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The Omega-Regular Unitary Dual of the Metaplectic Group of Rank 2

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This paper is dedicated to the dear memory of Professor Fokko du Cloux.

Abstract. In this paper we formulate a conjecture about the unitary dual of the metaplectic group. We prove this conjecture for the case of Mp(4,R). The result is a strengthening, for this case, of the following result by the third author: any unitary representation of a real reductive Lie group with strongly regular infinitesimal character can be obtained by cohomological induction from a one dimensional representation. Strongly regular representations are those whose infinitesimal character is at least as regular as that of the trivial representation. We are extending the result to representations with omega-regular infinitesimal character: those whose infinitesimal character is at least as regular as that of the oscillator representation. The proof relies heavily on Parthasarathy's Dirac operator inequality. In one exception we explicitly calculate the signature of an intertwining operator to establish nonunitarity. Some of the results on intertwining operators presented in section 5.2 are joint work of Dan M. Barbasch and the first author.

1. Introduction

This paper is based on a presentation by the third author at the 13th Conference of African American Researchers in the Mathematical Sciences (CAARMS13). The presentation was intentionally expository, aimed at non-experts in the field of representation theory. With this in mind, an introductory survey of the fundamental concepts underlying this work was provided. A brief extract of the original presentation appears in the appendix. We have limited the introductory remarks to a discussion about $SL(2,\mathbb{R})$, as some results relative to this group are paramount for understanding the main ideas of the paper.

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1.1. Classification of representations. Let G be a real reductive Lie group. Recall that in [**15**], Vogan gave a classification of all admissible irreducible representations of G. In fact, he gave a parametrization of all such representations containing any given irreducible representation of K as a lowest K-type. Here K is the maximal compact subgroup of G. More precisely, we have the following

PROPOSITION 1. (See [**15, 16**] for definitions and details). To a reductive Lie group G, a maximal compact subgroup K of G, and an irreducible representation μ of K, we can attach a subgroup $L_a = L_a(\mu)$ of G, a parabolic subalgebra $\mathfrak{q}_a = \mathfrak{l}_a + \mathfrak{u}_a \subseteq \mathfrak{g}$ and an $L_a \cap K$ representation μ^{L_a} such that there is a bijection

$$\mathcal{R}_{\mathfrak{q}_a} : \left\{ \begin{array}{c} (\mathfrak{l}_a, L_a \cap K) \text{ modules} \\ \text{with lowest } (L_a \cap K)\text{-type } \mu^{L_a} \end{array} \right\} \longmapsto \left\{ \begin{array}{c} (\mathfrak{g}, K) \text{ modules} \\ \text{with lowest } K\text{-type } \mu \end{array} \right\}.$$

Here $\mathfrak{g}$ and $\mathfrak{l}_a$ are the complexified Lie algebras of G and L_a, respectively. (We use similar notation for other groups and Lie algebras, and use the subscript 0 to denote real Lie algebras.)

This construction is called cohomological parabolic induction. We call $\mathcal{R}_{\mathfrak{q}}$ the cohomological induction functor. The representations on the left-hand side are minimal principal representations of the subgroup L_a. Proposition 1 essentially reduces the classification of irreducible admissible representations of G to minimal principal series of certain subgroups. If μ is the lowest K-type of a principal series representation of G then we have $L_a(\mu) = G$, and there is no reduction.

In the case of $G = SL(2, \mathbb{R})$ and $K = S^1$, $L_a(n) = K$ for $|n| \geq 2$ and $L_a(n) = G$ for $|n| \leq 1$. Here we have identified the irreducible representations of S^1 with integers in the usual way.

For unitary representations, we would like to have a statement similar to Proposition 1. In other words, we would like to have some way of classifying all the unitary representations containing a certain lowest K-type μ. This is known to be possible in some cases; in general, we have the following conjecture (see [**13**]).

CONJECTURE 1. (See [**13**]) To each representation μ of K, we can attach a subgroup L_u, a parabolic subalgebra $\mathfrak{q}_u$ and a representation μ^{L_u} of $L_u \cap K$ such that there is a bijection

$$R_{\mathfrak{q}_u} : \left\{ \begin{array}{c} \text{unitary } (\mathfrak{l}_u, L_u \cap K) \text{ modules} \\ \text{with lowest } (L_u \cap K)\text{-type } \mu^{L_u} \end{array} \right\} \longmapsto \left\{ \begin{array}{c} \text{unitary } (\mathfrak{g}, K) \text{ modules} \\ \text{with lowest } K\text{-type } \mu \end{array} \right\}.$$

In principle, μ^{L_u} is a representation for which there is no such reduction to a representation of a smaller group. However, in the case of $SL(2, \mathbb{R})$, even though we can realize the discrete series as cohomologically induced from one dimensional representations of the group T, it fits best into the general conjecture to make $L_u(n) = T$ for $|n| > 2$ and $L_u(n) = G$ for $|n| \leq 2$. This suggests that the non-reducing K-types are 0, ± 1 and ± 2. We want to have a bijection like this for any real reductive Lie group.

REMARK 1.1. As in this case, in general $\mathfrak{q}_u(\mu) \supsetneq \mathfrak{q}_a(\mu)$.

We will now provide examples of unitary representations that can be constructed from, in some sense, "smaller", or easier to understand representations of proper subgroups of G.

1.2. The $A_{\mathfrak{q}}$ representations. In this section we describe a family of unitary representations that are cohomologically induced from one-dimensional representations of a subgroup. We focus on representation that satisfy some regularity condition on the infinitesimal character. In [**12**], the strongly regular case was considered (see 1.2.1). In the present paper we consider a weakening of the regularity assumption for representations of the metaplectic groups $Mp\,(2n,\mathbb{R})$, $n=1,2$.

1.2.1. *Strongly regular case.* Let G be reductive. Let $\mathfrak{h}=\mathfrak{t}+\mathfrak{a}$ be a maximally compact Cartan subalgebra of $\mathfrak{g}$, with $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{k}$. For a weight $\phi\in\mathfrak{h}^*$, choose a positive root system from the set of roots positive on ϕ:

$$\Delta^+(\phi)\subseteq\{\alpha\in\Delta(\mathfrak{g},\mathfrak{t})\ \mid\ \langle\phi,\alpha\rangle\geq 0\}.$$

Then define

$$\rho_\phi=\rho\left(\Delta^+(\phi)\right)=\frac{1}{2}\sum_{\alpha\in\Delta^+(\phi)}\alpha.$$

Definition 1.2. Suppose $\phi\in\mathfrak{h}^*$ is real. We say that ϕ is strongly regular if $\langle\phi-\rho_\phi,\alpha\rangle\geq 0$ for all $\alpha\in\Delta^+(\phi)$.

1.2.2. $A_{\mathfrak{q}}(\lambda)$ *representations.* Recall that a theta stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ of $\mathfrak{g}$ is defined as the sum of the nonnegative root spaces for $ad\,(\xi)$ where ξ is an element of $i\mathfrak{t}_0$. The Levi subalgebra $\mathfrak{l}$ is the zero eigenspace and contains $\mathfrak{t}$. It is a reductive subalgebra of $\mathfrak{g}$. The sum of the positive eigenspaces is the nilradical $\mathfrak{u}$ of $\mathfrak{q}$. Let L be the Levi subgroup of G corresponding to $\mathfrak{l}$. Then $\mathfrak{l}_0$ is the Lie algebra of L. We construct a representation of G as follows.

Definition 1.3. For every one-dimensional representation $\mathbb{C}_\lambda$ of L satisfying

$$\langle\lambda|_{\mathfrak{t}},\alpha\rangle\geq 0\qquad\forall\,\alpha\in\Delta(\mathfrak{u})\tag{1.1}$$

we define $A_{\mathfrak{q}}(\lambda):=R_{\mathfrak{q}}(\mathbb{C}_\lambda)$.

Here $\Delta(\mathfrak{u})=\Delta(\mathfrak{u},\mathfrak{t})$. In general, for any $\mathfrak{t}$-invariant subspace $\mathfrak{s}\subseteq\mathfrak{g}$, we write $\Delta(\mathfrak{s})=\Delta(\mathfrak{s},\mathfrak{t})$ for the set of weights of $\mathfrak{t}$ in $\mathfrak{s}$ counted with multiplicities.

Remark 1.4. All $A_{\mathfrak{q}}(\lambda)$ representations constructed this way are nonzero, irreducible and unitary.

Proposition 2. [**12**] Suppose G is a real reductive Lie group and X is an irreducible Hermitian $(\mathfrak{g},K)$ module with a real, strongly regular infinitesimal character. Then X is unitary if and only if there is a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ and a one-dimensional representation $\mathbb{C}_\lambda$ of L satisfying (1.1) and such that

$$X\simeq A_{\mathfrak{q}}(\mathbb{C}_\lambda).$$

1.2.3. *The omega-regular case and the $A_{\mathfrak{q}}(\Omega)$ representations of $Mp(2n)$.* Let $G=Mp\,(2n)$, the connected double cover of the group $Sp\,(2n,\mathbb{R})$. Then $\mathfrak{g}=\mathfrak{sp}\,(2n)$. Representations of G may be divided into those which factor through $Sp(2n,\mathbb{R})$ ("nongenuine" ones), and those that do not, the "genuine" representations. The nongenuine representations of G are essentially the representations of the linear group; in particular, a nongenuine representation of G is unitary if and only if the corresponding representation of $Sp(2n,\mathbb{R})$ is.

In order to build a genuine $A_{\mathfrak{q}}(\lambda)$ representation of G, we need to start with a genuine one-dimensional representation of the Levi subgroup L corresponding to the Levi factor $\mathfrak{l}$ of a theta stable parabolic subalgebra of $\mathfrak{g}$. Such subgroups are (quotients of) products of factors isomorphic to smaller metaplectic groups and double covers of $U(p,q)$'s. Notice that the metaplectic group does not have any genuine one-dimensional representation, hence there are no $A_{\mathfrak{q}}(\lambda)$ representations for such $\mathfrak{q}$. We extend our definition to allow the oscillator representation ω, a minimal genuine unitary representation of $Mp(2m)$ on such factors. The infinitesimal character of the oscillator representation is not strongly regular, but satisfies a slightly weaker condition, which we call "omega-regular" (see Definition 2.1 for a precise definition). If we apply cohomological induction to representations of L of the form

$$\Omega = \mathbb{C}_\lambda \otimes \omega \tag{1.2}$$

then, by a construction analogous to the one of the $A_{\mathfrak{q}}(\lambda)$ representations, we obtain genuine irreducible unitary representations of G with ω-regular infinitesimal character, which we call $A_{\mathfrak{q}}(\Omega)$ representations (see Definition 2.4).

If we hope to list all unitary ω-regular representations of G, we must extend our definition of $A_{\mathfrak{q}}(\Omega)$ representations to the nongenuine case as well, since the representations $I_P(\delta_+, u)$ of $SL(2,\mathbb{R})$ with $\frac{1}{2} < u < 1$ ("complementary series") in Table 1 (6.8) are ω-regular and unitary, but not $A_{\mathfrak{q}}(\lambda)$ modules. We define a family of nongenuine ω-regular unitary representations of G, which we call Meta-$A_{\mathfrak{q}}(\lambda)$ representations, by allowing complementary series on any $Mp(2)$ factor of L, and relaxing condition 1.1 somewhat (see Definition 2.5).

Conjecture 2. Let G be $Mp\,(2n)$ and let X be a genuine irreducible representation of G with real infinitesimal character. Then X is ω-regular and unitary if and only if there are $\mathfrak{q}$, L and a genuine representation Ω of L as above such that

$$X \simeq A_{\mathfrak{q}}\,(\Omega)\,.$$

If X is nongenuine, then X is ω-regular and unitary if and only if X is a Meta-$A_{\mathfrak{q}}\,(\lambda)$ representation.

The main result of this paper is a proof of the conjecture for $Mp\,(2)$ and $Mp\,(4)$ (see Conjecture 3 and Theorem 2.8). The case of $Mp\,(2n)$ with $n \geq 3$ has many additional interesting and complicating features and will appear in a future paper.

The full unitary duals of $Mp(2)$ and $Sp(4,\mathbb{R})$ are well known (c.f. [**4**] and [**10**]); some basic results are reported for the sake of completeness. The most innovative part of this paper regards genuine representations of $Mp(4)$. The proof of the conjecture in this case requires more elaborate techniques. A synopsis follows. First, we determine the set of genuine representations of K which are lowest K-types of $A_{\mathfrak{q}}\,(\Omega)$ representations; for every representation of K in this list, we establish that there is a unique unitary irreducible representation of $Mp(4)$ with that lowest K-type. Then, we consider genuine representations of K which are *not* lowest K-types of $A_{\mathfrak{q}}\,(\Omega)$ representations, and we establish that any ω-regular representation of $Mp(4)$ containing those K-types is nonunitary. It turns out that Parthasarathy's Dirac operator inequality can be used to prove nonunitary for all but two representations. The last section of the paper is dedicated to proving that these two remaining K-types cannot occur in any unitary representation. The proof is based on an explicit calculation of the signature of the intertwining operator. Some results about intertwining operators are included in Section 5.2.

