Weyl Group Representations and Unitarity of Spherical Representations.

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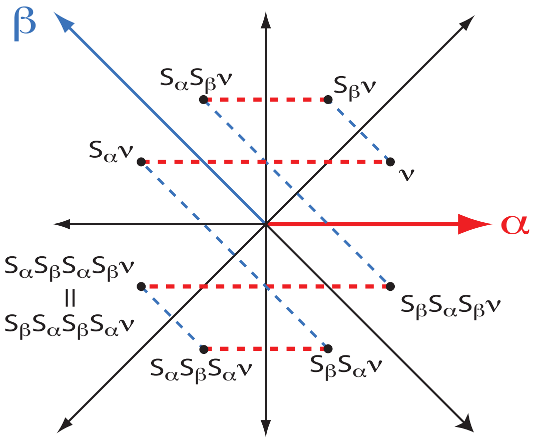
Windsor, October 23, 2008



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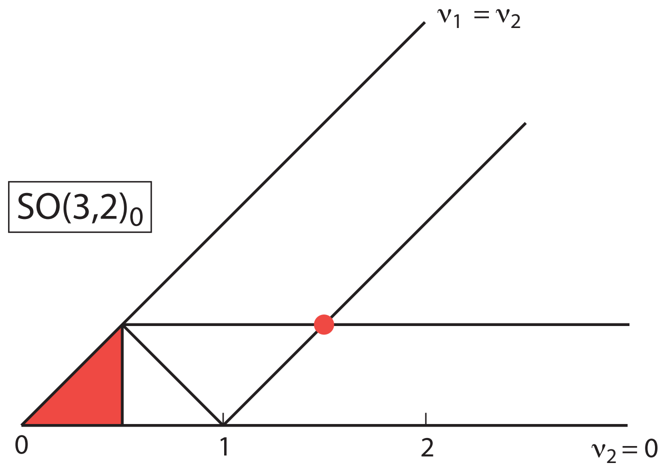


Spherical unitary dual of real split semisimple Lie groups

spherical unitary-dual of G	=	***equivalence classes of irreducible unitary spherical repr.s of G***	=	**?**

aim of the talk *Show how to compute this set using the Weyl group*

Plan of the talk

- **Preliminary notions**: root system of a split real Lie group
- **Define the unitary dual**
- **Examples** (finite and compact groups)
- **Spherical unitary dual of non-compact groups**
- **Petite K-types**
- **Real and p-adic groups: a comparison of unitary duals**
- **The example of $Sp(4)$**
- **Conclusions**

Lie Groups

A **Lie group** G is a group with a smooth manifold structure, such that the product and the inversion are smooth maps

Examples:

- the symmetryc group $\mathcal{S}_n$=$\{$bijections on $\{1, 2, \ldots, n\}\}$ ← *finite*
- the unit circle $\mathcal{S}^1 = \{z \in \mathbb{C} \colon \|z\| = 1\}$ ← *compact*
- $SL(2, \mathbb{R})$=$\{A \in M(2, \mathbb{R}) \colon \det A = 1\}$ ← *non-compact*

Root Systems

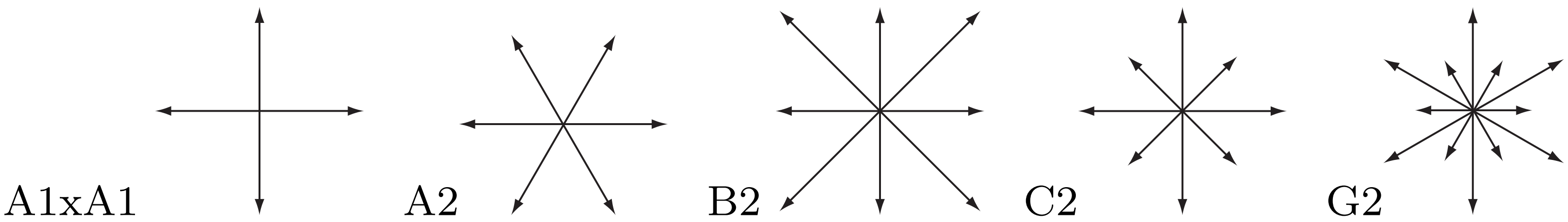
Let $V \simeq \mathbb{R}^n$ and let $\langle , \rangle$ be an inner product on V. If $v \in V$-$\{0\}$, let

$$\sigma_v : w \mapsto w - 2\frac{\langle v,w\rangle}{\langle v,v\rangle}v$$

be the reflection through the plane perpendicular to v.

A root system for V is a finite subset R of V such that

- **R spans V, and $0 \notin R$**
- **if $\alpha \in R$, then $\pm\alpha$ are the only multiples of α in R**
- **if $\alpha, \beta \in R$, then $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$**
- **if $\alpha, \beta \in R$, then $\sigma_\alpha(\beta) \in R$**

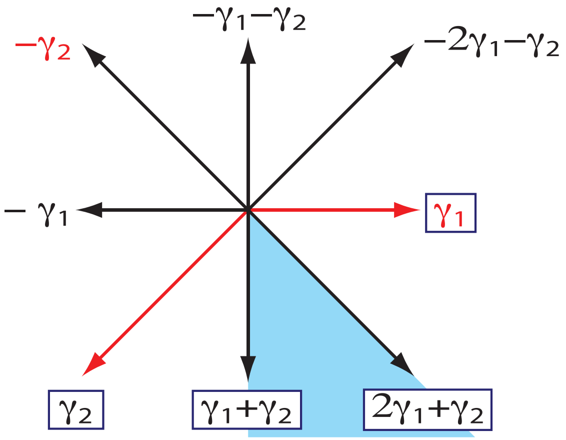


Simple roots

Let V be an n-dim.l vector space and let R be a root system for V.

