HW3 of Math 226 B

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1. (Gronwall's lemma). Suppose that φ is a nonnegative continuous function such that
$$\varphi(t) \leq a + b\int_0^t \varphi(s)\, ds, \qquad \text{for } t > 0,$$
where a and b are nonnegative constants. Prove that
$$\varphi(t) \leq a\, e^{bt}, \qquad \text{for } t > 0.$$

2. Let's consider solving the advection equation
$$u_t + au_x = 0, a > 0$$
by the upwinding scheme
$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{h}(U_j^n - U_{j-1}^n). \tag{1}$$
We can view (1) as a numerical method for the modified equation
$$v_t + av_x = \frac{1}{2}ah(1 - \frac{a\Delta t}{h})v_{xx}. \tag{2}$$
Compute the local truncation error and verify that it is $O(\Delta t^2) + O(h^2)$.

   This example says that the upwinding scheme is a second order scheme for the modified equation (with numerical viscosity). But since $v - u$ is of order h, the approximation to u is only first order.

3. Using von Neumann analysis to study the stability of the leapfrog scheme for solving the wave equation
$$u_{tt} = c^2 u_{xx}.$$
Namely we use three steps explicit scheme
$$\frac{U_j^{n+1} - 2U_j^n + U_h^{n-1}}{(\Delta t)^2} = c^2\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}.$$