1 Section 14

Problem 4. Find the order of the factor group $\mathbb{Z}_3 \times \mathbb{Z}_5/(\{0\} \times \mathbb{Z}_5)$.

Note that $\{0\} \times \mathbb{Z}_5$ is the same as the subgroup generated by the element $(0, 1)$. 0 has order 1 in $\mathbb{Z}_3$ and 1 has order 5 in $\mathbb{Z}_5$, so the element $(0, 1)$ has order $LCM(1, 5) = 5$ in $\mathbb{Z}_3 \times \mathbb{Z}_5$. So $\mathbb{Z}_3 \times \mathbb{Z}_5/(\{0\} \times \mathbb{Z}_5)$ has size $(3)(5)/5 = 3$.

Problem 8. Similar to number 4. The order of $(1, 1)$ is $LCM(11, 15) = (11)(15)$. So the factor group has size $(11)(15)/((11)(15)) = 1$. I.e. the factor group is trivial.

Problem 12. $\langle(1, 1)\rangle = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$. You must add $(3, 1)$ to itself twice before ending up in that set. So the coset in which $(3, 1)$ belongs has order 2 in the factor group.

Problem 14. Similar to number 12. You must add $(3, 3)$ to itself 8 times before landing in the set $\langle(1, 2)\rangle$, so its coset has order 8 in the factor group.

Problem 30. First note that if K is any finite group of size m, then for every $b \in K$, b^m is the identity (this is because by LaGrange's Theorem, the order of b divides m). Now since H is a normal subgroup of G of index m, we can form the factor group G/H, and this factor group has size m. Now let $a \in G$. By the above remark, $(aH)^m$ is the identity element of G/H, which is H. So $H = (aH)^m = (a^m)H$. Thus $a^m \in H$.

Problem 34. Fix any $g \in G$; we need to show that $g^{-1}Hg = H$. Consider the inner automorphism $\varphi_g : G \to G$ defined by $\varphi(x) = g^{-1}xg$. Then $\varphi \restriction H$ is an isomorphism between H and $\varphi[H]$, and so $\varphi[H]$ is a subgroup of G with the same size as H. By our assumption, this means $H = \varphi[H]$. But $\varphi[H]$ is just $g^{-1}Hg$.

2 Section 15

Problem 2. First, compute the size of the factor group $\mathbb{Z}_2 \times \mathbb{Z}_4/\langle(0, 2)\rangle$ as we did in problems 4 and 8 in section 14; you'll see that its size is 4. So by the Fundamental Theorem of Finitely Generated abelian groups, it's either isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the factor group has no elements of order 4, since for every $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_4$, $(a, b) + (a, b) = (2amod2, 2bmod2) = (0, 2bmod2) \in \langle(0, 2)\rangle$. So the answer must be $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Problem 6. $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(3, 3, 3)\rangle$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$. There are a couple of ways to show this:

If you happen to guess the answer correctly, you can prove it by noting that the map $\varphi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$ defined by $\varphi(a, b, c) = (amod3, b - a, c - a)$ is homomorphic, surjective, and its kernel is $\langle(3, 3, 3)\rangle$; thus by the Fundamental Homomorphism Theorem, the domain modulo the kernel is isomorphic to the range; i.e. $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(3, 3, 3)\rangle$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$.

Another way to see it is to examine the orders of elements of the factor group. Let K denote

$\langle(3,3,3)\rangle$, and note the following:

1. $(1,1,1)+K$ generates the *only* nontrivial finite cyclic subgroup of the factor group; and this subgroup has size 3.

2. $(1,0,0)+K$ and $(0,1,0)+K$ each generate infinite cyclic subgroups of the factor group, and furthermore the intersection of these two subgroups is trivial.

Now item 1 implies that if we write the factor groups as a cross product of cyclic groups, only one of the factors in the cross product will be finite, and furthermore it must be $\mathbb{Z}_3$. Item 2 implies that there must be at least two $\mathbb{Z}$ factors in the cross product. So far we know that the factor group is of the form $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z} \times$ (possibly more $\mathbb{Z}$ factors). Why are there no more $\mathbb{Z}$ factors? If there were, then the resulting cross product could not be generated by 3 elements; but we know the factor group can be generated by 3 elements, since its numerator can be generated by 3 elements. So the answer must be $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$.