The roots $\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2 \ldots \alpha_n} \in \boldsymbol{R}$ are called simple if

- **they are a basis of $\boldsymbol{V}$**
- **every root in $\boldsymbol{R}$ can written as $\sum_i a_i \alpha_i$, with all $a_i \geq 0$ or all $a_i \leq 0$.**



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Each choice of simple roots determines a set of ***positive roots***.

The Weyl group

Let R be a root system for V.

The Weyl group of the root system is the finite group of orthogonal transformations on V generated by the reflections through the simple roots.

[type A_2] $V=\{\underline{v} \in \mathbb{R}^3 | \sum_j v_j = 0\} \simeq \mathbb{R}^2$.

Simple roots: $\alpha = e_1 - e_2$, $\beta = e_2 - e_3$.

- $s_\alpha = s_{e_1 - e_2}$ acts $\underline{v} \in \mathbb{R}^3$ by switching the 1^{st} and 2^{nd} coordinate
- $s_\beta = s_{e_2 - e_3}$ acts $\underline{v} \in \mathbb{R}^3$ by switching the 2^{nd} and 3^{rd} coordinate.

$W = \langle s_\alpha, s_\beta \rangle =$ the symmetric group S_3 (permutations of $1, 2, 3$).

The root system of a real split Lie group

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For all $x \in G$, there is an inner automorphism $\mathrm{Int}(x)\colon G \to G$, $g \mapsto xgx^{-1}$.

The differential of $\mathrm{Int}(x)$ is a linear transformation on $\mathfrak{g}$, denoted by $\mathrm{Ad}(x)$. We extend $\mathrm{Ad}(x)$ to the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$, by linearity. The map $\mathrm{Ad}\colon G \to GL(\mathfrak{g}_{\mathbb{C}})$, $x \mapsto \mathrm{Ad}(x)$
is a representation of G, called the *adjoint representation*.

Assume that G is a **real split group** of rank n. Then G contains a subgroup $A \simeq (\mathbb{R}_{\geq 0})^n$ such that the operators $\{\mathrm{Ad}(g)\}_{g \in A}$ are simultaneously diagonalizable.

Decompose $\mathfrak{g}_{\mathbb{C}}$ in simultaneous eigenspaces for $\mathrm{Ad}(A)$. The *nonzero* eigenfunctions form **a root system**. The Weyl group is $\frac{N_K(A)}{C_K(A)}$.

Every "abstract" root system Δ appears as the root system of a real *split* semisimple Lie group G.

Δ	G	$K \subset G$ (maximal compact)
A_n	$SL(n+1,\mathbb{R})$	$SO(n+1)$
B_n	$SO(n+1,n)_0$	$SO(n+1) \times SO(n)$
C_n	$Sp(2n,\mathbb{R})$	$U(n)$
D_n	$SO(n,n)_0$	$SO(n) \times SO(n)$
G_2	G_2	$SU(2) \times SU(2)/\{\pm I\}$
F_4	F_4	$Sp(1) \times Sp(3)/\{\pm I\}$
E_6	E_6	$Sp(4)/\{\pm I\}$
E_7	E_7	$SU(8)/\{\pm I\}$
E_8	E_8	$Spin(16)/\{I,w\}$

Unitary Representations

Let G be a Lie group, $\mathcal{H}$ be a complex Hilbert space.

A representation of G on $\mathcal{H}$ is a group homomorphism

$$\pi\colon G \to \mathcal{B}(\mathcal{H}) = \{\text{bounded linear operators on } \mathcal{H}\}$$

such that the map $\pi\colon G \times \mathcal{H} \to \mathcal{H}$, $(g, v) \mapsto \pi(g)v$
is continuous.

π is called **unitary** if $\pi(g)$ is a unitary operator on $\mathcal{H}$, $\forall\, g \in G$

Examples:

- $\mathcal{H} = \mathbb{C}$, $\pi(g)v = v$ $\leftarrow$ *trivial representation*
- $\mathcal{H} = L^2(G, \mu)$, $\pi(g)f = f(\cdot\, g)$ $\leftarrow$ *right regular representation*

Irreducibility and Equivalence of Representations

A representation $(\pi, \mathcal{H})$ of G is **irreducible** if $\{0\}$ and $\mathcal{H}$ are the only *closed* *G-stable* subspaces. [W is G-stable if $\pi(G)W \subset W$]

Two representations $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ of G are **equivalent** if there is a bounded linear operator $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ (with bounded inverse) such that $T \circ \pi_1(g) = \pi_2(g) \circ T$, $\forall\, g \in G$

For finite-dimensional representations, define the **character** by

$$\chi_\pi\colon G \mapsto \mathbb{C},\ g \mapsto trace(\pi(g)).$$

If π_1, π_2 are finite-dimensional, then $\pi_1 \simeq \pi_2 \Leftrightarrow \chi_{\pi_1} = \chi_{\pi_2}$.

The Unitary Dual

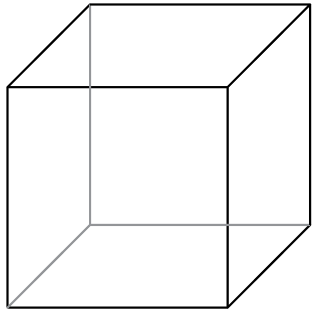
Let G be a Lie group.

Unitary Dual of G = Equivalence classes of irreducible **unitary** representations of G = ?

The symmetric group on n letters

$\mathcal{S}_n = \{$**permutations** on the set $\{1, 2, \ldots, n\}\}$

e.g. $\mathcal{S}_4$ ***is the group of symmetries of the cube (permutations of the diagonals).***



Every permutation can be written as a product of cycles:

$\sigma = (15)(24) \in \mathcal{S}_5$ is the bijection: $1 \leftrightarrow 5$; $2 \leftrightarrow 4$; $3 \circlearrowleft 3$.