The paper is organized as follows. In Section 2, we define our notation and state some preliminary facts and results. Sections 3, 4 and 5 contain the proof of

Conjecture 2 for $Mp\,(2)$ and $Mp\,(4)$. Section 3 is dedicated to $Mp(2)$, the remaining sections deal with $Mp(4)$.

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2. Preliminaries

2.1. Setup. Let $G = Mp(2n) = Mp(2n, \mathbb{R})$ be the metaplectic group, i. e., the connected double cover of the symplectic group $Sp(2n, \mathbb{R})$, and denote by

$$pr\colon Mp(2n) \to Sp(2n, \mathbb{R}) \tag{2.1}$$

the covering map. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, and let θ be the corresponding Cartan involution. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a theta stable parabolic subalgebra of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Then the Levi subgroup L of $Mp(2n)$ corresponding to $\mathfrak{l}$ is the inverse image under pr of a Levi subgroup of $Sp(2n, \mathbb{R})$ of the form

$$\prod_{i=1}^{r} U(p_i, q_i) \times Sp(2m, \mathbb{R}). \tag{2.2}$$

There is a surjection

$$\prod_{i=1}^{r} \widetilde{U}(p_i, q_i) \times Mp(2m) \to L, \tag{2.3}$$

where $\widetilde{U}(p_i, q_i)$ denotes the connected "square root of the determinant" cover of $U(p_i, q_i)$,

$$\widetilde{U}(p_i, q_i) \simeq \left\{(g, z) \in U(p_i, q_i) \times \mathbb{C}^{\times} : z^2 = \det(g)\right\}. \tag{2.4}$$

An irreducible admissible representation of L may be given by a representation

$$\bigotimes_{i=1}^{r} \pi_i \otimes \sigma, \tag{2.5}$$

where π_i is an irreducible admissible representation of $\widetilde{U}(p_i, q_i)$ for each i, and σ is an irreducible admissible representation of $Mp(2m)$. In order for this tensor product to descend to a representation of L, we must have that either all representations in the product are genuine, i. e., nontrivial on the kernel of the covering map, or all representations are nongenuine. In the first case, the representation σ of $Mp(2m)$ will then be genuine. In the second case, it will factor through $Sp(2m, \mathbb{R})$. With this in mind, we will often identify L with the product in (2.3), and a representation of L with a representation of the product.

In most cases, the representation $\pi = \bigotimes_{i=1}^{r} \pi_i$ we consider will be one-dimensional, and we denote it by $\mathbb{C}_\lambda$. The genuine representations of $Mp(2m)$ we consider will be the four oscillator representations $\omega_o^{\pm}$, $\omega_e^{\pm}$. Here $\omega^+ = \omega_e^+ + \omega_o^+$ denotes the holomorphic oscillator representation which is a sum of the even and odd constituents, and $\omega^- = \omega_e^- + \omega_o^-$ is its contragredient, the antiholomorphic oscillator representation of $Mp(2m)$. We will often refer to any of these four irreducible representations as "an oscillator representation of $Mp(2m)$".

The nongenuine representations of $Mp(2m)$ will be the trivial representation $\mathbb{C}$ or, in the case of $Mp(2)$, the unique spherical constituents of the spherical principal series representations J_ν with infinitesimal character ν satisfying $\frac{1}{2} \leq \nu \leq 1$ (the "complementary series representations"). Recall (see Table 1) that these representations are unitary, and J_1 is the trivial representation of $Mp(2)$.

Let $\mathfrak{t}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. Recall that $\mathfrak{t}$ is also a Cartan subalgebra for $\mathfrak{k}$, the complexified Lie algebra of $K \simeq \widetilde{U}(n)$, a maximal compact subgroup of G. Let $\Delta(\mathfrak{g}, \mathfrak{t}) \subseteq i\mathfrak{t}_0^*$ be the set of roots of $\mathfrak{t}$ in $\mathfrak{g}$. (Here, as everywhere else in the paper, we use the subscript 0 to denote real Lie algebras.) With respect to a standard parametrization, we can identify elements of $i\mathfrak{t}_0^*$ with n-tuples of real numbers. With this identification,

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm 2e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \colon 1 \leq i < j \leq n\}\,, \tag{2.6}$$

where e_i is the n-tuple with 1 in the ith position, and 0 everywhere else. Then the compact roots are

$$\Delta_k = \Delta(\mathfrak{k}, \mathfrak{t}) = \{\pm(e_i - e_j) \colon 1 \leq i < j \leq n\}\,. \tag{2.7}$$

We fix a system of positive compact roots

$$\Delta_k^+ = \{e_i - e_j \colon 1 \leq i < j \leq n\} \tag{2.8}$$

and write

$$\rho_c = \left(\frac{n-1}{2}, \frac{n-3}{2}, ..., \frac{-n+1}{2}\right), \tag{2.9}$$

one half the sum of the roots in Δ_k^+. We identify K-types, i. e., irreducible representations of K, with their highest weights which will be given by n-tuples of weakly decreasing integers (if nongenuine) or elements of $\mathbb{Z} + \frac{1}{2}$ (if genuine). The lowest K-types (in the sense of Vogan [**16**]) of $\omega_e^+, \omega_o^+, \omega_e^-$, and ω_o^- are

$$\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right), \tag{2.10}$$

$$\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right), \tag{2.11}$$

$$\left(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}\right), \text{ and} \tag{2.12}$$

$$\left(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2}\right) \tag{2.13}$$

respectively.

Using the Harish-Chandra map, we identify infinitesimal characters of admissible representations of G with (Weyl group orbits of) elements of $\mathfrak{t}^*$. Recall that the Weyl group $W(\mathfrak{g}, \mathfrak{t})$ acts on $\mathfrak{t}$ by permutations and sign changes. For example, the infinitesimal character γ_ω of any of the oscillator representations can be represented by the element

$$\left(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right); \tag{2.14}$$

we will often abuse notation by writing

$$\gamma_\omega = \left(n-\frac{1}{2}, n-\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right). \tag{2.15}$$

We fix a non-degenerate G- and θ-invariant symmetric bilinear form $<\ ,\ >$ on $\mathfrak{g}_0$, and we use the same notation for its various restrictions, extensions, and dualizations. In our parametrization of elements of $\mathfrak{t}$, this is the standard inner product

$$\left((a_1,\dots,a_n),(b_1,\dots,b_n)\right) = \sum_{i=1}^{n} a_i b_i. \tag{2.16}$$

2.2. Definitions and Conjecture.

Definition 2.1. Let $\gamma \in i\mathfrak{t}_0^*$. Choose a positive system $\Delta^+(\gamma) \subseteq \Delta(\mathfrak{g},\mathfrak{t})$ such that $\langle \alpha, \gamma\rangle \geq 0$ for all $\alpha \in \Delta^+(\gamma)$, and let γ_ω be the representative of the infinitesimal character of the oscillator representation which is dominant with respect to $\Delta^+(\gamma)$. We call γ *ω-regular* if the following regularity condition is satisfied:

$$\langle \alpha, \gamma - \gamma_\omega\rangle \geq 0 \quad \forall\, \alpha \in \Delta^+(\gamma). \tag{2.17}$$

We say that a representation of G is ω-regular if its infinitesimal character is.

Remark 2.2. The definition of ω-regular infinitesimal character is similar to the one of strongly regular infinitesimal character (c.f. [**12**]), but uses the infinitesimal character γ_ω of the oscillator representation instead of the infinitesimal character ρ of the trivial representation. Note that every strongly regular infinitesimal character is necessarily ω-regular. We will prove this result in the course of the proof of proposition 3.

Example 2.3. In $Mp(2)$, an infinitesimal character $\gamma = (k)$ is ω-regular if k is a real number such that $|k| \geq \frac{1}{2}$; it is strongly regular if $|k| \geq 1$.
In $Mp(4)$, an infinitesimal character $\gamma = (a,b)$ is ω-regular if a and b are both real, $||a|-|b|| \geq 1$, and $\min\{|a|,|b|\} \geq \frac{1}{2}$. It is strongly regular if, in addition, $\min\{|a|,|b|\} \geq 1$.

We will focus on two families of ω-regular representations: the $A_\mathfrak{q}(\Omega)$ and the *Meta-*$A_\mathfrak{q}(\lambda)$ representations of G, which we define below. In both cases, $\mathfrak{q} = \mathfrak{l} + \mathfrak{q}$ is a theta stable parabolic subalgebra of $\mathfrak{g}$ with

$$L = \prod_{i=1}^{r} \widetilde{U}(p_i,q_i) \times Mp(2m).$$

We write $\rho(\mathfrak{u})$ for one half the sum of the roots of $\mathfrak{u}$.

Definition 2.4. An $A_\mathfrak{q}(\Omega)$ representation is a genuine representation of G of the following form. Let $\mathbb{C}_\lambda$ be a genuine one-dimensional representation of $\left[\prod_{i=1}^{r} \widetilde{U}(p_i,q_i)\right]$ and let ω be an oscillator representation of $Mp(2m)$. Assume that the representation $\Omega = \mathbb{C}_\lambda \otimes \omega$ of L is in the good range for $\mathfrak{q}$, i. e., that the infinitesimal character γ^L of Ω is such that $\gamma^L + \rho(\mathfrak{u})$ is strictly dominant with respect to the roots of $\mathfrak{u}$. We define

$$A_\mathfrak{q}(\Omega) := \mathcal{R}_\mathfrak{q}(\Omega).$$

$\mathcal{R}_\mathfrak{q}$ denotes the right cohomological induction functor defined in [**15**] and [**16**].

Definition 2.5. A *Meta-*$A_{\mathfrak{q}}(\lambda)$ representation is a nongenuine representation X of G of the following form. Let $\mathbb{C}_\lambda$ be a nongenuine one-dimensional representation of $\left[\prod_{i=1}^{r}\widetilde{U}(p_i,q_i)\right]$ and let J_ν be the spherical constituent of the spherical principal series of $Mp(2m)$ with infinitesimal character ν. If $m \neq 1$ then we take $\nu = \rho$ so that $J_\nu = J_\rho$ is the trivial representation of $Mp(2m)$; if $m = 1$ then require that $\frac{1}{2} \leq \nu \leq 1$, so that J_ν is a complementary series of $Mp(2)$ if $\frac{1}{2} \leq \nu < 1$ and is the trivial representation if $\nu = 1$. Assume that $\mathbb{C}_\lambda \otimes J_\nu$ is in the good range for $\mathfrak{q}$. We define

$$X := \mathcal{R}_{\mathfrak{q}}(\mathbb{C}_\lambda \otimes J_\nu).$$

Remark 2.6. Every $A_{\mathfrak{q}}(\lambda)$ representation of $Mp(2n)$ in the good range is either an $A_{\mathfrak{q}}(\Omega)$ or a Meta-$A_{\mathfrak{q}}(\lambda)$ representation.
More explicitly, if X is a genuine $A_{\mathfrak{q}}(\lambda)$ representation in the good range, then we can consider X as an $A_{\mathfrak{q}}(\Omega)$ representation with $m = 0$ (note that, in this case, the Levi subgroup L does not contain any $Sp(2m)$ factor). If X is a nongenuine $A_{\mathfrak{q}}(\lambda)$ representation in the good range, then we can consider X as a Meta-$A_{\mathfrak{q}}(\lambda)$ representation with J_ν equal to the trivial representation of $Sp(2m)$ (for all m).

Proposition 3. The following properties hold:
1. (1) All $A_{\mathfrak{q}}(\Omega)$ and Meta-$A_{\mathfrak{q}}(\lambda)$ representations of G are nonzero, irreducible and unitary.
2. (2) If X is a Meta-$A_{\mathfrak{q}}(\lambda)$ representation with $\nu = \rho$, then X is an admissible $A_{\mathfrak{q}}(\lambda)$ in the sense of [**12**], and has strongly regular infinitesimal character.
3. (3) All Meta-$A_{\mathfrak{q}}(\lambda)$ representations of G are ω-regular.
4. (4) All $A_{\mathfrak{q}}(\Omega)$ representations of G are ω-regular.

The proof of this Proposition will be given at the end of this section.

Remark 2.7. Genuine $A_{\mathfrak{q}}(\lambda)$ representations in the good range are not necessarily strongly regular. For example, take $G = Mp(4)$, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ with $L \cong \widetilde{U}(1,1)$, so that $\rho(\mathfrak{u}) = \left(\frac{3}{2}, -\frac{3}{2}\right)$ and $\rho(\mathfrak{l}) = \left(\frac{1}{2}, \frac{1}{2}\right)$, and choose $\lambda = \left(-\frac{1}{2}, \frac{1}{2}\right)$. The module $A_{\mathfrak{q}}(\lambda)$ has lowest K-type

$$\mu = \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \left(-\frac{1}{2}, \frac{1}{2}\right) + (2, -2) = \left(\frac{3}{2}, -\frac{3}{2}\right) \tag{2.18}$$

and infinitesimal character

$$\gamma = \lambda + \rho(\mathfrak{l}) + \rho(\mathfrak{u}) = \left(\frac{3}{2}, -\frac{1}{2}\right) = \gamma_\omega. \tag{2.19}$$

Now we are ready to state our conjecture.

