1) Solve the IVP $x^2 y' + xy + 1 = 0$, $y(1) = 1$

This is a linear 1st order ODE, so we can use an integrating factor to solve. In standard form, this ODE is $y' + \frac{1}{x} y = -\frac{1}{x^2}$, so our integrating factor is $\mu = e^{\int \frac{1}{x} dx} = x$.
Multiplying on both sides: $xy' + y = -\frac{1}{x}$, which is the same as $\frac{d}{dx}[xy] = -\frac{1}{x}$.
Integrating both sides with respect to x, and we get $xy = -\ln|x| + C$. This is not continuous at $x = 0$ and our initial condition is positive, so we can drop the absolute value bars, and freely divide by x: $y = -x^{-1} \ln x + Cx^{-1}$. Applying the initial value, we see that $C = 1$, so our solution is $y = -x^{-1} \ln x + x^{-1}$.

2) Find a general solution of the equation $y'' - 3y' + 2y = 2^x$

First the homogeneous solutions: The auxiliary equation is
$m^2 - 3m + 2 = (m-2)(m-1)$, so the complementary solution is $y_c(x) = c_1 e^{2x} + c_2 e^x$.
To get the particular solution, we use variation of parameters:
$W = \begin{vmatrix} e^{2x} & e^x \\ 2e^{2x} & e^x \end{vmatrix} = -e^{3x}$, $W_1 = \begin{vmatrix} 0 & e^x \\ 2^x & e^x \end{vmatrix} = -2^x e^x$, $W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2^x \end{vmatrix} = 2^x e^{2x}$
$u_1' = \frac{W_1}{W} = 2^x e^{-2x}$, $u_2' = \frac{W_2}{W} = -2^x e^{-x}$. Both integrals resolve using integration by parts:
$u_1 = \int 2^x e^{-2x}\, dx = 2^x \left(-\frac{1}{2}\right) e^{-2x} - \int \left(-\frac{1}{2}\right) e^{-2x} 2^x \ln 2\, dx \to u_1 = \int 2^x e^{-2x}\, dx = \frac{-2^x e^{-2x}}{2 - \ln 2}$
Similarly, $u_2 = \frac{2^x e^{-x}}{1 - \ln 2}$, thus $y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^x + \frac{-2^x}{2 - \ln 2} + \frac{2^x}{1 - \ln 2}$

3) Find a general solution of the equation $x^2y'' - 4xy' + 6y = 2x^4 + x^2$

This is a Cauchy-Euler ODE, so we apply the necessary substitution: If we let $\tilde{y}(t) = y(e^t)$ then $x^2y'' - 4xy' + 6y = 2x^4 + x^2 \xrightarrow{x=e^t} \tilde{y}'' - 5\tilde{y}' + 6\tilde{y} = 2e^{4t} + e^{2t}$. The homogeneous equation has the auxiliary equation $m^2 - 5m + 6 = (m-2)(m-3) = 0$, so $\tilde{y}_c(x) = c_1e^{2t} + c_2e^{3t}$
For the particular solution, we apply the method of undetermined coefficients. A particular solution must be of the form $\tilde{y}_p(x) = Ate^{2t} + Be^{4t}$. The differential operator is $L = D^2 - 5D + 6$. $Ly_p = -Ae^{2t} + 2Be^{4t} = e^{2t} + 2e^{4t}$, so $A = -1$ and $B = 1$, so $\tilde{y}_p(x) = e^{4t} - te^{2t}$, so $\tilde{y}(x) = c_1e^{2t} + c_2e^{3t} + e^{4t} - te^{2t}$. Now, reversing the substitution using $t = \ln x$, we have $y(x) = c_1x^2 + c_2x^3 + x^4 - x^2\ln x$

4) Solve the following IVP $\begin{cases} x' = x + 2y + 1 \\ y' = 2x + y + e^t \end{cases}$ with $x(0) = \frac{3}{2}$ and $y(0) = -2$

Solving the first equation for $y = \frac{1}{2}(x' - x - 1)$, so $y' = \frac{1}{2}(x'' - x')$. Plugging this into the second equation, we have $\frac{1}{2}x'' - x' - \frac{3}{2}x = e^t - \frac{1}{2}$. The auxiliary equation for the homogeneous equation is $m^2 - 2m - 3 = (m-3)(m+1) = 0$ so $x_p(t) = c_1e^{3t} + c_2e^{-t}$.
Using undetermined coefficients, the particular solution is $x_p = Ae^t + B$. Our differential operator is $L = \frac{1}{2}D^2 - D - \frac{3}{2}$, so $Lx_p = -2Ae^t - \frac{3}{2}B = e^t - \frac{1}{2}$ so $A = -\frac{1}{2}$ and $B = \frac{1}{3}$.
This gives us $x(t) = c_1e^{3t} + c_2e^{-t} - \frac{1}{2}e^t + \frac{1}{3}$. We can now plug in and find $y(t) = c_1e^{3t} - c_2e^{-t} - \frac{2}{3}$. Now applying our initial conditions: $\frac{5}{3} = c_1 + c_2$ and $-\frac{4}{3} = c_1 - c_2$. Solving, we find that $\frac{3}{2} = c_2$ and $\frac{1}{6} = c_1$, so $x(t) = \frac{1}{6}e^{3t} + \frac{3}{2}e^{-t} - \frac{1}{2}e^t + \frac{1}{3}$ and $y(t) = \frac{1}{6}e^{3t} - \frac{3}{2}e^{-t} - \frac{2}{3}$.