Two permutations are conjugate $\Leftrightarrow$ same cycle structure.

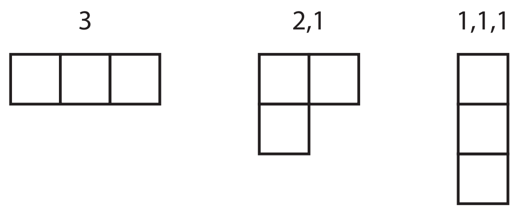
e.g. the partitions $(1324)(56)$ and $(1523)(46)$ are conjugate in $\mathcal{S}_6$.

Conjugacy classes in $S_n \longleftrightarrow$ partitions of n.

Irreducible representations of the symmetric group $\mathcal{S}_3$

Elements of S_3*:* $\underbrace{(123),(132)}_{\text{conjugate}},\underbrace{(12),(13),(23)}_{\text{conjugate}},\mathrm{Id}=(1)(2)(3).$

Conjugacy classes in S_3*:*



For every finite group G, the number of equivalence classes of irred. representations equals the number of conjugacy classes.

$\Rightarrow$ S_3 has 3 irreducible inequivalent representations.

- **The Trivial representation**: $\mathcal{H}_1 = \mathbb{C}$ and

$$\pi_1(\sigma)v = v \quad \forall\, \sigma \in \mathcal{S}_3,\ v \in \mathbb{C}.$$

- **The Sign representation**: $\mathcal{H}_2 = \mathbb{C}$ and

$$\pi_2(\sigma)v = \begin{cases} +v & \text{if } \sigma \text{ is even} \\ -v & \text{if } \sigma \text{ is odd.} \end{cases}$$

[*σ is even if it is a product of an even number of 2-cycles.*]

- **The Permutation representation**: $\mathcal{H} = \mathbb{C}^3$ and

$$\sigma \cdot (v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) \qquad \forall\, \sigma \in \mathcal{S}_3,\ \forall\, \underline{v} \in \mathbb{C}^3.$$

Not irreducible! $U = \langle 1, 1, 1\rangle$ and $W = U^{\perp} = \left\{ \underline{v} \in \mathbb{C}^3 \colon \sum_{i=1}^{3} v_i = 0 \right\}$ are (closed and) G-stable. Let π_3 be the restriction of σ to W.

π_1, π_2 and π_3 are *all* the irreducible repr.s of S_3, up to equivalence.

What about unitarity?

If G is a finite group, any representation (π, V) of G is unitarizable.

- Start from any inner product $(\cdot, \cdot)$ on V
- Construct a new inner product by averaging over the group:

$$\langle v, w \rangle \equiv \frac{1}{\#G} \sum_{g \in G} (\pi(g)v, \pi(g)w) \quad \forall\, v, w \in V.$$

- $\langle , \rangle$ is invariant under the action of G, so π unitary.

True for compact groups. Replace $\sum_{g \in G}$ by $\int_{g \in G} \cdots dg$.

The unitary dual of finite groups

Let G be a finite group.

- **The number of equivalence classes of irreducible representations equals the number of conjugacy classes.**
- **Every irreducible representation is finite-dimensional.**
- **Every irreducible representation is unitary.**
- **Two irreducible representations are equivalent if and only if they have the same character.**
- **The characters can be computed explicitly.**

This gives a complete classification. What we are still missing is an explicit model of the representations ...

The unitary dual of compact groups

Let G be a compact group.

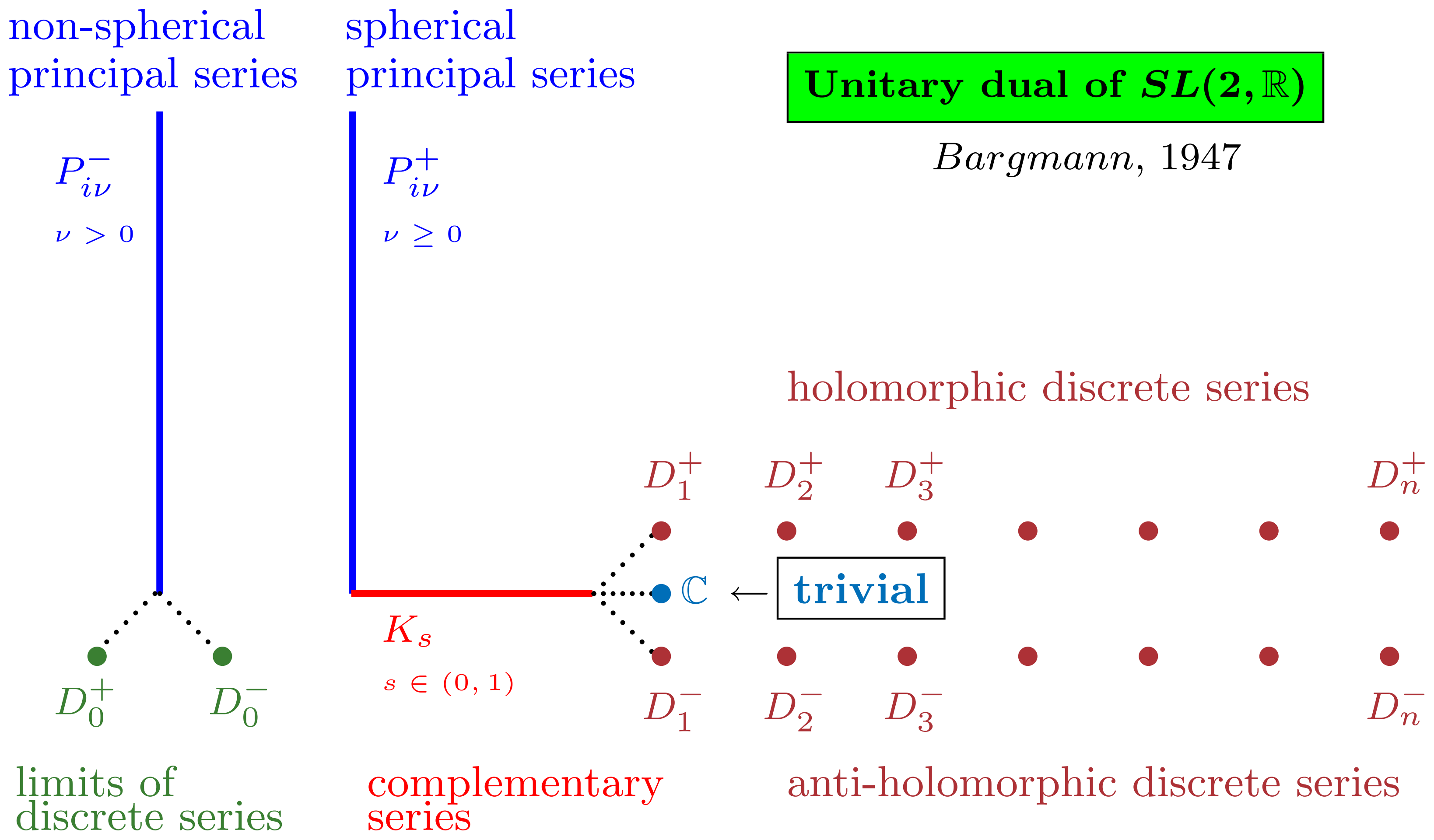
- **G has infinitely many irreducible inequivalent repr.s.**
- **Every irreducible representation is finite-dimensional.**
- **Every irreducible representation is unitary.**

