HW 5

Section 11

(6) In general, if $a \in G$ & $b \in H$, then the order of (a,b) in $G \times H$ is the LCM of the order of a in G and the order of b in H.
(If either a or b has infinite order, then so does (a,b)).

Now the order of 3 in $\mathbb{Z}_4$ is 4.
――― " ――― 10 in $\mathbb{Z}_{12}$ is 6
――― " ――― 9 in $\mathbb{Z}_{15}$ is 5

So the order of $(3,10,9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is $LCM(4,6,5) = 60$

(8) The highest possible order in $\mathbb{Z}_6 \times \mathbb{Z}_8$ is achieved by the element $(1,1)$; its order is $LCM(6,8) = 24$.
Similarly, the highest possible order in $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is achieved by the element $(1,1)$; its order is $LCM(12,15) = 60$.

10 Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has a minimal generating set consisting of 3 elements (namely $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$), to find the proper subgroups we can just look for those groups generated by 1 or 2 elements.

- subgroups minimally generated by 2 elements:

$\langle(1,0,0),(0,1,0)\rangle$ $\langle(1,1,0),(0,0,1)\rangle$ $\langle(1,1,0),(0,1,1)\rangle$

$\langle(1,0,0),(0,0,1)\rangle$ $\langle(1,0,1),(0,1,0)\rangle$

$\langle(0,1,0),(0,0,1)\rangle$ $\langle(0,1,1),(1,0,0)\rangle$

- subgroups minimally generated by 1 element (i.e. cyclic subgps):

$\langle(1,0,0)\rangle$, $\langle(0,1,0)\rangle$, $\langle(0,0,1)\rangle$, $\langle(1,1,0)\rangle$, $\langle(1,0,1)\rangle$, $\langle(0,1,1)\rangle$, $\langle(1,1,1)\rangle$.

18 $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} \cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_{2^3}$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_5$

$\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$

NOT the same.

So the groups are NOT isomorphic.

(24) $720 = 2^4 \cdot 3^2 \cdot 5$

$\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$\mathbb{Z}_{2^4} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

(26) $24 = 2^3 \cdot 3$, so there are 3 abelian groups of this order.

$25 = 5^2$. There are 2 abelian groups of this order.

Since 24 and 25 share NO common prime factors, by the Fundamental Theorem the number of abelian groups of order $(24)(25)$ is simply $3 \cdot 2 = 6$.

(47) $H := \{ a \in G \mid o(a) \leq 2 \}$. Assume G is abelian. Show $H \leq G$.

- <u>Closure</u>: Let $a, b \in H$. Then

$$(ab)^2 = abab \underset{\substack{G \text{ is} \\ \text{abelian}}}{=} aabb \underset{\text{b/c } a,b \in H}{=} ee = e$$

So $o(ab) \leq 2$. So $ab \in H$.

- <u>Identity</u>: $o(e) = 1$, so $e \in H$

- <u>Inverses</u>: Let $a \in H$. So $aa = e$; multiplying twice by a^{-1} gives $e = (a^{-1})(a^{-1})$.

  So $o(a^{-1}) \leq 2$. So $a^{-1} \in H$.

53 You don't need to assume G is commutative. If $|G| = p^k$ then any $a \in G$ has order which divides p^k, by LaGrange. The only numbers which divide p^k are powers of p.