INTRO GROUP THEORY (MATH 120 A)

Midterm (solutions)

Problem 1.

What is the order of a subgroup of S_8 generated by the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 8 & 7 & 6 & 2 & 1 \end{pmatrix}$?

Solution. The permutation σ can be represented as a product of disjoint cycles:

$$\sigma = (1348)(257)$$

The order of the cycle (1348) is 4, the order of the cycle (257) is 3, therefore the order of σ is $lcm(4, 3) = 12$. □

Problem 2.

Find all abelian groups (up to isomorphism) of order 4900.

Solution.

$$4900 = 7^2 \cdot 5^2 \cdot 2^2$$

The following groups are abelian, and have the order 4900:

- $\mathbb{Z}_{49} \times \mathbb{Z}_{25} \times \mathbb{Z}_4$
- $\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{25} \times \mathbb{Z}_4$
- $\mathbb{Z}_{49} \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_4$
- $\mathbb{Z}_{49} \times \mathbb{Z}_{25} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_4$
- $\mathbb{Z}_{49} \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{25} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Due to Fundamental Theorem of Finitely Generated Abelian Groups, all these groups are pairwise non-isomorphic, and any abelian group of order 4900 is isomorphic to one of the groups above. □

Problem 3.

Prove that any proper subgroup of any group of order 185 is abelian.

Solution.

$$185 = 37 \cdot 5$$

By Lagrange Theorem, any non-trivial proper subgroup of a group of order 185 must have an order that divides 185, i.e. 5 or 37. In both cases, this subgroup must be cyclic (since any group of a prime order is cyclic), hence it must be abelian (since any cyclic group is abelian). □

Problem 4.

Let σ be a permutation of an infinite set A. We sell say "σ **moves** $a \in A$" if $\sigma(a) \neq a$.

a) Let K be a set of all $\sigma \in S_A$ that move at most 5 elements of A. Is K a subgroup of S_A? Explain.

b) Let H be a set of all $\sigma \in S_A$ such that the number of elements moved by σ is finite. Show that H is a subgroup of S_A.

Solution.

a) K is not a subgroup of S_A. Indeed, take ten different elements of A, denote them $a_1, a_2, \ldots, a_{10}$. Consider permutations (cycles) $\sigma_1 = (a_1 a_2 a_3 a_4 a_5)$ and $\sigma_2 = (a_6 a_7 a_8 a_9 a_{10})$. Then $\sigma_1, \sigma_2 \in K$, but $\sigma_1 \circ \sigma_2 \notin K$. Therefore, K is not closed under the binary operation in S_A (composition).

b) If a permutation σ_1 moves a finite number of elements of A, and σ_2 moves a finite number of elements of A, then $\sigma_1 \circ \sigma_2$ also moves at most finite number of elements of A. Therefore, H is closed under the binary operation. Now let us check the group axioms G_1, G_2, G_3.

G_1) H is a subset of a group S_A, so the binary operation is associative.

G_2) Identity map $i \in S_A$ moves 0 elements of A, so $i \in H$.

G_3) If $\sigma \in H$ then σ moves only finite number of elements of A. Notice that σ^{-1} moves exactly the same elements as σ, therefore, σ^{-1} also moves only finite number of elements of A, and $\sigma^{-1} \in H$. □