MATH 120A / Fall 2007 / HW4 solutions 11/8/07

SECTION 9

2. The orbits of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$$

are $\{1, 5, 8, 7\}$, $\{2, 6, 3\}$, and $\{4\}$.

12. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$.
σ's representation as a product of disjoint cycles: (13478652).
As a product of transpositions (answer is not unique): (12)(15)(16)(18)(17)(14)(13)

18. Find the maximum possible order for an element of S_{15}. Every element of S_{15} can be represented as a disjoint product of cycles; and the order of a disjoint product of cycles is the least common multiple of the cycle lengths. So you have to find which type of disjoint product yields the highest LCM (equivalently, the lowest GCD). Note that in S_{15} the sum of the cycle lengths of a disjoint product of cycles is at most 15. So you have to find a list of numbers whose sum is ≤ 15 and which has the largest possible LCM. The list 3, 5, 7 yields the highest LCM in this case, which is 105 (i.e. a disjoint product of the form (a b c)(d e f g h)(i j k l m n o) has order 105, and this is the largest possible order in S_{15}).

32. Let A be an infinite set, and let K be the set of all $\sigma \in S_A$ which move at most 50 elements of A. Is K a subgroup of S_A? No, K is not closed. Let $a_1, a_2, ..., a_{50}, a_{51}, a_{52}$ be a list of 52 distinct elements of A; this can be done since A is infinite. Let $\sigma := (a_1 a_2 ... a_{50})$, and let $\tau := (a_{51} a_{52})$. Then σ and τ are both in K, but $\sigma \circ \tau$ moves 52 elements, so it is not in K.

SECTION 10

4. There are 4 cosets of $\langle 4 \rangle$ in $\mathbb{Z}_{12}$; each of them has several "names":
$\langle 4 \rangle = 0 + \langle 4 \rangle = 4 + \langle 4 \rangle = 8 + \langle 4 \rangle = \{0, 4, 8\}$
$1 + \langle 4 \rangle = 5 + \langle 4 \rangle = 9 + \langle 4 \rangle = \{1, 5, 9\}$
$2 + \langle 4 \rangle = 6 + \langle 4 \rangle = 10 + \langle 4 \rangle = \{2, 6, 10\}$
$3 + \langle 4 \rangle = 7 + \langle 4 \rangle = 11 + \langle 4 \rangle = \{3, 7, 11\}$

12. $\langle 3 \rangle$ has 8 elements in $\mathbb{Z}_{24}$, so its index in $\mathbb{Z}_{24}$ is $24/8 = 3$. (If G is a finite group and H is a subgroup, then the index of H in G equals $|G|/|H|$).
16. Since μ is a disjoint product of cycles of length 4 and 2, its order is $LCM(4, 2) = 4$. So the index of $\langle \mu \rangle$ in S_6 is $6!/4$.
37. Assume G is a group with at least 2 elements, and that G has no proper nontrivial subgroups. Show that G is finite and of prime order.
Proof: Pick some non-identity element $a \in G$; we can do this because we assume G has at

least 2 elements. Then a generates G (if not, then $\langle a \rangle$ would be a proper nontrivial subgroup of G, and we're assuming no such subgroups exist). So G is cyclic and thus either isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some intenger n. It can't be isomorphic to $\mathbb{Z}$, since that group has proper nontrivial subgroups (e.g. the subgroup $2\mathbb{Z}$). So G must be isomorphic to some $\mathbb{Z}_n$; in particular, it is finite. Also, n must be prime; if not, there is some $1 < d < n$ which divides n, and so $\langle d \rangle$ would be a proper nontrivial subgroup of $\mathbb{Z}_n$, which we're assuming G doesn't have.