Conjecture 3. Let X be an irreducible admissible representation of $Mp(2n)$. Then X is ω-regular and unitary if and only if X is either an $A_{\mathfrak{q}}(\Omega)$ or a Meta-$A_{\mathfrak{q}}(\lambda)$.

Theorem 2.8. *Conjecture 3 is true for* $n = 1$ *and* $n = 2$.

The proof of Theorem 2.8 will occupy most of the remainder of this paper. Before proving Proposition 3 (and other facts about $A_{\mathfrak{q}}(\Omega)$ and Meta-$A_{\mathfrak{q}}(\lambda)$ representations), we need to collect a few results on cohomological parabolic induction.

Fix a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$, and a (Levi) subgroup $L = N_G(\mathfrak{q})$. The cohomological parabolic induction functor $\mathfrak{R}_{\mathfrak{q}}$, defined in [**16**, Def. 6.3.1], maps

$(\mathfrak{l},L\cap K)$ modules to $(\mathfrak{g},K)$ modules. Its restriction to K, denoted by $\mathfrak{R}_{\mathfrak{q}}^{K}$, maps $(L\cap K)$-modules to K-modules.

PROPOSITION 4. ([**17**, Lemma 6.5]). Let W be an irreducible representation of $L\cap K$, and let μ^L be a highest weight of W. Set

$$\mu = \mu^L + 2\rho\left(\mathfrak{u}\cap\mathfrak{p}\right).$$

If μ is dominant for K, then every irreducible constituent of $\mathfrak{R}_{\mathfrak{q}}^{K}(W)$ has highest weight μ. Otherwise, $\mathfrak{R}_{\mathfrak{q}}^{K}(W)=0$.

PROPOSITION 5. ([**17**, Theorems 1.2, 1.3] and [**6**, Theorem 10.44]). Suppose that the group $L=N_G(\mathfrak{q})$ meets every component of G and that $\mathfrak{h}\subseteq\mathfrak{l}$ is a Cartan subalgebra. Let Y be an $(\mathfrak{l},L\cap K)$ module, and let $\gamma^L\in\mathfrak{h}^*$ be a weight associated to the infinitesimal character of Y. Then

(1) The weight $\gamma=\gamma^L+\rho(\mathfrak{u})\in\mathfrak{h}^*$ is attached to the infinitesimal character of the representation $\mathfrak{R}_{\mathfrak{q}}Y$.
(2) If Y is in the good range for $\mathfrak{q}$, that is

$$\operatorname{Re}\left\langle\gamma^L+\rho\left(\mathfrak{u}\right),\alpha\right\rangle>0\quad\forall\,\alpha\in\Delta(\mathfrak{u}),\tag{2.20}$$

the following additional properties hold:
(a) If Y is irreducible and unitary, then $\mathfrak{R}_{\mathfrak{q}}Y$ is irreducible, non zero and unitary.
(b) The correspondence

$$\delta^L\longmapsto\delta=\delta^L+2\rho\left(\mathfrak{u}\cap\mathfrak{p}\right)$$

gives a bijection between lowest $(L\cap K)$-types of Y and lowest K-types of $\mathfrak{R}_{\mathfrak{q}}Y$. In fact, every such expression for δ is dominant for K.

REMARK 2.9. If the inequality in equation 2.20 is not strict, then the induced module $\mathfrak{R}_{\mathfrak{q}}Y$ may be zero or not unitary, and some of the lowest $(L\cap K)$-types δ^L may give rise to weights for K that are not dominant.

We now give the proof of Proposition 3.

PROOF. Part (1) of Proposition 3 follows directly from Proposition 5, because both Ω and $\mathbb{C}_\lambda\otimes J_\nu$ are assumed to be in the good range for $\mathfrak{q}$.

For the second part, write γ^L for the infinitesimal character of $Z=\mathbb{C}_\lambda\otimes J_\nu$. Assume Z is in the good range for $\mathfrak{q}$, so that $\gamma^L+\rho\left(\mathfrak{u}\right)$ is strictly dominant for the roots of $\Delta(\mathfrak{u})$, and choose $\nu=\rho$. Note that Z has infinitesimal character

$$\gamma^L=\lambda+\rho(\mathfrak{l})$$

for some choice of positive roots $\Delta^+(\mathfrak{l})\subset\Delta(\mathfrak{l},\mathfrak{t})$. By Proposition 5, the infinitesimal character of $X=\mathfrak{R}_{\mathfrak{q}}Z$ is

$$\gamma=\gamma^L+\rho\left(\mathfrak{u}\right)=\lambda+\rho(\mathfrak{l})+\rho\left(\mathfrak{u}\right)=\lambda+\rho.$$

Here ρ is one half the sum of the roots in $\Delta^+(\mathfrak{g})=\Delta^+(\mathfrak{l})\cup\Delta\left(\mathfrak{u}\right)$. We want to prove that γ is strongly regular.
If $\alpha\in\Delta^+\left(\mathfrak{l}\right)$, then

$$\langle\gamma-\rho,\alpha\rangle=\langle\lambda,\alpha\rangle=0\tag{2.21}$$

because λ is the differential of a one-dimensional representation of L.
If α is a simple root in $\Delta(\mathfrak{u})$, then $\langle \rho, \alpha\rangle = 1$ or 2, depending on whether α is short or long, and $\langle \gamma, \alpha\rangle > 0$ (by the "good range" condition). So

$$\begin{cases} \langle \gamma - \rho, \alpha\rangle > -1 & \text{if } \alpha \text{ is short} \\ \langle \gamma - \rho, \alpha\rangle > -2 & \text{if } \alpha \text{ is long.} \end{cases}$$

Now, because λ is the differential of a nongenuine character, the inner product $\langle \lambda,\, \alpha\rangle$ has integer values for all roots in $\Delta^+(\mathfrak{g})$:

$$\langle \gamma - \rho,\, \alpha\rangle = \langle \lambda,\, \alpha\rangle \in \mathbb{Z} \quad \forall\, \alpha \in \Delta^+(\mathfrak{g}).$$

Notice that $\langle \lambda,\, \alpha\rangle$ is an even integer if α is long. Then

$$\langle \gamma - \rho,\, \alpha\rangle = \langle \lambda,\, \alpha\rangle \geq 0 \tag{2.22}$$

for every (simple) root in $\Delta^+(\mathfrak{u})$. Combining this result with equation 2.21 we find that

$$\langle \gamma - \rho, \alpha\rangle = \langle \lambda, \alpha\rangle \geq 0 \quad \forall\, \alpha \in \Delta^+(\mathfrak{g}) = \Delta^+(\mathfrak{l}) \cup \Delta\,(\mathfrak{u})\,. \tag{2.23}$$

Hence our Meta-$A_{\mathfrak{q}}(\lambda)$ representation is admissible in the sense of [**12**] (c.f. (1.1)). Equation 2.23 also shows that $\gamma - \rho$ (and therefore γ) is weakly dominant with respect to the positive root system $\Delta^+(\mathfrak{g})$, so γ lies in the Weyl chamber of ρ, and we can take $\Delta^+(\gamma) = \Delta^+(\mathfrak{g})$ (c.f. Definition 2.1 and the remark following it). We conclude that γ is strongly regular.

For the third part of Proposition 3, we must show that every Meta-$A_{\mathfrak{q}}(\lambda)$ representation is ω-regular. If $\nu = \rho$, this result is not hard to prove. Indeed, strong regularity easily implies ω-regularity: assume that γ is strongly regular and let γ_ω be the representative of the infinitesimal character of the oscillator representation which is in the Weyl chamber of ρ. Then for every simple root $\alpha \in \Delta^+(\mathfrak{g})$, we have

$$\langle \gamma_\omega, \alpha\rangle = 1. \tag{2.24}$$

In particular, $\langle \gamma_\omega, \alpha\rangle \leq \langle \rho, \alpha\rangle$, so we get

$$\langle \gamma - \gamma_\omega,\, \alpha\rangle \geq \langle \gamma - \rho,\, \alpha\rangle \geq 0,$$

proving that γ is omega-regular.
Now assume that $\nu \neq \rho$. Then J_ν is the irreducible quotient of a complementary series representation of $Mp(2)$ with $\frac{1}{2} \leq \nu < 1$. The infinitesimal character of J_ν is equal to ν, and the restriction of $\lambda+\ \rho(\mathfrak{l})$ to $\mathfrak{t} \cap \mathfrak{sp}(2)$ is 1, so the infinitesimal characters of Z and $X = \mathcal{R}_{\mathfrak{q}}(Z)$ can be written as

$$\gamma^L = \lambda + \rho(\mathfrak{l}) + (\nu - 1), \tag{2.25}$$

and

$$\gamma = \lambda + \rho + (\nu - 1) \tag{2.26}$$

respectively. Assume that γ^L is in the good range, and note that because $\frac{1}{2} \leq \nu < 1$ and λ is integral, this condition is equivalent to requiring that $\lambda + \rho(\mathfrak{l})$ be in the good range. The same argument used in the second part of the proof shows that λ is weakly dominant with respect to $\Delta^+(\mathfrak{g})$. Then for all simple roots δ we have $\langle \rho,\, \delta\rangle \geq 1$, $\langle (\nu - 1),\, \delta\rangle \geq -1$ and

$$\langle \gamma,\, \delta\rangle = \langle \lambda + \rho + (\nu - 1),\, \delta\rangle \geq 0 + 1 - 1 = 0.$$

This proves that γ lies in the Weyl chamber determined by ρ, so we can take $\Delta^+(\gamma) = \Delta^+(\mathfrak{g})$, and γ is ω-regular if and only if

$$\langle \gamma - \gamma_\omega,\, \alpha\rangle \geq 0 \quad \forall\, \alpha \in \Delta^+(\mathfrak{g}).$$

It is sufficient to restrict the attention to the simple roots that are not orthogonal to $(\nu - 1)$: if $\langle (\nu-1),\, \alpha\rangle = 0$, then the proof for the previous case ($\nu = 1$) goes through, so $\langle \gamma - \gamma_\omega,\, \alpha\rangle \geq 0$ by the previous argument.
There are two simple roots in $\Delta^+(\mathfrak{g})$ not orthogonal to $(\nu - 1)$: a long root $\beta_L \in \Delta^+(\mathfrak{l})$ satisfying

$$\langle \nu - 1,\, \beta_L\rangle = 2\nu - 2,$$

and a short root $\beta_S \in \Delta(\mathfrak{u})$ satisfying

$$\langle \nu - 1,\, \beta_S\rangle = 1 - \nu.$$

Because β_L is simple and long, $\langle \rho, \beta_L\rangle = 2$ and $\langle \gamma_\omega, \beta_L\rangle = 1$ (from 2.24). Then

$$\begin{aligned}\langle \gamma - \gamma_\omega,\, \beta_L\rangle &= \langle \lambda + (\rho - \gamma_\omega) + (\nu - 1),\, \beta_L\rangle = \\ &= \langle \lambda, \beta_L\rangle + \langle \rho - \gamma_\omega, \beta_L\rangle + \langle (\nu-1), \beta_L\rangle = \\ &= 0 + (2-1) + (2\nu - 2) = 2\nu - 1 \geq 0.\end{aligned}$$

Similarly, because β_S is simple and short, we have $\langle \rho - \gamma_\omega, \beta_S\rangle = 1 - 1 = 0$, and

$$\langle \gamma - \gamma_\omega,\, \beta_S\rangle = \langle \lambda,\, \beta_S\rangle + \langle (\nu - 1),\, \beta_S\rangle = \langle \lambda,\, \beta_S\rangle + (1-\nu) > \langle \lambda,\, \beta_S\rangle \geq 0.$$

This proves that γ is ω-regular.