The irreducible repr.s of compact connected semisimple groups are known. They are classified by highest weight.

We have formulas for the character and the dimension of each representations. What we are missing is an explicit construction.

What about the non-compact group $SL(2,\mathbb{R})$?

$SL(2,\mathbb{R})$ has many interesting unitary irreducible representations, but only one is finite-dimensional: the trivial representation!



The unitary dual of non-compact groups

G: real reductive group, e.g. $SL(n,\mathbb{R})$, $SO(n,\mathbb{R})$, $Sp(n)$ or any closed subgroup of $GL(n,\mathbb{C})$ stable under $A \mapsto \overline{(A^t)^{-1}}$.

- **Not every irreducible representation is unitary**
- **Non-trivial irreducible unitary repr.s are infinite diml**

$\Rightarrow$ *finding the unitary dual of non-compact groups is much harder!*

The full unitary dual is only known for

- $SL(2,\mathbb{R}) \leftarrow$ Bargmann, 1947
- $GL(n,\mathbb{R})$, $G_2 \leftarrow$ Vogan, 1986, 1994
- complex classical groups $\leftarrow$ Barbasch, 1989

Spherical unitary dual of real split semisimple Lie groups

G: *a real split semisimple Lie group*

K: *a maximal compact subgroup of* G

π: *an irreducible representation of* G *on a Hilbert space* $\mathcal{H}$.

π is spherical if $\mathcal{H}$ contains a vector which is fixed by K.

spherical unitary-dual of G = ***equivalence classes of irreducible unitary spherical repr.s of G*** = **?**

Unitary representations and $(\mathfrak{g}, K)$-modules

Harish-Chandra introduced a tool that allows to study unitary repr.s using algebra instead of analysis: "the notion of $(\mathfrak{g}, K)$*-module".*

A $(\mathfrak{g}, K)$-module V is a $\mathbb{C}$-vector space carrying an action of the Lie algebra $\mathfrak{g}=(\mathfrak{g}_0)_{\mathbb{C}}$ <u>and</u> an action of the maximal compact subgroup K, with some compatibility conditions.

A $(\mathfrak{g}, K)$-module is **unitary** if it has a positive definite non-zero Hermitian form which is invariant under the actions of $\mathfrak{g}$ and K.

unitary dual of G = ***equivalence classes of irreducible unitary $(\mathfrak{g}, K)$-modules***

Unitary dual of G

$\|$

unitary irreducible $(\mathfrak{g}, K)$*-modules* $\subseteq$ *Hermitian irreducible* $(\mathfrak{g}, K)$*-modules* $\subseteq$ *irreducible* $(\mathfrak{g}, K)$*-modules*

$\searrow$ **?** $\searrow$ *known* $\downarrow$ *known*

- In 1973, Langlands proved that every irreducible $(\mathfrak{g}, K)$-module is a **“Langlands quotient”**.
- In 1976, Knapp and Zuckerman understood which irreducible Langlands quotients are Hermitian.

find the unitary dual = ***explain which Langlands quotients have a pos. definite invariant form***

Spherical Langlands quotients

(with real infinitesimal character)

- G: *real split* semisimple Lie group
- $K \subset G$ maximal compact subgroup
- $P = MAN$ minimal parabolic subgroup

Here M is a finite abelian group and $A \simeq (\mathbb{R}_{\geq 0})^n$, with n=rank(G).

G	K	$P = MAN$	M	A
$SL_n(\mathbb{R})$	$SO_n(\mathbb{R})$	upper triangular matrices	diagonal matrices $a_{i,i}=\pm 1$	diagonal matrices $a_{i,i} > 0$

Spherical Langlands quotients are a 1-parameter family of irred. spherical repr.s of G. The parameter lies in a cone inside $(\mathfrak{a})^* \simeq \mathbb{R}^n$.

Spherical Langlands Quotients

- *Fix a strictly dominant linear functional* $\nu\colon \mathfrak{a} \longrightarrow \mathbb{R}$

  ν is an element in a cone inside the vector space $(\mathfrak{a})^* \simeq \mathbb{R}^n$.

