3.1. Energy.

The ohmic power losses per unit volume is

$$\frac{dP_{\text{loss}}}{dV} = J \cdot \boldsymbol{E}.$$

Electromagnetic energy flowing into a region will partially increase the stored energy in that region and partially dissipate into heat in the form of $J \cdot \boldsymbol{E}$.

More physics in this section.

References

[1] S. J. Orfanidis. *Electromagnetic Waves and Antennas*. Online, 2011.