Finally, we prove part 4 of Proposition 3. Assume that $\Omega = \mathbb{C}_\lambda \otimes \omega$ is in the good range for $\mathfrak{q}$. Recall that ω is an oscillator representation of $Mp(2m) \subset L$. Set $\mathfrak{l}_1 = \mathfrak{sp}\,(2m, \mathbb{C}) \subset \mathfrak{l}$ and

$$\Lambda_{\mathfrak{l}_1} = \left\{\beta \in \Delta^+\left(\mathfrak{l}_1, \mathfrak{t}\right) : \beta \text{ is long}\right\}.$$

Then

- ω has infinitesimal character $\gamma_\omega^{\mathfrak{l}_1} = \rho(\mathfrak{l}_1) - \frac{1}{2}\rho\left(\Lambda_{\mathfrak{l}_1}\right)$
- Ω has infinitesimal character $\gamma^L = \lambda + \rho(\mathfrak{l}) - \frac{1}{2}\rho\left(\Lambda_{\mathfrak{l}_1}\right)$
- $A_{\mathfrak{q}}(\Omega)$ has infinitesimal character $\gamma = \gamma^L + \rho(\mathfrak{u}) = \lambda + \rho - \frac{1}{2}\rho\left(\Lambda_{\mathfrak{l}_1}\right).$

We need to show that γ is ω-regular, i.e.

$$\langle \gamma - \gamma_\omega^{\mathfrak{g}},\, \alpha\rangle \geq 0 \quad \forall \alpha \in \Delta^+(\gamma). \tag{2.27}$$

Here $\gamma_\omega^{\mathfrak{g}}$ is an infinitesimal character for an oscillator representation of G. We can write

$$\gamma_\omega^{\mathfrak{g}} = \rho - \frac{1}{2}\rho\left(\Lambda_{\mathfrak{g}}\right)$$

with

$$\Lambda_{\mathfrak{g}} = \left\{\beta \in \Delta^+\left(\mathfrak{g}, \mathfrak{t}\right) : \beta \text{ is long}\right\}.$$

So equation 2.27 is equivalent to:

$$\left\langle \lambda + \frac{1}{2}\rho\left(\Lambda_{\mathfrak{g}}\right) - \frac{1}{2}\rho\left(\Lambda_1\right),\, \alpha \right\rangle \geq 0 \quad \forall \alpha \in \Delta^+(\mathfrak{g}). \tag{2.28}$$

Choose $w \in W(\mathfrak{g}, \mathfrak{t})$ such that $w\Delta^+(\mathfrak{g})$ is the standard positive system of roots:

$$w\Delta^+(\mathfrak{g}) = \left\{2e_i \colon 1 \leq i \leq n\right\} \cup \left\{e_i \pm e_j \colon 1 \leq i < j \leq n\right\}. \tag{2.29}$$

Then

$$w\lambda = \Big(\underbrace{\lambda_1,\dots,\lambda_1}_{p_1+q_1},\underbrace{\lambda_2,\dots,\lambda_2}_{p_2+q_2},\dots,\underbrace{\lambda_r,\dots,\lambda_r}_{p_r+q_r},\underbrace{0,\dots,0}_{m}\Big) \tag{2.30}$$

for some $\lambda_i \in \mathbb{Z}+\frac{1}{2}$ (since $\mathbb{C}_\lambda$ is genuine), and

$$w\gamma = \Big(\underbrace{\lambda_1,\dots,\lambda_1}_{p_1+q_1},\underbrace{\lambda_2,\dots,\lambda_2}_{p_2+q_2},\dots,\underbrace{\lambda_r,\dots,\lambda_r}_{p_r+q_r},\underbrace{-\frac{1}{2},\dots,-\frac{1}{2}}_{m}\Big) + w\rho \tag{2.31}$$

with $w\rho = (n, n-1, \dots, 2, 1)$. By assumption, Ω is in the good range for $\mathfrak{q}$, so γ is strictly dominant for the roots of $\mathfrak{u}$. We also have

$$\langle w\gamma,\, w\alpha\rangle > 0 \quad \forall \alpha \in \Delta^+(\mathfrak{u}). \tag{2.32}$$

If α is a positive simple root for $\mathfrak{u}$, then $w\alpha$ can be of the form

$$w\alpha = \begin{cases} e_i - e_{i+1} & \text{for } i = \sum_{k=1}^{l}(p_k+q_k)\ ,\ 1 \le l \le r-1, \\ e_{n-m} - e_{n-m+1} & \text{if } m > 0, \\ 2e_n & \text{if } m = 0. \end{cases}$$

Equation 2.32 implies that

$$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r \ge -\frac{1}{2}. \tag{2.33}$$

Then $w\gamma$ is weakly dominant with respect to $w\Delta^+(\mathfrak{g})$ (and of course γ is weakly dominant with respect to $\Delta^+(\mathfrak{g})$). Therefore, we can choose $\Delta^+(\gamma) = \Delta^+(\mathfrak{g})$.
Conjugating $\gamma - \gamma^{\mathfrak{g}}_{\omega}$ in a similar way, we find

$$w(\gamma-\gamma^{\mathfrak{g}}_{\omega}) = \Big(\underbrace{\lambda_1+\frac{1}{2},\dots,\lambda_1+\frac{1}{2}}_{p_1+q_1},\underbrace{\lambda_2+\frac{1}{2},\dots,\lambda_2+\frac{1}{2}}_{p_2+q_2},\dots,\underbrace{\lambda_r+\frac{1}{2},\dots,\lambda_r+\frac{1}{2}}_{p_r+q_r},\underbrace{0,\dots,0}_{m}\Big).$$

Notice that the entries of $w(\gamma - \gamma^{\mathfrak{g}}_{\omega})$ are weakly decreasing and nonnegative (by 2.33). Hence $w(\gamma - \gamma^{\mathfrak{g}}_{\omega})$ is weakly dominant with respect to the roots in $w\Delta^+(\mathfrak{g})$:

$$\langle w(\gamma - \gamma^{\mathfrak{g}}_{\omega}),\, w\alpha\rangle \ge 0 \quad \forall\, \alpha \in \Delta^+(\mathfrak{g}).$$

Equivalently,

$$\langle \gamma - \gamma^{\mathfrak{g}}_{\omega},\, \alpha\rangle \ge 0 \quad \forall\, \alpha \in \Delta^+(\mathfrak{g})$$

and γ is ω-regular. This concludes the proof of Proposition 3. □

2.3. Some Facts. The lowest K-types of the $A_{\mathfrak{q}}(\Omega)$ and Meta-$A_{\mathfrak{q}}(\lambda)$ representations will play a very important role in the rest of this paper.
Recall that the lowest K-types of a representation are those that are minimal with respect to the Vogan norm

$$\|\mu\| = (\mu + 2\rho_c, \mu + 2\rho_c)\,, \tag{2.34}$$

and that any irreducible representation has only finitely many lowest K-types.
It turns out that every $A_{\mathfrak{q}}(\Omega)$ and Meta-$A_{\mathfrak{q}}(\lambda)$ representation admits a unique lowest K-type, which is computed in the following proposition.

PROPOSITION 6. In the setting of Definitions 2.4 and 2.5, let $\rho(\mathfrak{u}\cap\mathfrak{p})$ be one half the sum of the noncompact roots of $\mathfrak{u}$. Then

(1) The $A_{\mathfrak{q}}(\Omega)$ representation $\mathcal{R}_{\mathfrak{q}}(\Omega)$ has a unique lowest K-type:

$$\mu = \mu(\mathfrak{q},\Omega) = \mu^L + 2\rho(\mathfrak{u}\cap\mathfrak{p}) \tag{2.35}$$

with μ^L the unique lowest $L\cap K$-type of Ω.

(2) The Meta-$A_{\mathfrak{q}}(\lambda)$ representation $\mathcal{R}_{\mathfrak{q}}(\mathbb{C}_\lambda \otimes J_\nu)$ has a unique lowest $L\cap K$-type:

$$\mu = \lambda \otimes 0 + 2\rho(\mathfrak{u}\cap\mathfrak{p}). \tag{2.36}$$

As usual, we have identified K-types and $L\cap K$-type with their highest weights.

PROOF. Both results follow from Proposition 5, because Ω and $\mathbb{C}_\lambda \otimes J_\nu$ are in the good range for $\mathfrak{q}$. □

To prove that certain representations are nonunitary, we will rely heavily on the following useful result.

PROPOSITION 7. (Parthasarathy's Dirac Operator Inequality [**8**], [**19**]) Let X be a unitary representation of G with infinitesimal character γ, and let μ be a K-type occurring in X. Choose a positive system $\Delta^+(\mathfrak{g},\mathfrak{t}) \subseteq \Delta(\mathfrak{g},\mathfrak{t})$ of roots containing our fixed Δ_k^+, and let ρ_n, ρ_c be one half the sums of the noncompact and compact roots in $\Delta^+(\mathfrak{g},\mathfrak{t})$, respectively. Choose $w \in W_{\mathfrak{k}}$, the Weyl group of $\mathfrak{k}$, so that $w(\mu-\rho_n)$ is dominant with respect to Δ_k^+. Then

$$(w(\mu-\rho_n)+\rho_c, w(\mu-\rho_n)+\rho_c) \geq (\gamma,\gamma)\,. \tag{2.37}$$

We will often refer to the Parthasarathy's Dirac Operator Inequality as "PDOI".

If X is an irreducible admissible representation of $Mp(2n)$ and X^* its contragredient representation, then X and X^* have the same properties; in particular, X is unitary, ω-regular, an $A_{\mathfrak{q}}(\Omega)$, a Meta-$A_{\mathfrak{q}}(\lambda)$, a discrete series representation, finite dimensional, one-dimensional, etc. if and only if X^* is. We will sometimes use this symmetry to reduce the number of cases to be considered. Note that the K-types which occur in X^* are precisely those dual to the K-types occurring in X.

PROPOSITION 8. Let μ be an irreducible representation of $\widetilde{U}(n)$ with highest weight

$$\lambda = (a_1, a_{2,\dots,}a_n)\,. \tag{2.38}$$

Then the contragredient representation μ^* of μ has highest weight

$$\xi = (-a_n, -a_{n-1}, \cdots - a_2, -a_1)\,. \tag{2.39}$$

PROOF. Let μ be realized on the finite dimensional vector space V. Realize μ^* on the dual space V^*. The weights of μ^* are easily seen to be the opposite of the weights of μ: if $\{v_{\lambda_1}, v_{\lambda_2}, \dots, v_{\lambda_r}\}$ is a basis of V consisting of weight vectors corresponding to the weights $\lambda_1,\dots,\lambda_r$, then the dual basis $\left\{v^*_{\lambda_1}, v^*_{\lambda_2}, \dots, v^*_{\lambda_r}\right\}$ of V^* is a set of weight vectors corresponding to the weights $-\lambda_1,\dots,-\lambda_r$.

Define $-\xi = (a_n, a_{n-1},\dots,a_1)$, with $\lambda = (a_1, a_{2,\dots,}a_n)$ the highest weight for μ.

$-\xi$ is an extremal weight of μ (because is Weyl group conjugate to λ), and is the lowest weight of μ (because it is antidominant with respect to our fixed set of positive roots). Then $\zeta - (-\xi)$ is a a sum of positive roots, for every weight ζ of μ. Equivalently, $\xi - (-\zeta)$ is a a sum of positive roots, for every weight $(-\zeta)$ of μ^*, hence ξ is the highest weight of μ^*. $\square$

2.4. Langlands Classification and Lowest K-Types. Our proof of Theorem 2.8 proceeds by K-types: for each K-type μ and each ω-regular irreducible unitary representation π with lowest K-type μ, we show that π must be either an $A_{\mathfrak{q}}(\Omega)$ or a Meta-$A_{\mathfrak{q}}(\lambda)$ representation. Therefore, it is important to know which representations contain a given μ as a lowest K-type, and what are the possible infinitesimal characters for such representations.
Because all ω-regular infinitesimal characters are in particular real, we will assume from now on that all infinitesimal characters have this property.

To determine the set of representations with a given infinitesimal character, we use the Langlands Classification, which is a construction equivalent to Vogan's construction from Proposition 1, but uses real parabolic induction instead of cohomological parabolic induction. According to the Langlands Classification (c.f. [**5**], [**17**]), every irreducible admissible representation of G occurs as an irreducible quotient $\overline{X}(\sigma,\nu)$ of an induced representation

$$I_P(\sigma,\nu) = \mathrm{Ind}_P^G(\sigma\otimes\nu\otimes 1), \tag{2.40}$$

where $P = MAN$ is a cuspidal parabolic subgroup of G, σ a discrete series representation of M, and ν a character of A. (We are abusing notation and using ν to denote both the character and its differential.) If the infinitesimal character of the representation is regular, as it always is in our setting, then $I_P(\sigma,\nu)$ has a unique irreducible quotient. Inducing data give rise to equivalent irreducible representations if and only if they are conjugate by G.

We now give a more specific description of the data for irreducible representations of $Mp(4)$ with real regular infinitesimal character (see also [**1**], [**9**]); for $Mp(2)$, the situation is similar, yet much simpler.
Irreducible representations of $Mp(4)$ are in one-one correspondence with triples

$$(MA,\, \sigma,\, \nu) \tag{2.41}$$

as follows. There are four conjugacy classes of cuspidal parabolic subgroups, given by their Levi factors

$$MA = \begin{cases} Mp(4) & \text{with } M = Mp(4),\ A = \{1\} \\ Mp(2)\times\widetilde{GL}(1,\mathbb{R}) & \text{with } M = Mp(2)\times\mathbb{Z}/4\mathbb{Z},\ A = \mathbb{R} \\ \widetilde{GL}(2,\mathbb{R}) & \text{with } M = Mp(2)^{\pm},\ A = \mathbb{R} \\ \widetilde{GL}(1,\mathbb{R})^2 & \text{with } M = (\mathbb{Z}/4\mathbb{Z})^2,\ A = \mathbb{R}^2 \end{cases} \tag{2.42}$$

(in the second and fourth case, MA and M are actually quotients by a subgroup of order 2 of this product). The group $\widetilde{GL}(2,\mathbb{R})$ above is the split double cover of $GL(2,\mathbb{R})$. The discrete series σ may be given by its Harish-Chandra parameter and by a character of $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/4\mathbb{Z})^2$. The parameter ν can be conjugated into a positive number or a pair of positive numbers (recall that we are only considering representations with real regular infinitesimal character, so ν is real and, if A is nontrivial, ν is nonzero).