- *Form the principal series* $I_P(\nu) = \mathrm{Ind}_{MAN}^G(triv \otimes \nu \otimes triv)$

  $I_P(\nu) = \{F\colon G \to \mathbb{C}\colon F|_K \in L^2,\ F(xman) = e^{-(\nu+\rho)log(a)}F(x),\ \forall\, man\}$ and G acts by left translation.

- *Take the unique irreducible quotient* $L_P(\nu)$ *of* $I_P(\nu)$

  There is an intertwining operator

  $$A(\nu)\colon I_P(\nu) \longrightarrow I_{\bar{P}}(\nu),\ F \mapsto \int_{\bar{N}} F(x\bar{n})\, d\bar{n}$$

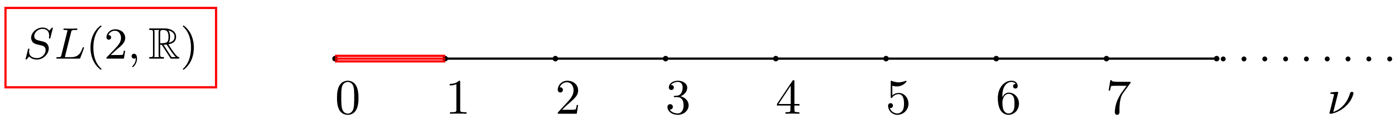
  such that $\boxed{L(\nu) \equiv \frac{I_P(\nu)}{\mathrm{Ker}\ A(\nu)}}$.

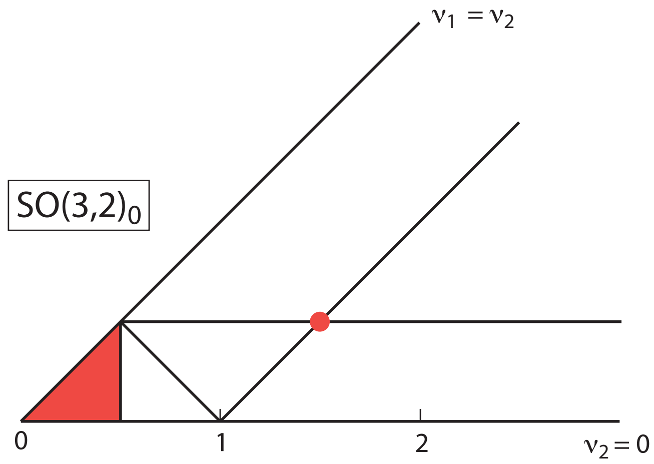
- *If* $L(\nu)$ *is Hermitian, the form is induced by the operator* $A(\nu)$.

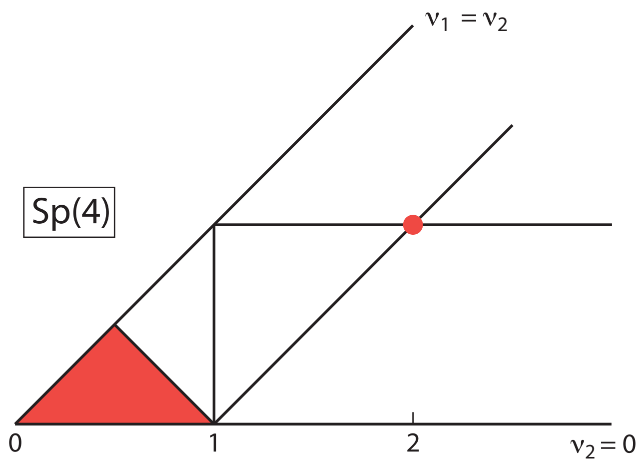
Finding the Spherical Unitary Dual

Finding all the parameters ν such that $A(\nu)$ is positive semidefinite

Hard Problem: the set of unitary parameters is very small!







Studying the signature of the operator $A(\nu)$

- $A(\nu)$ acts on the principal series, which is infinite dimensional.
- Restrict $A(\nu)$ to the isotypic component of each irreducible repr. of K that appears in the principal series (*"K-types"*)

$\Downarrow$

You get one finite-dim.l operator $A_\mu(\nu)$ for each spherical K-type μ

The Langlands quotient $L(\nu)$ is unitary if and only if the operator $A_\mu(\nu)$ is semidefinite for all $\mu \in \widehat{K}$

Hard problem

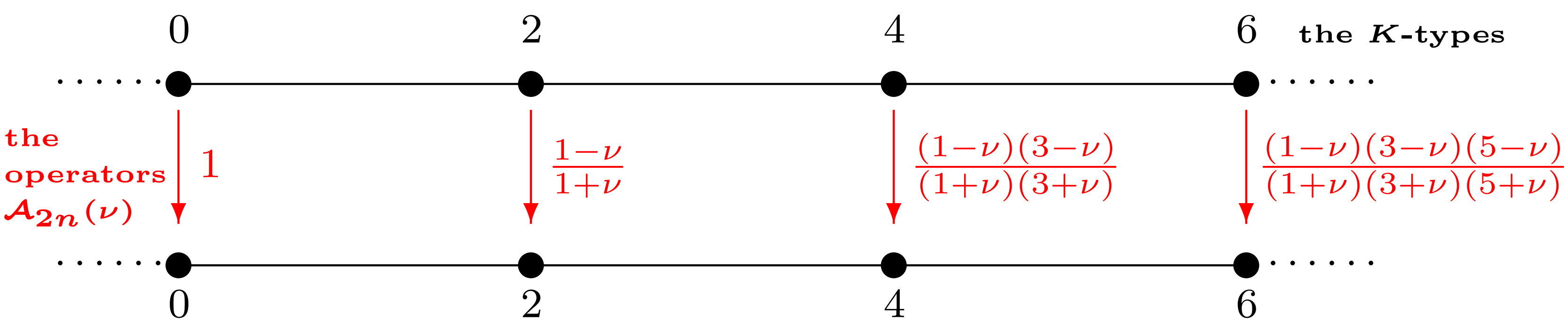
- *Need to compute infinitely many operators*
- *The operators can be quite complicated*

The easiest example: $SL(2,\mathbb{R})$

$G = SL(2,\mathbb{R}),\ K = SO(2,\mathbb{R}) \simeq S^1,\ \widehat{K} = \mathbb{Z}\ (\chi_n \colon e^{i\theta} \mapsto e^{in\theta}),\ \nu > 0.$

There is one operator $A_{2n}(\nu)$ for each spherical K-type $2n$.

