Math 3D Solution
Quiz 2

1) *(6 pts)* Find an explicit solution for the initial value problem $x^2 \frac{dy}{dx} = y - xy$, with $y(1) = -1$

$$x^2 \frac{dy}{dx} = y - xy \to x^2 \frac{dy}{dx} = y(1-x) \to \int \frac{dy}{y} = \int \frac{(1-x)}{x^2} dx$$

The left hand side $\int \frac{dy}{y} = \ln|y| + C_1$

The right hand side: $\int \frac{(1-x)}{x^2} dx = \int \left(\frac{1}{x^2} - \frac{x}{x^2} \right) dx = -x^{-1} - \ln|x| + C_2$.

Now isolate y. First, let $C_3 = C_2 - C_1$.

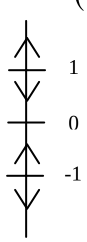
$$\ln|y| = -x^{-1} - \ln|x| + C_3 \to e^{\ln|y|} = e^{-\frac{1}{x} + \ln|x|^{-1} + C_3} \to |y| = e^{-\frac{1}{x}} e^{\ln|x|^{-1}} e^{C_3} = e^{-\frac{1}{x}} \frac{1}{|x|} e^{C_3}$$

$e^{C_3} > 0$, but when we remove the absolute value around y, we can now get positive or negative values. $y = 0$ is a solution of the differential equation, so our constant can be any real number, including 0. We'll call this final constant C.

Finally note that this solution is only valid on intervals where it is continuous, so (for solutions other than the trivial solution) the solution is only valid on either $(-\infty, 0)$ or $(0, \infty)$. This final restriction allows us to remove the absolute value bars around x term, leaving us with $y(x) = \frac{C}{xe^{1/x}}$. Solving for C, $y(1) = -1 = \frac{C}{e}$ so $C = -e$. Thus $y(x) = \frac{-e}{xe^{1/x}}$ is the explicit solution on $(0, \infty)$.

2) *(4 pts)* Find the critical points and draw the phase portrait $\frac{dy}{dx} = y^3 - y$. Classify each critical point as stable, unstable or semi-stable.

$\frac{dy}{dx} = y^3 - y = y(y^2 - 1) = y(y+1)(y-1)$ so the critical points are $y = 0$, $y = 1$ and $y = -1$. We break the real line at these critical points, resulting in the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$; choosing a value from each interval, we find:



Interval	dy/dx
$(1, \infty)$	+
$(0, 1)$	−
$(-1, 0)$	+
$(-\infty, -1)$	−

so -1 and 1 are unstable (repellers), and 0 is stable (an attracter).