Recall that to every K-type μ we can assign a Vogan parameter $\lambda_a = \lambda_a(\mu) \in \mathfrak{t}^*$ as follows (c.f. [**16**]): choose a representative of ρ such that $\mu + 2\rho_c$ is weakly dominant with respect to ρ. Then

$$\lambda_a = p(\mu + 2\rho_c - \rho), \tag{2.43}$$

where p denotes the projection onto the positive Weyl chamber determined by ρ. The Vogan parameter λ_a then gives the Harish-Chandra parameter of σ, for any representation $\overline{X}(\sigma,\nu)$ with lowest K-type μ. This determines the (conjugacy class of the) Levi subgroup MA as well. Write

$$\lambda_a = (a,b) \tag{2.44}$$

with $a \geq b$.

(1) If $|a| \neq |b|$ and both are nonzero, then $MA = Mp(4)$, and λ_a is the Harish-Chandra parameter of a discrete series of G.

(2) If $a > b = 0$ then $MA = Mp(2) \times \widetilde{GL}(1,\mathbb{R})$. In this case, $A = \mathbb{R}$ so ν is just a positive number. The infinitesimal character of the corresponding representation is

$$\gamma = (a,\nu). \tag{2.45}$$

The character of $\mathbb{Z}/4\mathbb{Z}$ is also uniquely determined by μ. An analogous statement holds if $a = 0 > b$; then we have $\gamma = (\nu, b)$.

(3) If $a = -b \neq 0$ then $MA = \widetilde{GL}(2,\mathbb{R})$. Also in this case, $A = \mathbb{R}$ and ν is a positive number. The infinitesimal character of the corresponding representation is

$$\gamma = (a+\nu, -a+\nu). \tag{2.46}$$

(4) If $a = b = 0$, then we say that μ is a fine K-type. In this case, the representation is a principal series, and has infinitesimal character

$$\gamma = (\nu_1, \nu_2). \tag{2.47}$$

The case $a = b \neq 0$ does not occur; the parameter λ_a must be such that its centralizer L_a in G is a quasisplit Levi subgroup [**16**].

3. The Group Mp(2)

Let $G = Mp\,(2,\mathbb{R})$ be the connected double cover of

$$Sp\,(2,\mathbb{R}) = \left\{ g \in GL\,(2,\mathbb{R}) : g^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \tag{3.1}$$

Note that $Sp\,(2,\mathbb{R})$ equals $SL\,(2,\mathbb{R})$. The Lie algebra of G is

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a,b,c \in \mathbb{R} \right\}, \tag{3.2}$$

and the maximal compact Cartan subalgebra of $\mathfrak{g}_0$ is

$$\mathfrak{t}_0 = \mathfrak{k}_0 = \left\{ \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} : t \in \mathbb{R} \right\}. \tag{3.3}$$

The maximal compact subgroup of $Sp\,(2,\mathbb{R})$ is $SO\,(2) \simeq U\,(1)$, hence the maximal compact subgroup K of $Mp\,(2,\mathbb{R})$ is isomorphic to $\widetilde{U}(1)$. We identify $\widehat{K}$ with $\frac{1}{2}\mathbb{Z}$, as follows: write

$$K \simeq \left\{ (g,z) \in U(1) \times \mathbb{C}^\times : z^2 = g \right\}. \tag{3.4}$$

Then the character of K corresponding to the half integer a is given by

$$(g,z) \mapsto z^{2a}. \tag{3.5}$$

This character is genuine if and only if $a \in \mathbb{Z} + \frac{1}{2}$.

We are interested in ω-regular unitary representations of G. It turns out that they are all obtained by either complementary series or cohomological parabolic induction from a Levi subgroup L of a theta stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. This subalgebra $\mathfrak{q}$ is related to the subalgebra $\mathfrak{q}_a$ defined in Proposition 1.

Proposition 9. If $G = Mp\,(2,\mathbb{R})$ and X is an irreducible unitary representation of G with ω-regular infinitesimal character, then either

(1) $X \simeq A_{\mathfrak{q}}\,(\Omega)$ for some θ-stable parabolic subalgebra $\mathfrak{q}$ and some representation Ω of L, as in Definition 2.4, or
(2) X is isomorphic to a Meta-$A_{\mathfrak{q}}\,(\lambda)$ representation, as in Definition 2.5.

Proof. Let X be an irreducible unitary representation of G with ω-regular infinitesimal character, and let $\mu = (a) \in \frac{1}{2}\mathbb{Z}$ be a lowest K-type for X. We prove that X is either an $A_q(\Omega)$ or a Meta-$A_q(\lambda)$ representation. Recall that, for $Mp(2)$, the $A_q(\Omega)$ representations are the oscillator representations and the genuine discrete series. The Meta-$A_q(\lambda)$ representations are the nongenuine discrete series and the complementary series J_ν, with $\frac{1}{2} \leq \nu \leq 1$ (if $\nu = 1$ then J_ν is trivial representation).

First assume that $a \in \frac{1}{2}\mathbb{Z} \setminus \left\{0, \pm\frac{1}{2}, \pm 1\right\}$. Vogan's classification of irreducible admissible representations ([**15**]) implies that if X has lowest K-type $\mu = (a)$ in $\frac{1}{2}\mathbb{Z} \setminus \left\{0, \pm\frac{1}{2}, \pm 1\right\}$, then X is a discrete series representation with Harish-Chandra parameter $\lambda = a - sgn\,(a) \neq 0$. Hence X is an $A_q(\Omega)$ representation if genuine, and a Meta-$A_q(\lambda)$ representation if nongenuine. We notice that in this case

$$X = A_{\mathfrak{q}}\,(\lambda) \tag{3.6}$$

with $\mathfrak{q}$ the Borel subalgebra determined by λ. Because $|\lambda| \geq \frac{1}{2}$, X is ω-regular (see Example 2.3).

We are left with the cases $a \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}$. First assume $a = \pm 1$, and choose $\rho_n = \pm 1$. Since $\rho_c = 0$, Vogan's classification of admissible representations gives that X has infinitesimal character $\gamma = (\lambda_a, \nu) = (0, \nu)$. In this case, for w trivial, we have

$$\langle w(\mu - \rho_n) + \rho_c,\, w(\mu - \rho_n) + \rho_c \rangle = 0 \tag{3.7}$$

so the Parthasarathy's Dirac operator inequality ("PDOI", cf. Proposition 7) yields that if $\nu \neq 0$, then X is nonunitary. We conclude that there are no irreducible unitary ω-regular representations of G with lowest K-type $\mu = \pm 1$.

Next, assume $a = \pm\frac{1}{2}$. Note that $\mu = \pm\frac{1}{2}$ is the lowest K-type of an even oscillator representation ω, and that $\omega = A_{\mathfrak{q}}\,(\Omega)$ with $\mathfrak{q} = \mathfrak{g}$ and $\Omega = \omega$. We will show that the oscillator representations are the only irreducible unitary ω-regular representations X of G containing $\mu = \pm\frac{1}{2}$ as their lowest K-type.
Choose $\mu = \pm\frac{1}{2}$, and $\rho_n = \pm 1$ with the same sign as μ. For w trivial, we get

$$\langle w(\mu - \rho_n) + \rho_c,\, w(\mu - \rho_n) + \rho_c \rangle = \frac{1}{4}. \tag{3.8}$$

So PDOI implies that X is nonunitary if its infinitesimal character γ satisfies $\langle \gamma, \gamma \rangle > \frac{1}{4}$. On the other hand, X is not ω-regular if $\langle \gamma, \gamma \rangle < \frac{1}{4}$. Hence any representation X of G (with lowest K-type $\pm\frac{1}{2}$) which is both ω-regular and unitary must satisfy $\langle \gamma, \gamma \rangle = \frac{1}{4}$.

Because $\mu = \left(\pm\frac{1}{2}\right)$ is fine, and X contains μ as a lowest K-type, X must be induced from a representation $\delta \otimes \nu$ of $P = MAN$, where

$$M \simeq \widetilde{\mathbb{Z}_2} \simeq \mathbb{Z}_4. \tag{3.9}$$

The infinitesimal character γ of X is given by $(0, \nu)$, and the condition $\langle \gamma, \gamma \rangle = \frac{1}{4}$ implies $\nu = \pm\frac{1}{2}$. The two choices are conjugate by the Weyl group, hence give equivalent representations; we assume $\nu = \frac{1}{2}$. Next, we prove that the choice of δ is also uniquely determined by μ. This is an easy application of Frobenius reciprocity: if μ is contained in $X = \operatorname{Ind}(\delta \otimes \nu)$, then δ is contained in the restriction of μ to M. With our identification (3.4), we can write

$$M = \{(1, \pm 1), (-1, \pm i)\} \tag{3.10}$$

and the restriction of a K-type $\mu = (b)$ to M is the character

$$\widetilde{U}(1) \to \mathbb{C}^{\times},\ (g, z) \mapsto z^{2b}.$$

The K-types $\mu = \frac{1}{2}$ and $\mu = -\frac{1}{2}$ restrict to the characters $(g, z) \mapsto z$ and $(g, z) \mapsto z^{-1}$ respectively. Then δ must be the identity M-type $(x \mapsto x)$ if $\mu = \frac{1}{2}$, and the inverse M-type $(x \mapsto x^{-1})$ if $\mu = -\frac{1}{2}$. Note that, in both cases, $X = \operatorname{Ind}(\delta \otimes \frac{1}{2})$ is an oscillator representation.

Lastly, we assume $a = 0$. If X is an irreducible unitary ω-regular representation containing the trivial K-type, then $X = J_\nu$ for some value of ν (these are the only spherical representations of $Mp(2)$). Note that PDOI implies that $|\nu| \leq 1$ and the ω-regular condition requires that $|\nu| \geq \frac{1}{2}$, hence $\frac{1}{2} \leq \nu \leq 1$. So X is a complementary series (and a Meta-$A_q(\lambda)$ representation). $\square$

4. The Group Mp(4)

4.1. The Structure of Mp(4). We realize the Lie algebra $\mathfrak{g}_0 = \mathfrak{sp}(4, \mathbb{R})$ of $Mp(4)$ and the Lie algebra $\mathfrak{k}_0$ of its maximal compact subgroup as

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{12} & b_{22} \\ \hline c_{11} & c_{12} & -a_{11} & -a_{21} \\ c_{12} & c_{22} & -a_{12} & -a_{22} \end{array} \right) : a_{i,j}, b_{i,j}, c_{i,j} \in \mathbb{R} \right\} \tag{4.1}$$

and

$$\mathfrak{k}_0 = \left\{ Y = \left(\begin{array}{cc|cc} 0 & a & x & z \\ -a & 0 & z & y \\ \hline -x & -z & 0 & a \\ -z & -y & -a & 0 \end{array} \right) : a, x, y, z \in \mathbb{R} \right\}. \tag{4.2}$$

The maximal compact subgroup $K \simeq \widetilde{U}(2)$ of $G = Mp(4)$ is isomorphic to the subgroup

$$U = \{(g, z) \in U(2) \times U(1) \colon \det(g) = z^2\}$$

of $U(2) \times U(1)$. Identifying $\mathfrak{k}_0$ with the Lie algebra of U gives a map

$$\iota \colon \mathfrak{k}_0 \to \mathfrak{u}(2) \oplus \mathfrak{u}(1),\ Y \mapsto \iota(Y) = \left[\begin{pmatrix} xi & a + zi \\ -a + zi & yi \end{pmatrix}, \frac{x + y}{2} i \right], \tag{4.3}$$

where Y is the element of $\mathfrak{k}_0$ as in (4.2). We denote the exponentiated map $K \to U$ by ι as well. Let $\mathfrak{a}_0$ be the diagonal CSA of $\mathfrak{g}_0$, $A = \exp(\mathfrak{a}_0)$, and $M = Cent_K(A)$. Here exp denotes the exponential map in $Mp(4)$. Also let

$$a = \exp \left(\begin{array}{cc|cc} 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \qquad b = \exp \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ \hline 0 & 0 & 0 & 0 \\ 0 & -\pi & 0 & \end{array} \right), \tag{4.4}$$

$$x = \exp \left(\begin{array}{cc|cc} 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \\ \hline -\pi & 0 & 0 & 0 \\ 0 & -\pi & 0 & 0 \end{array} \right) \qquad y = \exp \left(\begin{array}{cc|cc} 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & -\pi \\ \hline -\pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{array} \right). \tag{4.5}$$

Then

$$\iota(a) = \exp \left[\begin{pmatrix} i\pi & 0 \\ 0 & 0 \end{pmatrix}, \frac{\pi}{2} i \right] = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, i \right] \tag{4.6}$$

$$\iota(b) = \exp \left[\begin{pmatrix} 0 & 0 \\ 0 & i\pi \end{pmatrix}, \frac{\pi}{2} i \right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, i \right] \tag{4.7}$$

$$\iota(x) = \exp \left[\begin{pmatrix} i\pi & 0 \\ 0 & i\pi \end{pmatrix}, \pi i \right] = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, -1 \right] \quad \text{and} \tag{4.8}$$

$$\iota(y) = \exp \left[\begin{pmatrix} \pi i & 0 \\ 0 & -\pi i \end{pmatrix}, 0 \right] = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right]. \tag{4.9}$$

We notice that $pr(a) = diag(-1, 1, -1, 1) = pr(a^{-1})$, $x = ab$ and $y = ab^{-1}$. Set

$$z := a^2 = \exp \left(\begin{array}{cc|cc} 0 & 0 & 2\pi & 0 \\ 0 & 0 & 0 & 0 \\ \hline -2\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = \exp \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi \\ \hline 0 & 0 & 0 & 0 \\ 0 & -2\pi & 0 & 0 \end{array} \right) = b^2. \tag{4.10}$$