Each operator acts by a scalar.



$L(\nu)$ is unitary $\Leftrightarrow$ $A_{2n}(\nu)$ pos. semidefinite $\forall\, n$ $\Rightarrow$ $0 \leq \nu \leq 1$

The operators $A_\mu(\nu)$ for real split groups

$A_\mu(\nu)$ decomposes as a product of operators corresponding to simple reflections: $\boxed{A_\mu(\nu) = \prod_{\alpha\ simple} A_\mu(s_\alpha, \lambda)}$.

The α-factor $A_\mu(s_\alpha, \lambda)$ depends on the decomposition of μ with respect to the $SO(2)$-subgroup attached to α

$$\mu|_{SO(2)_\alpha} = \bigoplus_{n\in\mathbb{Z}} V^\alpha(n) \Rightarrow (\mu^*)^M = \bigoplus_{m\in\mathbb{N}} \mathrm{Hom}_M(V^\alpha(2m) \oplus V^\alpha(-2m), \mathbb{C}).$$

$A_\mu(s_\alpha, \lambda)$ acts on the m^{th} piece by the scalar $\boxed{\frac{(1-\xi)(3-\xi)\cdots(2m-1-\xi)}{(1+\xi)(3+\xi)\cdots(2m-1+\xi)}}$, with $\xi = \langle\check{\alpha}, \lambda\rangle$. *Main difficulty: keep track of the decompositions* $\mu|_{SO(2)_\alpha}$.

The idea of petite K-types *(Barbasch, Vogan)*

To get necessary *and sufficient* conditions for unitarity $\rightsquigarrow$ compute infinitely many complicated operators (one for each K-type μ).

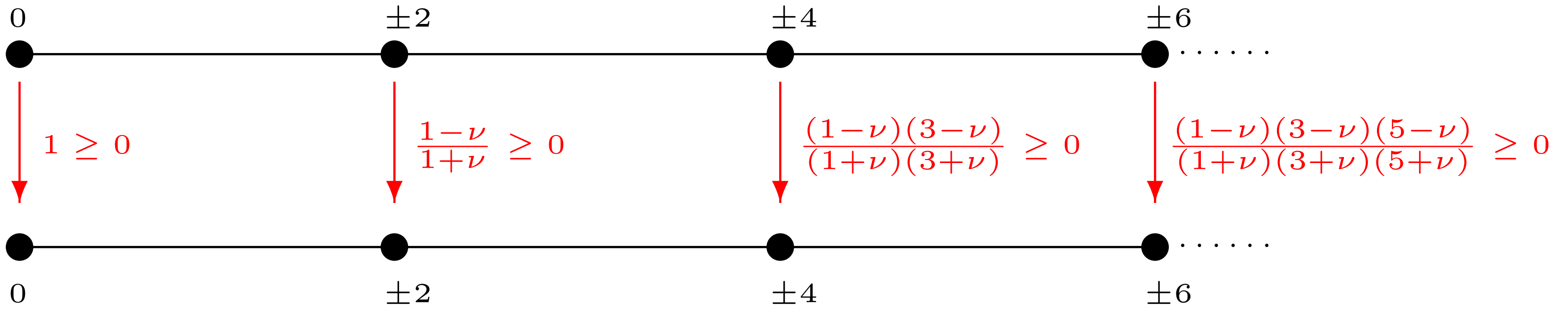
Instead . . .

Choose a small set of "petite" K-types where computations are easy. Compute *only* these finitely many easy operators. $\rightsquigarrow$ **necessary conditions for unitarity**

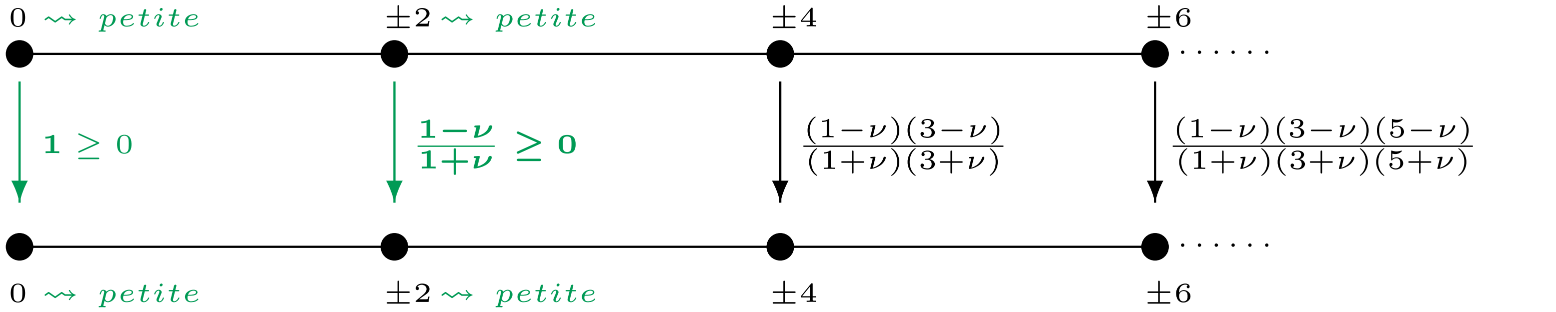
This method is often enough to rule out large non-unitarity regions.