5) Solve the IVP $y'' + y = \begin{cases} 0 & 0 \le t < 2\pi \\ \sin t & t \ge 2\pi \end{cases}$ with $y(0) = 1$ and $y'(0) = 0$.

Approach #1: Laplace Transforms:
The ODE is $y'' + y = U(t-2\pi)\sin t$. Note $\sin t$ is periodic (period 2π), so this can also be written $y'' + y = \mathcal{U}(t-2\pi)\sin(t-2\pi)$. Apply the Laplace transform to both sides:

$$s^2 Y(s) - s + Y(s) = \frac{e^{-2\pi s}}{1+s^2} \to Y(s)\left(s^2+1\right) = \frac{e^{-2\pi s}}{1+s^2} + s \to Y(s) = \frac{e^{-2\pi s}}{\left(1+s^2\right)^2} + \frac{s}{1+s^2}$$

We need to reverse the Laplace Transform to get the answer, but the $\frac{1}{\left(1+s^2\right)^2}$ term is a problem. The form of the denominator suggests that this is related to a derivative of a trig function: Examine $\mathcal{L}\{t\cos t\} = \frac{d}{ds}\frac{s}{1+s^2} = \frac{s^2-1}{\left(1+s^2\right)^2}$. This is close, but not correct: Subtracting another trig function helps the numerator:

$\mathcal{L}\{t\cos t - \sin t\} = \frac{s^2-1}{\left(1+s^2\right)^2} - \frac{1}{1+s^2} = \frac{-2}{\left(1+s^2\right)^2}$, so finally we have it:

$\mathcal{L}\left\{-\frac{1}{2}(t\cos t - \sin t)\right\} = \frac{1}{\left(1+s^2\right)^2}$. Thus, the answer is:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{\left(1+s^2\right)^2} + \frac{s}{1+s^2}\right\} = -\mathcal{U}(t-2\pi)\frac{1}{2}\big((t-2\pi)\cos(t-2\pi) - \sin(t-2\pi)\big) + \cos(t)$$

so

$$y(t) = -\mathcal{U}(t-2\pi)\frac{1}{2}\big((t-2\pi)\cos(t) - \sin(t)\big) + \cos(t)$$

5) Approach #2: Solution Using Smoothness
We have two behaviors for our input function; we solve them independently, and then assemble them under the assumption that the solution should be smooth:
Interval $[0,2\pi)$: $y''+y=0$ is homogeneous. It has the auxiliary equation $m^2+1=0$, whose roots are $m=\pm i$, so $y(t)=c_1\sin t+c_2\cos t$. The provided initial values are in the interval associated with this behavior, so $y(0)=1=c_2$ and $0=y'(0)=c_1$, so $y(t)=\cos t$. This gives us the new conditions at 2π : $y(2\pi)=1$ and $y'(2\pi)=0$

Interval $[2\pi,\infty)$: $y''+y=\sin t$ has the complementary function (as above) $y_c(t)=c_1\sin t+c_2\cos t$. Using the method of variation of parameters, we find that
$W=\begin{vmatrix}\sin t & \cos t\\ \cos t & -\sin t\end{vmatrix}=-1$, $W_1=\begin{vmatrix}0 & \cos t\\ \sin t & -\sin t\end{vmatrix}=-\cos t\sin t$, $W_2=\begin{vmatrix}\sin t & 0\\ \cos t & \sin t\end{vmatrix}=\sin^2 t$
$u_1'=\frac{W_1}{W}=\cos t\sin t$, $u_2'=\frac{W_2}{W}=-\sin^2 t$:
$u_1=\int\cos t\sin t\,dt=-\frac{1}{2}\cos^2 t$ and
$u_2=\int-\sin^2 t\,dt=-\int\frac{1}{2}\left(1-\cos(2t)\right)dt=-\frac{1}{2}t+\frac{1}{4}\sin(2t)$, thus
$y_p(x)=-\frac{1}{2}\cos^2 t\sin t-\frac{1}{2}t\cos t+\frac{1}{4}\sin(2t)\cos t$, yielding the solution
$y(t)=c_1\sin t+c_2\cos t-\frac{1}{2}\cos^2 t\sin t-\frac{1}{2}t\cos t+\frac{1}{4}\sin(2t)\cos t$.
Applying the (new) initial conditions: $y(2\pi)=c_2-\pi=1$ so $c_2=1+\pi$
Similarly, we can take the derivative of this function and find $y'(2\pi)=c_1-\frac{1}{2}=0$ so $c_1=\frac{1}{2}$. We are left $y(t)=\frac{1}{2}\sin t+(1+\pi)\cos t-\frac{1}{2}\cos^2 t\sin t-\frac{1}{2}t\cos t+\frac{1}{4}\sin(2t)\cos t$

Our answer is thus

$$y(t)=\begin{cases}\cos t & 0\le t<2\pi\\ \frac{1}{2}\sin t+(1+\pi)\cos t-\frac{1}{2}\cos^2 t\sin t-\frac{1}{2}t\cos t+\frac{1}{4}\sin(2t)\cos t & t\ge 2\pi\end{cases}$$

This simplifies to:

$$y(t)=\cos t-\mathcal{U}(t-2\pi)\frac{1}{2}\left[(t-2\pi)\cos t-\sin t\right]$$

(This is the same answer we got the other way)