Then $Z(Mp(4)) = \{e, z, x, y\} = \langle x, y \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, with e the identity, and

$$M = \{e, z, x, y, a, a^{-1}, b, b^{-1}\} = \langle a, b \rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_2. \tag{4.11}$$

Let

$$\mathfrak{t}_0 = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \varphi \\ \hline -\theta & 0 & 0 & 0 \\ 0 & -\varphi & 0 & 0 \end{array} \right) : \theta, \varphi \in \mathbb{R} \right\} \subseteq \mathfrak{k}_0, \tag{4.12}$$

a fundamental Cartan subalgebra. Then

$$\iota(\mathfrak{t}_0) = \left\{ \left[\begin{pmatrix} \theta i & 0 \\ 0 & \varphi i \end{pmatrix}, \frac{\theta + \varphi}{2} i \right] : \theta, \varphi \in \mathbb{R} \right\} \tag{4.13}$$

and the corresponding Cartan subgroup T is given by

$$\iota(T) = \left\{ t_{\theta,\varphi} = \left[\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}, e^{\frac{\theta+\varphi}{2} i} \right] : \theta, \varphi \in \mathbb{R} \right\}. \tag{4.14}$$

A weight $\mu = \left(\frac{k}{2}, \frac{l}{2}\right)$ corresponds to the character of T given by

$$t_{\theta,\varphi} \mapsto e^{\frac{k\theta + l\varphi}{2} i}. \tag{4.15}$$

We identify K-types with their highest weights. The K-type

$$\mu = \left(\frac{k}{2}, \frac{l}{2}\right) \tag{4.16}$$

is genuine if and only if both k and l are odd, and nongenuine if they are both even. In each case, the K-type has dimension $\frac{k-l}{2}+1$, the other weights being

$$\left(\frac{k}{2}-j, \frac{l}{2}+j\right), \text{ for } 1 \leq j \leq \frac{k-l}{2}. \tag{4.17}$$

We will check the unitarity of (ω-regular) representations of $Mp(4)$ by partitioning the set of $\widetilde{U}(2)$-types μ in a suitable way, and considering for each family the set of ω-regular representations which have such a μ as a lowest K-type. The symmetry considerations at the end of Section 2.3 and Proposition 8 reduce the $\widetilde{U}(2)$-types we need to consider to those of the form

$$\mu = \left(\frac{k}{2}, \frac{l}{2}\right) \text{ with } k \geq |l|\,. \tag{4.18}$$

4.2. The Genuine Case: *K*-Types. We partition the genuine K-types into those that are lowest K-types of $A_{\mathfrak{q}}(\Omega)$ representations, and those that are not. In order to construct an $A_{\mathfrak{q}}(\Omega)$ module, we must start with a theta stable subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Any such algebra is of the form

$$\mathfrak{q} = \mathfrak{q}(\xi) = \mathfrak{l}(\xi) + \mathfrak{u}(\xi),$$

where $\xi \in i\mathfrak{t}_0^*$,

$$\mathfrak{l}(\xi) = \mathfrak{t} + \sum_{\langle\alpha,\xi\rangle=0} \mathfrak{g}_\alpha,$$
$$\mathfrak{u}(\xi) = \sum_{\langle\alpha,\xi\rangle>0} \mathfrak{g}_\alpha.$$

We get 10 theta stable parabolic algebras $\mathfrak{q}_i = \mathfrak{q}(\xi_i)$, with

$$\begin{array}{ll}
\xi_1 = (0,0) & \xi_6 = (0,-1) \\
\xi_2 = (2,1) & \xi_7 = (-1,-2) \\
\xi_3 = (2,-1) & \xi_8 = (1,-2) \\
\xi_4 = (1,-1) & \xi_9 = (1,1) \\
\xi_5 = (1,0) & \xi_{10} = (-1,-1).
\end{array}$$

The corresponding Levi factors L_i are

$$\begin{array}{ll}
L_1 = Mp(4) & L_6 = \widetilde{U}(0,1) \times Mp(2) \\
L_2 = \widetilde{U}(1,0) \times \widetilde{U}(1,0) & L_7 = \widetilde{U}(0,1) \times \widetilde{U}(0,1) \\
L_3 = \widetilde{U}(1,0) \times \widetilde{U}(0,1) & L_8 = \widetilde{U}(0,1) \times \widetilde{U}(1,0) \\
L_4 = \widetilde{U}(1,1) & L_9 = \widetilde{U}(2,0) \\
L_5 = \widetilde{U}(1,0) \times Mp(2) & L_{10} = \widetilde{U}(0,2).
\end{array}$$

Some of these Levi subgroups are, of course, pairwise identical; however, the corresponding nilpotent parts of the parabolic subalgebras are different. We express these differences in our notation for the L_i as above. We notice that

- any $A_{\mathfrak{q}}(\Omega)$ representation with $L = L_7$ is dual to one with $L = L_2$. Similarly for the pairs $\{L_6, L_5\}$, $\{L_8, L_3\}$, and $\{L_{10}, L_9\}$;

- any $A_{\mathfrak{q}}(\Omega)$ representation with $L = L_9$ is a discrete series which may also be constructed with $L = L_2$ (see [**6**]).

Hence we may restrict our attention to the cases

$$L_1 = Mp(4), \tag{4.19}$$

$$L_2 = \widetilde{U}(1,0) \times \widetilde{U}(1,0), \tag{4.20}$$

$$L_3 = \widetilde{U}(1,0) \times \widetilde{U}(0,1), \tag{4.21}$$

$$L_4 = \widetilde{U}(1,1) \text{ and} \tag{4.22}$$

$$L_5 = \widetilde{U}(1,0) \times Mp(2). \tag{4.23}$$

With $L_1 = Mp(4)$, the $A_{\mathfrak{q}}(\Omega)$ modules we obtain are the four oscillator representations with lowest K-types

$$\Lambda_1 = \left\{ \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{3}{2}\right) \right\}. \tag{4.24}$$

Now consider $L_2 = \widetilde{U}(1,0) \times \widetilde{U}(1,0)$. Then

$$\rho(\mathfrak{u}) = (2,1) \tag{4.25}$$

$$\rho(\mathfrak{l}) = (0,0) \tag{4.26}$$

$$\rho(\mathfrak{u} \cap \mathfrak{p}) = \left(\frac{3}{2}, \frac{3}{2}\right). \tag{4.27}$$

Set

$$\lambda = (\lambda_1, \lambda_2) \tag{4.28}$$

with $\lambda_i \in \mathbb{Z} + \frac{1}{2}$. For λ to be in the good range for $\mathfrak{q}$, the parameter

$$\lambda + \rho(\mathfrak{l}) + \rho(\mathfrak{u}) = (\lambda_1 + 2, \lambda_2 + 1) \tag{4.29}$$

must be strictly dominant for the roots $\{2e_1, 2e_2, e_1 \pm e_2\}$ of $\mathfrak{u}$. This says that

$$\lambda_1 + 2 > \lambda_2 + 1 > 0 \quad \Leftrightarrow \quad \lambda_1 \geq \lambda_2 \geq -\frac{1}{2}. \tag{4.30}$$

We obtain lowest K-types of the form

$$\mu = \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = (\lambda_1 + 3, \lambda_2 + 3), \tag{4.31}$$

which belong to the set

$$\Lambda_2 = \left\{ (r,s) \colon r \geq s \geq \frac{5}{2} \right\}. \tag{4.32}$$

For $L_3 = \widetilde{U}(1,0) \times \widetilde{U}(0,1)$, we have essentially two choices for $\mathfrak{u}$, corresponding to

$$\rho(\mathfrak{u}) = (2,-1) \quad \text{and} \quad \rho(\mathfrak{u}) = (1,-2). \tag{4.33}$$

The collection of $A_{\mathfrak{q}}(\Omega)$ modules obtained with the second choice are easily seen to be the contragredient modules of those obtained with the first choice, so we may assume that

$$\rho(\mathfrak{u}) = (2,-1) \tag{4.34}$$

$$\rho(\mathfrak{l}) = (0,0) \tag{4.35}$$

$$\rho(\mathfrak{u}\cap\mathfrak{p}) = \left(\frac{3}{2},-\frac{1}{2}\right). \tag{4.36}$$

This time $\lambda = (\lambda_1,\lambda_2)$ is in the good range if the parameter

$$\lambda + \rho(\mathfrak{l}) + \rho(\mathfrak{u}) = (\lambda_1+2,\lambda_2-1) \tag{4.37}$$

is strictly dominant for the roots $\{2e_1,-2e_2,e_1\pm e_2\}$, i. e.,

$$\lambda_1+2 > -(\lambda_2-1) > 0 \quad\Leftrightarrow\quad \lambda_1 \geq -\lambda_2 \geq -\frac{1}{2}. \tag{4.38}$$

We get lowest K-types of the form

$$\mu = (\lambda_1+3,\lambda_2-1)\,, \tag{4.39}$$

which give rise to the collection

$$\Lambda_3 = \left\{(r,s) : s \leq -\frac{1}{2}, r \geq -s+2\right\}. \tag{4.40}$$

For $L_4 = \widetilde{U}(1,1)$, we have

$$\rho(\mathfrak{u}) = \left(\frac{3}{2},-\frac{3}{2}\right) \tag{4.41}$$

$$\rho(\mathfrak{l}) = \left(\frac{1}{2},\frac{1}{2}\right) \tag{4.42}$$

$$\rho(\mathfrak{u}\cap\mathfrak{p}) = (1,-1)\,. \tag{4.43}$$

Set

$$\lambda = (\lambda_1,-\lambda_1) \tag{4.44}$$

with $\lambda_1 \in \mathbb{Z}+\frac{1}{2}$. Then λ is in the good range if

$$\lambda + \rho(\mathfrak{l}) + \rho(\mathfrak{u}) = (\lambda_1+2,-\lambda_1-1) \tag{4.45}$$

is strictly dominant for the roots $\{2e_1,-2e_2,e_1-e_2\}$ of $\mathfrak{u}$. Given that λ_1 is half-integral, this condition is equivalent to

$$\lambda_1 \geq -\frac{1}{2}. \tag{4.46}$$

We obtain lowest K-types of the form

$$\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p}) = (\lambda_1+2,-\lambda_1-2)\,, \tag{4.47}$$

which belong to the set

$$\Lambda_4 = \left\{(r,-r) : r \geq \frac{3}{2}\right\}. \tag{4.48}$$

Finally, let $L_5 = \widetilde{U}(1,0)\times Mp(2)$. In this case we have

$$\rho(\mathfrak{u}) = (2,0) \tag{4.49}$$

$$\rho(\mathfrak{l}) = (0,1) \tag{4.50}$$

$$\rho(\mathfrak{u}\cap\mathfrak{p}) = \left(\frac{3}{2},\frac{1}{2}\right). \tag{4.51}$$

We consider representations of L_5 of the form $\Omega = \mathbb{C}_\lambda \otimes \omega$, with ω an oscillator representation of $Mp(2)$. If ω is an *odd* oscillator representations, then ω is a discrete series of $Mp(2)$; the corresponding $A_{\mathfrak{q}}(\Omega)$ representation is a discrete series of $Mp(4)$ and has already been considered. Hence we may assume that ω is an *even* oscillator representation of $Mp(2)$.
The infinitesimal character of $\Omega = \mathbb{C}_\lambda \otimes \omega$ is $\left(\lambda, \frac{1}{2}\right)$; this is in the good range if

$$\left(\lambda, \frac{1}{2}\right) + (2,0) = \left(\lambda + 2, \frac{1}{2}\right) \tag{4.52}$$

is strictly dominant with respect to $\Delta(\mathfrak{u}) = \{2e_1, e_1 \pm e_2\}$, i. e.,

$$\lambda \geq -\frac{1}{2}. \tag{4.53}$$

The lowest $L\cap K$-type of Ω is then either

$$\left(\lambda, \frac{1}{2}\right) \quad \text{or} \quad \left(\lambda, -\frac{1}{2}\right) \tag{4.54}$$

depending on whether ω is even holomorphic or even antiholomorphic. Adding $2\rho(\mathfrak{u} \cap \mathfrak{p})$ we get the following set of lowest K-types

$$\Lambda_5 = \left\{\left(r, \frac{3}{2}\right) : r \geq \frac{5}{2}\right\} \cup \left\{\left(r, \frac{1}{2}\right) : r \geq \frac{5}{2}\right\}. \tag{4.55}$$

REMARK 4.1. The sets Λ_i for $1 \leq i \leq 5$ list all the genuine K-types (r,s) which occur as lowest K-types of $A_{\mathfrak{q}}(\Omega)$ modules, and satisfy $r \geq |s|$.

This leaves us with the following set Σ of genuine K-types which are NOT lowest K-types of $A_{\mathfrak{q}}(\Omega)$ representations:

$$\Sigma = \left\{\left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right)\right\} \cup \left\{(r, -r+1) \colon r \geq \frac{3}{2}\right\}. \tag{4.56}$$

4.3. The Set Σ of Non-$A_{\mathfrak{q}}(\Omega)$ Lowest K-Types. We show that any ω-regular representation with a lowest K-type in the set Σ must be nonunitary. In most cases this is done using PDOI.