Spherical Petite K-types for $SL(2,\mathbb{R})$: $n = 0, \pm 2$

N+S. conditions for unitarity : $A_{2n}(\nu)$ *pos. semidefinite* $\forall\, n$



N. conditions for unitarity : $A_{2n}(\nu)$ *pos. semidefinite if* $n{=}0, \pm 2$



In both cases you get $\nu > 1$. *You rule out the same region!*

Spherical "petite" K-types for split real groups

A spherical K-type μ is "**petite**" if -for every root α- the restriction of μ to the $SO(2)$ subgroup attached to α only contains the $SO(2)$-types $n = 0, \pm 1, \pm 2, \pm 3$.

If $G{=}SL(2n, \mathbb{R})$, $K{=}SO(2n, \mathbb{R})$. K-types are parameterized by:

$$(a_1, \ldots, a_n) \in \mathbb{Z}^n \quad | \quad a_1 \geq a_2 \geq \ldots a_{n-1} \geq |a_n|.$$

The *spherical petite* K-types are:

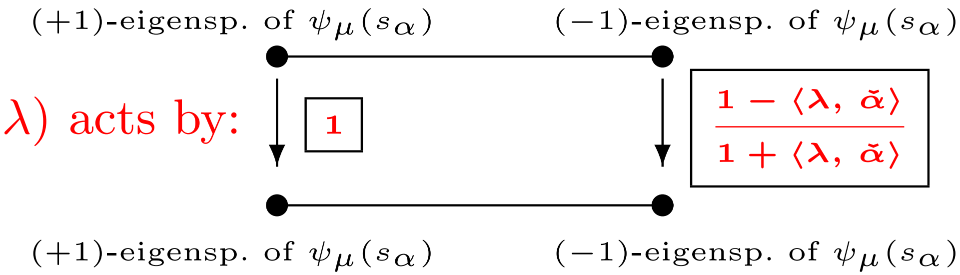
- $(0, 0, \ldots, 0) \rightsquigarrow$ the trivial representation of $SO(2n, \mathbb{R})$
- $(\underbrace{2, 2, \ldots, 2}_{k<n}, 0, \ldots, 0) \rightsquigarrow$ the representation $Sym^2(\Lambda^k \mathbb{C}^{2n})$
- $(2, 2, \ldots, 2, \pm 2) \rightsquigarrow$ the two irreducibles pieces of $Sym^2(\Lambda^n \mathbb{C}^{2n})$.

What makes spherical petite K-types so special?

Let μ be a spherical K-type. The operator $A_\mu(\nu)$ acts on $(\mu^*)^M$. This space carries a representation ψ_μ of the Weyl group.

If μ is petite, the operator $A_\mu(\nu)$ only depends on the W-type ψ_μ and can be computed with Weyl group calculations.

$A_\mu(\nu) = \prod_{\alpha \text{ simple}} A_\mu(s_\alpha, \lambda)$. $A_\mu(s_\alpha, \lambda)$ acts by:



P-adic Groups

If μ is petite, the formula for $A_\mu(\nu)$ coincides with the formula for a similar operator for a split p-adic group.

So we can use petite K-types to explore the relation between the spherical unitary duals of real and split groups.

Fix ν dominant. Denote by $L(\nu)$ the spherical module for the real split group G, and by $L(\nu)^{p\text{-}adic}$ the one for the corresponding p-adic group.

- In the real case, there is an operator $A_\mu(\nu)$ for each $\mu \in \widehat{K}$.
  $L(\nu)$ is unitary $\Leftrightarrow$ $A_\mu(\nu)$ is positive semidefinite $\forall\, \mu \in \widehat{K}$.
- In the p-adic case, there is an operator $R_\tau(\nu)$ for each $\tau \in \widehat{W}$.
  $L(\nu)^{p\text{-}adic}$ is unitary $\Leftrightarrow$ $R_\tau(\nu)$ is positive semidefinite $\forall\, \tau \in \widehat{W}$.
  It is enough to consider only the "relevant" W-types.
- For each relevant W-type τ there is a petite K-type μ such that "the p-adic operator" $R_\tau(\nu)$ = "the real operator" $A_\mu(\nu)$.

$L^G(\nu)$ is unitary ==================> $L(\nu)^{p\text{-}adic}$ is unitary

$\Updownarrow$ $\Updownarrow$

$A_\mu(\nu) \geq 0$ $\forall \mu \in \widehat{K}$ $\Rightarrow$ $A_\mu(\nu) \geq 0$ $\forall \mu$ petite $\Rightarrow$ $R_\tau(\nu) \geq 0$ $\forall \tau$ relevant $\Leftrightarrow$ $R_\tau(\nu) \geq 0$ $\forall \tau \in \widehat{W}$

Embedding of unitary duals

This matching of intertwining operators between real and p-adic groups gives an embedding of unitary duals:

spherical unitary dual of G	$\subseteq$	spherical unitary dual of $G^{p\text{-}adic}$

[Barbasch]: The inclusion is an equality for classical groups

Corollary: **A spherical Langlands quotient for a real split classical group is unitary if and only if the operator $R_\tau(\nu)$ is positive semidefinite for every relevant W-type τ.**

The spherical unitary dual of $Sp(4)$

If $G = Sp(4)$, $K = U(2)$. The root system is of type C_2:

$$\Delta^+ = \{e_1 - e_2,\ e_1 + e_2, 2e_1, 2e_2\}.$$

The Weyl group consists of all permutations and sign changes of the coordinates of $\mathbb{R}^2$, and is generated by the simple reflections:

$s_{e_1-e_2}$ which switches the two coordinates, and

s_{2e_2} which changes sign to the second coordinate.

