HW #6

Section 13

(2) $\phi : \mathbb{R} \to \mathbb{Z}$, $x \mapsto$ the greatest integer $\le x$.

NOT a HOM: $\phi(1.5) + \phi(1.5) = 1 + 1 = 2$

but $\phi(1.5 + 1.5) = \phi(3) = 3$.

(6) $\phi : (\mathbb{R}, +) \to (\mathbb{R}^*, \cdot)$ $\quad x \mapsto 2^x$.

$$\phi(x+y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y).$$

Yes, ϕ is a HOM.

(8) $\phi : G \to G$, $g \mapsto g^{-1}$. (G is arbitrary group).

Not necessarily a homomorphism: pick any non-abelian group G and some pair $a,b \in G$ which don't commute. (ie $ab \neq ba$).

Then also

$$\begin{array}{ccc} a^{-1}b^{-1} & \neq & b^{-1}a^{-1} \\ \| & & \| \\ \phi(a)\phi(b) & & (ab)^{-1} \\ & & \| \\ & & \phi(ab). \end{array}$$

(Note ϕ is a homomorphism iff G is abelian).

(12) $\phi : (M_n, +) \longrightarrow (\mathbb{R}, +)$

$$A \mapsto \det A.$$

ϕ is NOT a homomorphism because

$$\phi\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \phi\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) = 1,$$

but $\phi\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) + \phi\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = 0 + 0 = 0.$

(14) $\phi : (GL_n(\mathbb{R}), \cdot) \longrightarrow (\mathbb{R}, +)$

$$A \mapsto tr(A).$$

ϕ is NOT a homomorphism. Let I be the $n \times n$ identity matrix; note $I \in GL_n(\mathbb{R})$.

$$\phi(I \cdot I) = tr(I) = n$$

$$\neq$$

$$\phi(I) + \phi(I) = n + n = 2n$$

(18) $\phi : \mathbb{Z} \to \mathbb{Z}_{10} \quad \phi(1) = 6.$

Since 1 generates $\mathbb{Z}$, there is at most one way to extend to a homomorphism; namely,

$$\phi(n) = 6n \bmod 10.$$

(And this is in fact a homomorphism).

$\ker\phi = \{n \in \mathbb{Z} \mid 6n \bmod 10 = 0\} = \{n \in \mathbb{Z} \mid 10 \text{ divides } 6n\} =$

$= \{ n \in \mathbb{Z} \mid 5 \text{ divides } 3n \} = \{ n \in \mathbb{Z} \mid 5 \text{ divides } n \} = 5\mathbb{Z}$

So $\ker \phi = 5\mathbb{Z}$

$\phi(18) = 6 \cdot 18 \mod 10 = 8$

22 $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $\quad \phi(1,0) = 3$ & $\phi(0,1) = -5$.

Since every $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ is of the form

$m(1,0) + n(0,1)$, the homomorphism must

send (m,n) to $m \cdot 3 + n(-5)$.

i.e. $\phi(m,n) = 3m - 5n$.

Notice that if $3m - 5n = 0$, then $3|n$ & $5|m$. It follows that $(m,n) = k(5,3)$ for some $k \in \mathbb{Z}$.

$\ker \phi = \{ (m,n) \in \mathbb{Z} \times \mathbb{Z} \mid 3m - 5n = 0 \} = \langle (5,3) \rangle$

$\phi(-3,2) = 3(-3) - 5(2) = -9 - 10 = -19$

50 $\phi : G \to H$ group hom.

- Assume $\phi[G]$ is abelian. Let $x, y \in G$. Then

$$\phi(xyx^{-1}y^{-1}) \underset{\substack{\phi \text{ is} \\ \text{HOM}}}{=} \phi(x)\phi(y)\phi(x^{-1})\phi(y^{-1}) \underset{\substack{\phi[G] \\ \text{abelian}}}{=} \phi(x)\phi(x^{-1})\phi(y)\phi(y^{-1})$$

$$\underset{\phi \text{ is HOM}}{=} \phi(x)\phi(x)^{-1}\phi(y)\phi(y)^{-1} = e \cdot e = e. \text{ So } xyx^{-1}y^{-1} \in \ker \phi.$$

- Assume $\forall x, y \in G$ $\phi(xyx^{-1}y^{-1}) = e$. Then for arbitrary $\phi(x), \phi(y)$

in $\phi[G]$: $e = \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1}$

$\phi(y) = \phi(x)\phi(y)\phi(x)^{-1}$ (Multiplied on right by $\phi(y)$)

$\phi(y)\phi(x) = \phi(x)\phi(y)$ (Multiplied on right by $\phi(x)$). So $\phi[G]$ is abelian.