First consider

$$\mu = \left(\frac{3}{2}, \frac{3}{2}\right). \tag{4.57}$$

Choose

$$\rho_n = \left(\frac{3}{2}, \frac{3}{2}\right) = \mu \tag{4.58}$$

and w trivial. Then

$$\langle w(\mu - \rho_n) + \rho_c,\, w(\mu - \rho_n) + \rho_c\rangle = \langle \rho_c,\, \rho_c\rangle = \frac{1}{2} \tag{4.59}$$

because $\rho_c = \left(\frac{1}{2}, -\frac{1}{2}\right)$. Notice that any ω-regular infinitesimal character γ satisfies

$$\langle \gamma,\, \gamma\rangle \geq \langle \gamma_\omega,\, \gamma_\omega\rangle = \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{5}{2} > \frac{1}{2}, \tag{4.60}$$

hence the Parthasarathy's Dirac operator inequality (2.37) fails. This proves that any ω-regular representation with lowest K-type $\mu = \left(\frac{3}{2}, \frac{3}{2}\right)$ is nonunitary.

Now let

$$\mu = (r, -r+1) \tag{4.61}$$

with $r \geq \frac{3}{2}$. Choose

$$\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right) \tag{4.62}$$

and w trivial. Then

$$w(\mu - \rho_n) + \rho_c = (r-1, -r+1)\,. \tag{4.63}$$

and

$$\langle w(\mu - \rho_n) + \rho_c,\, w(\mu - \rho_n) + \rho_c\rangle = 2(r-1)^2. \tag{4.64}$$

A representation with this lowest K-type has Vogan parameter

$$\lambda_a = p(\mu + 2\rho_c - \rho). \tag{4.65}$$

Since

$$\rho = (2, -1), \tag{4.66}$$

we get

$$\lambda_a = (r-1, -r+1). \tag{4.67}$$

Hence the corresponding standard module is induced from a parabolic subgroup $P = MAN$ with $MA \simeq GL(2, \mathbb{R})$, and the infinitesimal character is of the form

$$\gamma = (r - 1 + \nu, -r + 1 + \nu) \tag{4.68}$$

for some number ν (see section 2.4).
In order for γ to be ω-regular, ν must be real with $|\nu| \geq \frac{1}{2}$. Recall from Section 2.4 that we may conjugate ν to be positive. Then we have $\nu \geq 1/2$, and

$$\langle \gamma,\, \gamma\rangle = (r-1+\nu)^2 + (r-1-\nu)^2 > 2(r-1)^2 \tag{4.69}$$

Because the PDOI fails, such a representation is nonunitary.

Finally consider the K-type

$$\mu = \left(\frac{1}{2}, -\frac{1}{2}\right). \tag{4.70}$$

If we choose

$$\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right) \tag{4.71}$$

and $w = -1$ (the long Weyl group element), then

$$w(\mu - \rho_n) + \rho_c = \left(\frac{3}{2}, \frac{1}{2}\right), \tag{4.72}$$

and

$$\langle w(\mu - \rho_n) + \rho_c,\, w(\mu - \rho_n) + \rho_c\rangle = \frac{5}{2} = \langle \gamma_\omega,\, \gamma_\omega\rangle. \tag{4.73}$$

PDOI implies that any unitary ω-regular representation containing this K-type must have infinitesimal character γ_ω. It is easy to check that other choices for ρ_n do not give any better estimate. So it remains to show that the irreducible representations with lowest K-type $\mu = \left(\frac{1}{2}, -\frac{1}{2}\right)$ and infinitesimal character γ_ω (these are two principal series representations which are dual to each other) are nonunitary. We do this in Section 5, by explicitly computing the signature of a Hermitian form.

4.4. Uniqueness of Representations with $A_{\mathfrak{q}}(\Omega)$ Lowest K-types. In this section we prove that there are no ω-regular unitary representations of G which have an $A_{\mathfrak{q}}(\Omega)$ lowest K-type, but are not $A_{\mathfrak{q}}(\Omega)$ representations. We do this case by case, considering in turn the K-types listed in the sets Λ_i in Section 4.2. For each K-type μ, we show that there is only one ω-regular irreducible representation with lowest K-type μ and such that μ satisfies PDOI. We rely heavily on the Langlands classification and Vogan's lowest K-type ideas as outlined in Section 2.4.

4.4.1. *The Set* Λ_1. Suppose $\mu \in \Lambda_1$. We only have to consider the cases $\mu = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\mu = \left(\frac{3}{2}, \frac{1}{2}\right)$ (the other K-types in Λ_1 are dual to these, by Proposition 8). Using PDOI with $\rho_n = \left(\frac{3}{2}, \frac{3}{2}\right)$ and $w = 1$, we can easily see that every unitary ω-regular representation π of G containing either of these two lowest K-types has infinitesimal character γ_ω.

The K-type $\left(\frac{1}{2}, \frac{1}{2}\right)$ is one-dimensional. In this case, the uniqueness of π follows directly from a result of Zhu [**20**] which states that a representation with scalar lowest K-type is uniquely determined by its infinitesimal character.

It remains to show that there is a unique irreducible representation with lowest K-type $\mu = \left(\frac{3}{2}, \frac{1}{2}\right)$ and infinitesimal character γ_ω. We compute the Vogan-parameter λ_a associated to μ. We have

$$\mu + 2\rho_c = \left(\frac{5}{2}, -\frac{1}{2}\right); \tag{4.74}$$

we choose $\rho = (2, -1)$ to get

$$\mu + 2\rho_c - \rho = \left(\frac{1}{2}, \frac{1}{2}\right). \tag{4.75}$$

This parameter is not in the Weyl chamber determined by ρ, so we project it and obtain

$$\lambda_a = \left(\frac{1}{2}, 0\right). \tag{4.76}$$

So the corresponding Levi factor is

$$MA = Mp(2) \times \widetilde{GL}(1, \mathbb{R}), \tag{4.77}$$

with the discrete series with Harish-Chandra parameter $\frac{1}{2}$ on the first factor, and a character $\chi_{\varepsilon,\nu}$ on the second. In order to obtain infinitesimal character γ_ω on the induced representation, we must have $\nu = \frac{3}{2}$, and the sign ε is uniquely determined by the lowest K-type. So there is indeed only one such representation, which must then be the odd oscillator representation.

4.4.2. *The Sets* Λ_2 *and* Λ_3. Since the elements of these two sets are lowest K-types of $A_{\mathfrak{q}}(\lambda)$ representations with L compact, they are lowest K-types of discrete series. In this case, the representation is determined uniquely and there is nothing to prove.

4.4.3. *The Set* Λ_4. Now suppose that $\mu \in \Lambda_4$. Then

$$\mu = (a, -a) \quad \text{with} \quad a \geq \frac{3}{2}. \tag{4.78}$$

The corresponding Vogan parameter can easily be computed:

$$\lambda_a = \left(a - \frac{1}{2}, -a + \frac{1}{2}\right), \tag{4.79}$$

so we get $MA = \widetilde{GL}(2, \mathbb{R})$. The representation depends only on a continuous parameter $\nu \geq 0$, and has infinitesimal character

$$\gamma = \left(a - \frac{1}{2} + \nu, -a + \frac{1}{2} + \nu\right). \tag{4.80}$$

Notice that for γ to be ω-regular, we must have $\nu \geq \frac{1}{2}$. Set $\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right)$; then

$$\mu - \rho_n + \rho_c = (a - 1, -a) \tag{4.81}$$

and PDOI gives

$$\langle \gamma, \gamma \rangle \leq a^2 + (a - 1)^2. \tag{4.82}$$

It is now easy to check that this condition implies $\nu \leq \frac{1}{2}$. Hence we must have $\nu = \frac{1}{2}$, and we are done with this case.

4.4.4. *The Set* Λ_5. Now consider K-types of the form $\mu = \left(a, \frac{3}{2}\right)$ and $\mu = \left(a, \frac{1}{2}\right)$ with $a \geq \frac{5}{2}$. In both cases, the Vogan algorithm yields

$$\lambda_a = (a - 1, 0), \tag{4.83}$$

so the corresponding Levi subgroup is $MA = Mp(2) \times \widetilde{GL}(1, \mathbb{R})$. The representation of $Mp(2)$ is the discrete series representation with Harish-Chandra parameter $a-1$; on $\widetilde{GL}(1, \mathbb{R})$ we have a character $\chi_{\varepsilon,\nu}$. The sign ε is uniquely determined by the lowest K-type ($\mu = \left(a, \frac{3}{2}\right)$ or $\mu = \left(a, \frac{1}{2}\right)$), hence any representation with lowest K-type μ is uniquely determined by its infinitesimal character

$$\gamma = (a - 1, \nu). \tag{4.84}$$

It remains to check that the unitarity and ω-regularity conditions determine ν uniquely. Note that in order for γ to be ω-regular, we must have $\nu \geq \frac{1}{2}$.
Now using $\rho_n = \left(\frac{3}{2}, \frac{3}{2}\right)$ for $\mu = \left(a, \frac{3}{2}\right)$, and $\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right)$ for $\mu = \left(a, \frac{1}{2}\right)$, we obtain

$$\mu - \rho_n + \rho_c = \left(a - 1, \mp\frac{1}{2}\right). \tag{4.85}$$

By PDOI, γ must have length no greater than this parameter, so $\nu \leq \frac{1}{2}$. This forces $\nu = \frac{1}{2}$, proving the uniqueness.

4.5. The Nongenuine Case. In this section, we show that any nongenuine ω-regular unitary representation of $Mp(4)$ is a Meta-$A_{\mathfrak{q}}(\lambda)$ representation.

4.5.1. *Nongenuine K-types.* As for the genuine case, we partition the nongenuine K-types into families, and consider each family in turn. Note that nongenuine K-types are irreducible representations of $U(2)$, parameterized by pairs of integers. Since the calculations are very similar to those performed in the genuine case, we omit many of the details. Partitioning the lowest K-types of Meta-$A_{\mathfrak{q}}(\lambda)$ representations according to the Levi factors L_1 through L_5 of possible parabolic subalgebras as in Section 4.2, we get the families

$$\Lambda_1 = \{(0,0)\}\,, \tag{4.86}$$

$$\Lambda_2 = \{(r,s) : r \geq s \geq 3\}\,, \tag{4.87}$$

$$\Lambda_3 = \{(r.s) : s \leq -1,\, r \geq -s+2\}\,, \tag{4.88}$$

$$\Lambda_4 = \{(r,-r) : r \geq 2\}\,, \tag{4.89}$$

$$\Lambda_5 = \{(r,1) : r \geq 3\}\,. \tag{4.90}$$

The nongenuine K-types which are not lowest K-types of Meta-$A_{\mathfrak{q}}(\lambda)$ representations are therefore part of the set

$$\Sigma = \{(r,2) : r \geq 2\} \cup \{(r,0) : r \geq 1\} \cup \{(r+1,-r) : r \geq 1\} \cup \{(2,1)\,,(1,1)\,,(1,-1)\}\,.$$

4.5.2. *The K-Types in Σ.* For every K-type μ in the set Σ we can show, using PDOI, that there is no ω-regular unitary representation which has μ as one of its lowest K-types. For example, for

$$\mu = (2,1) \tag{4.91}$$

we can use $\rho_n = \left(\frac{3}{2}, \frac{3}{2}\right)$ to see that

$$\mu - \rho_n + \rho_c = (1,-1)\,, \tag{4.92}$$

a weight whose length is strictly less than that of any ω-regular infinitesimal character. Because PDOI fails, an ω-regular representation with lowest K-type μ cannot be unitary. For a second example, let's consider

$$\mu = (r,0) \tag{4.93}$$

with $r \geq 1$. The corresponding Vogan-parameter is

$$\lambda_a = (r-1,0)\,. \tag{4.94}$$

If $r = 1$, then $\lambda_a = (0,0)$, so any representation with lowest K-type $\mu = (r,0)$ is a principal series. In this case, with $\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right)$ and w the long element of the Weyl group, we get

$$w(\mu - \rho_n) + \rho_c = (1,-1)\,, \tag{4.95}$$

once again a weight whose length is less than that of γ_ω.
If $r \geq 2$, then the corresponding Levi subgroup is $MA = Mp(2) \times \widetilde{GL}(1,\mathbb{R})$, and the infinitesimal character is

$$\gamma = (r-1,\nu)\,. \tag{4.96}$$

The ω-regularity condition forces $\nu \geq \frac{1}{2}$. If we choose $\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right)$ and $w = 1$, we get

$$w(\mu - \rho_n) + \rho_c = (r-1,0)\,, \tag{4.97}$$

a weight of length strictly smaller than γ. In either case, we conclude there is no ω-regular unitary representation with lowest K-type $(r,0)$.

The calculation for the remaining K-types in Σ is similar; we leave the details to the diligent reader.

4.5.3. *The Lowest K-Types of Meta-$A_{\mathfrak{q}}(\lambda)$'s.* As for the genuine case, it remains to show that every ω-regular unitary representation with a lowest K-type of a Meta-$A_{\mathfrak{q}}(\lambda)$ is indeed a Meta-$A_{\mathfrak{q}}(\lambda)$. The proof for K-types in Λ_2, Λ_3, and Λ_4 is very similar to the corresponding one in the genuine case, so we omit it.