Irreducible representations of W are parameterized by pairs of partitions. The relevant W-types are:

$$(2) \times (0),\ (1) \times (1),\ (0) \times (2),\ (1,1) \times (0).$$

Set $\nu = (a, b)$. The intertwining operator $R_\tau(\nu)$ admits a decomposition of the form:

$$R_\tau(s_{e_1-e_2},(-b,-a))R_\tau(s_{2e_2},(-b,a))R_\tau(s_{e_1-e_2},(a,-b))R_\tau(s_{2e_2},(a,b))$$

The factors are computed using the formula:

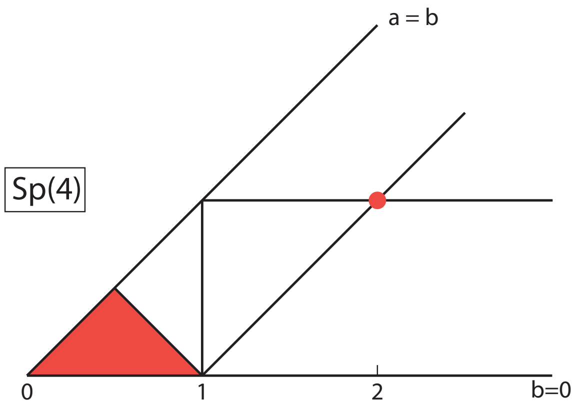
$$\boxed{R_\tau(s_\alpha,\lambda) = \frac{Id+\langle\check{\alpha},\lambda\rangle\tau(s_\alpha)}{1+\langle\check{\alpha},\lambda\rangle}}.$$

We need to know $\tau(s_\alpha)$. Here is an explicit description of the representations τ :

τ	dim	$\tau(s_{e_1-e_2})$	$\tau(s_{2e_2})$
$(2)\times(0)$	1	1	1
$(11)\times(0)$	1	-1	1
$(0)\times(2)$	1	1	-1
$(1)\times(1)$	2	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

relevant W-type τ	the operator $R_\tau(\nu)$
$(2)\times(0)$	1
$(1,1)\times(0)$	$\frac{1-(a-b)}{1+(a-b)}\,\frac{1-(a+b)}{1+(a+b)}$
$(0)\times(2)$	$\frac{1-a}{1+a}\,\frac{1-b}{1+b}$
$(1)\times(1)$	*trace* $2\frac{1+a^2-a^3b-b^2+ab+ab^3}{(1+a)(1+b)[1+(a-b)][1+(a+b)]}$ *det* $\frac{1-a}{1+a}\,\frac{1-b}{1+b}\,\frac{1-(a-b)}{1+(a-b)}\,\frac{1-(a+b)}{1+(a+b)}$

$L(\nu)$ is unitary $\Leftrightarrow$ These 4 operators are positive semidefinite. . .

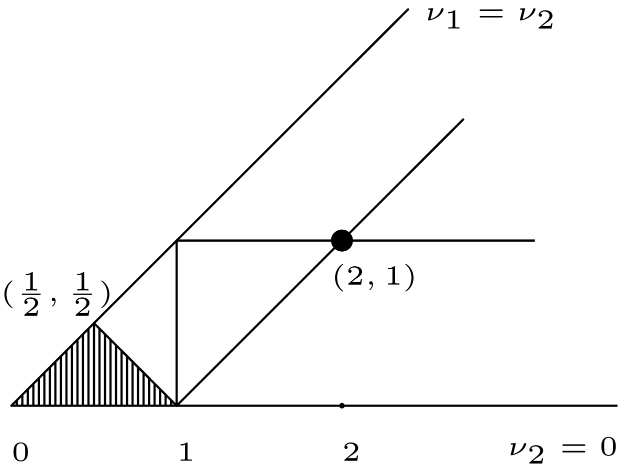


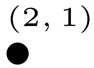
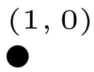
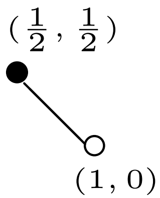
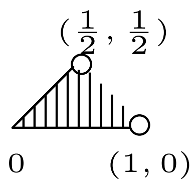
Another look at the spherical unitary dual of $Sp(4,\mathbb{R})$

The spherical unitary dual of the real split group $Sp(4,\mathbb{R})$ coincides with the spherical unitary dual of the p-adic split group of type C_2.

Hence it is a union of complementary series attached to the various nilpotent orbits in $\check{g} = \mathfrak{so}(5)$.

$Sp(4,\mathbb{R})$



	$\check{\mathcal{O}} = (5)$
	$\check{\mathcal{O}} = (3,1,1)$
	$\check{\mathcal{O}} = (2,2,1)$
	$\check{\mathcal{O}} = (1,1,1,1,1)$

Conclusions

*The spherical unitary dual of a **p-adic split group** is known. It is a union of complementary series attached to the various nilpotent orbits in the complex dual Lie algebra.*

If G is a **real split classical group**, the spherical unitary dual of G coincides the one of the corresponding p-adic group.

If G is a **real split exceptional group**, the matching is still a conjecture. However, we know the existence of an embedding:

spherical unitary dual of G^{real}	$\subseteq$	spherical unitary dual of $G^{p\text{-}adic}$

This inclusion provides interesting necessary conditions for the unitarity of spherical modules for the real group.

The proof relies on the notion of "spherical petite K-type".

Generalization

It is possible to generalize the notion of "petite K-types" to the context of non-spherical principal series, and derive simiilar inclusions.

Using non-spherical petite K-types one can relate the unitarity of a non-spherical Langlands quotient for a real split group G with the unitarity of a spherical Langlands quotient for a (different) p-adic group G^L.

G	G^L
$Mp(2n)$	$SO(p+1,p) \times SO(q+1,q)$, with $p+q=n$
$Sp(2n)$	$Sp(2p) \times Sp(2q)$, with $p+q=n$
F_4	$Sp(8); SO(4,3) \times SL(2)$
...	...