MATH 120A / Fall 2007 / HW8 solutions 11/30/07

SECTION 14

Problem 26. Assume G is abelian and T is the torsion subgroup of G (i.e. T is all the elements of finite order). Show T is a normal subgroup of G and that G/T is torsion-free (i.e. has no elements of finite order except the identity).

T is a normal subgroup:

- Closure: Let $a, b \in T$. Say they have orders m and n respectively. Then since G is abelian, $(ab)^{mn} = a^{mn}b^{mn} = e$, so ab has finite order and so is in T.
- T obviously contains the identity element.
- Inverses: Let a have finite order m. So then a^{-1} also has order m.
- Normality: G is abelian, so every subgroup is normal.

G/T is torsion free: Pick a non-identity element gT of the factor group G/T (so g is *not* an element of T; i.e. g has infinite order). We need to show that gT does not have finite order. Suppose to the contrary that it has finite order n. Then $T = (gT)^n = (g^n)T$, so $g^n \in T$; i.e. g^n has finite order; say it has order k. But then $g^{nk} = e$, so g has finite order, a contradiction.

SECTION 15

Problem 12 $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(3, 3, 3)\rangle$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$. There are a couple of ways to show this:

If you happen to guess the answer correctly, you can prove it by noting that the map $\phi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$ defined by $\phi(a, b, c) = (amod3, b - a, c - a)$ is homomorphic, surjective, and its kernel is $\langle(3, 3, 3)\rangle$; thus by the Fundamental Homomorphism Theorem, the domain modulo the kernel is isomorphic to the range; i.e. $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(3, 3, 3)\rangle$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$.

Another way to see it is to examine the orders of elements of the factor group. Let K denote $\langle(3, 3, 3)\rangle$, and note the following:

(1) $(1, 1, 1) + K$ generates the *only* nontrivial finite cyclic subgroup of the factor group; and this subgroup has size 3.

(2) $(1, 0, 0) + K$ and $(0, 1, 0) + K$ each generate infinite cyclic subgroups of the factor group, and furthermore the intersection of these two subgroups is trivial.

Now item 1 implies that if we write the factor groups as a cross product of cyclic groups, only one of the factors in the cross product will be finite, and furthermore it must be $\mathbb{Z}_3$. Item 2 implies that there must be at least two $\mathbb{Z}$ factors in the cross product. So far we know that the factor group is of the form $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z} \times$ (possibly more $\mathbb{Z}$ factors). Why are there no more $\mathbb{Z}$ factors? If there were, then the resulting cross product could not be generated by 3 elements; but we know the factor group can be generated by 3 elements, since its numerator can be generated by 3 elements. So the answer must be $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$.

Problem 14 $Z(\mathbb{Z}_3 \times S_3) = Z(\mathbb{Z}_3) \times Z(S_3) = \mathbb{Z} \times \{id\}$. $C(\mathbb{Z}_3 \times S_3) = C(\mathbb{Z}_3) \times C(S_3)$. Since $\mathbb{Z}_3$ is abelian, its commutator subgroup is trivial. To see what $C(S_3)$ is requires more work. Note that A_3 is a normal subgroup of S_3, and $S_3/(A_3)$ is isomorphic to $\mathbb{Z}_2$; in particular it is abelian. So by Theorem 15.20, $C(S_3)$ is a subgroup of A_3. By LaGrange's Theorem this leaves 2 possibilities: $C(S_3)$ is either trivial, or all of A_3. It can't be trivial, since, for example, the permutation $(12)(23)(12)^{-1}(23)^{-1}$ is a nontrivial commutator. So $C(S_3) = A_3$.

Problem 30 a) If G is abelian, then $Z(G) = G$. b) If G is simple and non-abelian: since G is simple and $Z(G)$ is always a normal subgroup of G, then $Z(G)$ is either trivial or all of G. It can't be all of G, since G is not abelian. So $Z(G)$ must be trivial.

Problem 26 $U/\langle\zeta_n\rangle$ is isomorphic to U. Consider the map $\phi : U \to U$ defined by $\phi(z) = z^n$. This is homomorphic, maps onto U, and its kernel is the collection of the nth roots of unity; i.e. its kernel is $\langle\zeta_n\rangle$. So by the Fundamental Homomorphism theorem, $U/\langle\zeta_n\rangle$ is isomorphic to U.

Problem 28 One example is $mathbbZ/2\mathbb{Z}$. $\mathbb{Z}$ has no element of finite order > 1, but the factor group itself is finite.

Another example where the factor group is not even finite: $\mathbb{Q}/\mathbb{Z}$ (under addition). This factor group is not finite, but every element has finite order.

SECTION 16

Problem 11 One direction: Assume G acts faithfully on X, and assume a, b are elements of G with the same action on X. Then for every $x \in X$, $ax = bx$; so for every $x \in X$, $b^{-1}ax = x = ex$. So the element $b^{-1}a$ has the same action on X that e does. Since G acts faithfully, this means that $b^{-1}a = e$; i.e. $a = b$.
Other direction: Assume G does not act faithfully on X; this means there is some non-identity element $a \in G$ which acts trivially on X. But e also acts trivially on X, so we have 2 distinct elements with the same action on X.

Problem 12

- Closure: Let $a, b \in G_Y$. Let $y \in Y$. Then $(ab)y = a(by) = ay = y$, so $ab \in G_Y$.
- Identity: e fixes everything in X, so it fixes everything in Y.
- Inverses: Let $a \in G_Y$ and let $y \in Y$. So $ay = y$. Not let a^{-1} act on both sides of that equation to get $y = a^{-1}y$. So a^{-1} fixes y.