HOMEWORK 1 OF MATH 226C: TRANSPORT EQUATIONS

The following are selected homework problems from the text book: J.W. Thomas. *Numerical Partial Differential Equations: Finite Difference Methods*.

1. Develop a symbolic code to check the consistency and stability of the two-level difference scheme in the form of

$$u^{n+1}(x) = \sum_p a_p u^n(x - ph)$$

for the heat equation and the transport equation. Use the code to analze

(1) (HW5.3.3 in p.216) The 1-D forward Beam-Warming scheme

$$\begin{aligned} u_i^* &= u_i^n - R(u_i^n - u_{i-1}^n), \\ u_i^{n+1} &= \frac{1}{2}\left[u_i^n + u_i^* - R(u_i^* - u_{i-1}^*) - R(2u_i^n - u_{i-1}^n - u_{i+1}^n)\right], \end{aligned}$$

for solving

$$u_t + au_x = 0.$$

(2) (HW5.8.5 in p.247) The 2-D forward alternative direction scheme

$$\begin{aligned} u_{ij}^{n+1/2} &= u_{ij}^n - \frac{R_x}{2}(u_{i+1,j}^n - u_{i-1,j}^n) + \frac{R_x^2}{2}(2u_{i,j}^n - u_{i-1,j}^n - u_{i+1,j}^n) \\ u_{ij}^{n+1} &= u_{ij}^{n+1/2} - \frac{R_y}{2}(u_{i,j+1}^{n+1/2} - u_{i,j-1}^{n+1/2}) + \frac{R_y^2}{2}(2u_{i,j}^{n+1/2} - u_{i,j-1}^{n+1/2} - u_{i,j+1}^{n+1/2}), \end{aligned}$$

for solving

$$u_t + au_x + bu_y = 0.$$

Here recall that $R = R_x = a\Delta t/\Delta x$ and $R_y = b\Delta t/\Delta y$.

2. Verify the upwinding scheme for solving $u_t + au_x = 0$ is equivalent to the centered difference scheme for solving $u_t + au_x - \frac{ah}{2}u_{xx} = 0$. Namely we add numerical viscosity to stabilize the scheme.

3. (HW5.9.1 in p.257) We consider upwinding scheme for convection-diffusion and convection-dominated problem

$$u_t + au_x - \nu u_{xx} = 0.$$

The first order derivative u_x is discretized by a unwinding scheme and the second order derivative u_{xx} by centered difference scheme.

Let $R = a\Delta t/\Delta x$ and $r = \nu\Delta t/\Delta x^2$. Show that the above difference scheme is stable if and only if

$$\begin{cases} R \le 2r \le 1 + R & \text{for } a < 0 \\ R \le 2r \le 1 + R & \text{for } a > 0. \end{cases}$$

We can also apply Lax-Wendroff scheme for the hyperbolic part and centered difference for the diffusion part. Prove that the resulting scheme is stable if and only if

$$r + R^2/2 \le 1/2.$$

4. (HW5.6.10 in p.232) Consider the following initial-boundary-value problem

$$
\begin{aligned}
u_t - 2u_x = 0, &\quad x \in (0,1), t > 0 \\
u(x,0) = 1 + \sin 2\pi x, &\quad x \in [0,1] \\
u(1,t) = 1.&
\end{aligned}
$$

Code the problem using the explicit Lax-Wendroff scheme with the following numerical boundary conditions at $x = 0$.

(1) $u_0^n = 1$;
(2) $u_0^n = u_1^n$;
(3) $u_0^{n+1} = u_0^n - R(u_1^n - u_0^n)$;
(4) $u_0^{n+1} + u_1^{n+1} + R(u_1^{n+1} - u_0^{n+1}) = u_0^n + u_1^n - R(u_1^n - u_0^n)$.