For $\mu = (0,0)$, we need to show that the trivial representation is the only ω-regular unitary spherical representation of $Mp(4)$. Here, we refer to the results of [**3**] or [**10**].

We are left with the K-types of the form

$$\mu = (r,1) \text{ with } r \geq 3. \tag{4.98}$$

These are lowest K-types of Meta-$A_{\mathfrak{q}}(\lambda)$ representations $\mathcal{R}_{\mathfrak{q}}(\mathbb{C}_\lambda \otimes J_\nu)$ (cf. Definition 2.5) with $L = L_5 = \widetilde{U}(1,0) \times Mp(2)$, $\lambda = r-3$, and J_ν a spherical complementary series of $SL(2,\mathbb{R})$ with infinitesimal character $\frac{1}{2} \leq \nu \leq 1$. Such a representation has infinitesimal character

$$(\lambda,\nu) + \rho(\mathfrak{u}) = (r-3,\nu) + (2,0) = (r-1,\nu)\,. \tag{4.99}$$

We prove that every ω-regular unitary representation X with lowest K-type $\mu = (r,1)$ is of the form $\mathcal{R}_{\mathfrak{q}}(\mathbb{C}_\lambda \otimes J_\nu)$, with $\lambda = r-3$ and $\frac{1}{2} \leq \nu \leq 1$.
The Vogan parameter associated to μ is

$$\lambda_a = (r-1,0) \tag{4.100}$$

as in (4.94) above. Hence X has infinitesimal character

$$\gamma = (r-1,\nu) \tag{4.101}$$

for some positive number ν. The representation X is uniquely determined by the value of ν, so must be of the form $\mathcal{R}_{\mathfrak{q}}(\mathbb{C}_\lambda \otimes J_\nu)$, with $\lambda = r-3$. It remains to prove that ν belongs to the appropriate range. Applying PDOI to μ with $\rho_n = \left(\frac{3}{2}, -\frac{1}{2}\right)$, we find that the infinitesimal character γ of X can not be greater than

$$(r-1,1)\,. \tag{4.102}$$

Therefore $\nu \leq 1$. The ω-regularity condition forces $\nu \geq \frac{1}{2}$, so we are done.

Consequently, we have

Proposition 10. If X is an irreducible ω-regular unitary nongenuine representation of $Mp(4)$ then X is a Meta-$A_{\mathfrak{q}}(\lambda)$ representation.

5. Nonunitarity of the Mystery Representation X_M

In this section we finish the proof of Theorem 2.8 by showing that the two representations of $Mp(4)$ with lowest K-type $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and infinitesimal character $\left(\frac{3}{2}, \frac{1}{2}\right)$ are not unitary (cf. Section 4.3). Since the PDOI is satisfied in this case, we have to use a different method; i. e., we must construct the intertwining operator giving rise to the invariant hermitian form and show that the form is indefinite. We discuss this theory in Section 5.2 in some detail.

5.1. The Representation X_M. We consider the irreducible representations of $Mp(4)$ with lowest K-type $\left(\frac{1}{2},-\frac{1}{2}\right)$ and infinitesimal character $\left(\frac{3}{2},\frac{1}{2}\right)$, and prove that they are nonunitary. From now on we identify elements of K with their images under the map ι (cf. (4.3)).

The K-type $\left(\frac{1}{2},-\frac{1}{2}\right)$ is fine, so any representation with this lowest K-type is a constituent of a principal series, i. e., an induced representation of the form

$$I_P(\delta,\nu)=\mathrm{Ind}_P^G(\delta\otimes\nu\otimes 1), \tag{5.1}$$

where $P=MAN$ is a minimal parabolic subgroup of G, M and A are as in Section 4.1, δ is a character of M, and ν is the character of A with differential $\left(\frac{3}{2},\frac{1}{2}\right)$. Recall that, by Frobenius reciprocity, the K-type

$$\mu_\delta=\left(\frac{1}{2},-\frac{1}{2}\right) \tag{5.2}$$

occurs in the principal series $I_P(\delta,\nu)$ if (and only if) the character δ is a summand of the restriction of μ_δ to M. We look at this restriction.
The K-type μ_δ contains the weights

$$\left(\frac{1}{2},-\frac{1}{2}\right)\ \text{ and }\ \left(-\frac{1}{2},\frac{1}{2}\right); \tag{5.3}$$

they correspond to the M-characters

$$\delta_1(a)=+i=-\delta_1(b) \tag{5.4}$$

$$\delta_2(a)=-i=-\delta_2(b) \tag{5.5}$$

respectively. The two elements

$$a=\left[\begin{pmatrix}-1&0\\0&1\end{pmatrix},i\right]\ \text{ and }\ b=\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix},i\right] \tag{5.6}$$

are the generators of M. Then $I_P(\delta,\nu)$ contains μ_δ if and only if δ equals δ_1 or δ_2. We conclude that there are exactly two irreducible representations of $Mp(4)$ with lowest K-type μ_δ and infinitesimal character $\left(\frac{3}{2},\frac{1}{2}\right)$: they are the irreducible constituents of the principal series $I_P(\delta_1,\nu)$ and $I_P(\delta_2,\nu)$ containing μ_δ. We denote these representations by $\overline{X}(\delta_1,\nu)$ and $\overline{X}(\delta_2,\nu)$.

It is not hard to see that $\overline{X}(\delta_1,\nu)$ and $\overline{X}(\delta_2,\nu)$ are contragredient of each other (hence one is unitary if and only if the other one is). We sometimes refer to either one of these two representations as the "Mystery Representation" X_M, and write δ for either one of the two characters δ_i.

The purpose of this section is to prove that the Mystery representation is nonunitary. First we observe that X_M is Hermitian (i. e., it is equivalent to its Hermitian dual and hence carries a nondegenerate G-invariant Hermitian form). By [**17**], it is sufficient to prove that there exists an element w of the Weyl group $W(G,A)$ taking (P,δ,ν) to $\left(\bar{P},\delta,-\overline{\nu}\right)$, with $\bar{P}$ the opposite parabolic. Note that $-\overline{\nu}=-\nu$ because ν is real.
We claim that the long Weyl group element $w_0=s_{2e_1}s_{2e_2}$ has the desired properties. Indeed, w_0 takes P to its opposite, ν to its negative, and fixes δ:

$$(w_0\cdot\delta)(m)=\delta(\sigma_0^{-1}m\sigma_0)=\delta(m)\quad\forall m\in M. \tag{5.7}$$

Here σ_0 denotes a representative for w_0 in K:

$$\sigma_0=\sigma_{2e_1}\sigma_{2e_2}=\left[\begin{pmatrix}i&0\\0&1\end{pmatrix},e^{\frac{\pi}{4}i}\right]\left[\begin{pmatrix}1&0\\0&i\end{pmatrix},e^{\frac{\pi}{4}i}\right]=\left[\begin{pmatrix}i&0\\0&i\end{pmatrix},i\right] \tag{5.8}$$

(note that σ_0 is diagonal, hence it commutes with all elements of M).

The nondegenerate invariant Hermitian form on X_M is unique up to a constant. We will explicitly construct a Hermitian intertwining operator that induces the form. Proving that X_M is nonunitary is then equivalent to showing that this intertwining operator is indefinite.

5.2. Intertwining Operators and Unitarity of Principal Series. In this section, we review the theory of intertwining operators for principal series of (double covers of) real split groups. The results presented at the beginning of the section are well known in the literature (c.f. [**5**], [**16**], [**2**] and [**7**]) and are reported here for completeness. The content of the last part is more innovative. The idea, explained at the end of the section, of using the operator $l_\mu(w_0, \nu)$ to look at several principal series at the same time, is due to Barbasch and Pantano, and has not appeared in any published work yet.

5.2.1. *(Formal) Intertwining Operators for Principal Series.* Let G be a (possibly trivial) two-fold cover of the split real form of a connected (simple and simply connected) reductive algebraic group. By allowing the cover to be trivial, we intend to discuss split linear groups and their nonlinear double cover simultaneously.

Choose a minimal parabolic subgroup $P = MAN$ of G. For every irreducible representation (δ, V^δ) of M and every character ν of A, we write

$$I_P(\delta, \nu) = \operatorname{Ind}_P^G(\delta \otimes \nu \otimes 1)$$

for the induced representation of G. This is normalized induction, so $I_P(\delta, \nu)$ is unitary when $\delta \otimes \nu \otimes 1$ is. The representation space for $I_P(\delta, \nu)$ is denoted by $\mathcal{H}_P(\delta, \nu)$, and consists of functions

$$f\colon G \to V^\delta \tag{5.9}$$

whose restriction to K is square integrable, such that

$$f(gman) = e^{-(\nu+\rho)\log a}\delta(m)^{-1} f(g) \tag{5.10}$$

for all $g \in G$, and $man \in P$. The action of G on these functions is by left translation.

For every element w of the Weyl group, there is a *formal* intertwining operator

$$\mathcal{A}_P(w, \delta, \nu)\colon I_P(\delta, \nu) \to I_P(w\delta, w\nu). \tag{5.11}$$

Notice that $\mathcal{A}_P(w, \delta, \nu)$ satisfies all the intertwining properties, but is defined by an integral that may not converge for all values of ν. (See [**5**] for details.)
For all $F\colon G \to V^\delta$ in $\mathcal{H}_P(\delta, \nu)$, we set:

$$\mathcal{A}_P(w, \delta, \nu)f\colon G \to V^{w\delta},\ g \mapsto \int_{\overline{N}\cap(wNw^{-1})} f(gw\overline{n})\, d\overline{n}. \tag{5.12}$$

To reduce the computation of $\mathcal{A}_P(w, \delta, \nu)$ to a finite-dimensional problem, we restrict the operator to the various K-types appearing in the principal series. For every $(\mu, E_\mu) \in \widehat{K}$, we obtain an operator

$$\widetilde{a}_\mu(w, \delta, \nu)\colon\ \operatorname{Hom}_K(\mu, \operatorname{Ind}_P(\delta, \nu)) \to \operatorname{Hom}_K(\mu, \operatorname{Ind}_P(w\delta, w\nu)) \tag{5.13}$$

by composition on the range. Note that the restriction of $\operatorname{Ind}_P(\delta, \nu)$ to K is independent of ν and equal to the induced representation $\operatorname{Ind}_M^K \delta$. Similarly, $\operatorname{Ind}_P(w\delta, w\nu)$ restricts to $\operatorname{Ind}_M^K w\delta$. Then we can interpret $\widetilde{a}_\mu(w, \delta, \nu)$ as an operator

$$\widetilde{a}_\mu(w, \delta, \nu)\colon\ \operatorname{Hom}_K(\mu, \operatorname{Ind}_M^K \delta) \to \operatorname{Hom}_K(\mu, \operatorname{Ind}_M^K w\delta). \tag{5.14}$$

By Frobenius reciprocity, we obtain an operator

$$a_\mu(w,\delta,\nu)\colon \operatorname{Hom}_M(\mu,\delta) \to \operatorname{Hom}_M(\mu, w\delta) \tag{5.15}$$

for every $\mu \in \widehat{K}$. An easy computation shows that

$$(a_\mu(w,\delta,\nu)T)(v) = \int_{\overline{N}\cap(\sigma N \sigma^{-1})} e^{-(\nu+\rho)\log \underline{a}(\overline{n})} T\left(\mu\left(\sigma \underline{k}\left(\overline{n}\right)\right)^{-1} v\right) d\overline{n} \tag{5.16}$$

for all $T \in \operatorname{Hom}_M(\mu,\delta)$ and all $v \in E_\mu$. (See [**7**] for details.) Here σ denotes a representative for w in K, and $\underline{k}(g)\underline{a}(g)\underline{n}(g)$ denotes the Iwasawa decomposition of an element g in $G = KAN$.

We are going to break down the operator $a_\mu(w,\delta,\nu)$ so that its computation becomes manageable. The factorization

$$\mathcal{A}_P(w_1w_2,\delta,\nu) = \mathcal{A}_P(w_1, w_2\delta, w_2\nu)\mathcal{A}_P(w_2,\delta,\nu) \tag{5.17}$$

holds for any pair of Weyl group elements satisfying the condition

$$l(w_1w_2) = l(w_1) + l(w_2).$$

Here l denotes the length function on W. It follows that any formal operator $\mathcal{A}_P(w,\delta,\nu)$ can be decomposed as a product of operators corresponding to simple root reflections. The operators $a_\mu(w,\delta,\nu)$ inherit a similar decomposition. In particular, if

$$w = s_{\alpha_r} s_{\alpha_{r-1}} \cdots s_{\alpha_1} \tag{5.18}$$

is a *minimal* decomposition of w as a product of simple reflections, then we can factor $a_\mu(w,\delta,\nu)$ as a product of operators of the form

$$a_\mu(s_{\alpha_i},\delta_{i-1},\nu_{i-1})\colon \operatorname{Hom}_M(\mu,\delta_{i-1}) \to \operatorname{Hom}_M(\mu, s_{\alpha_i}\delta_{i-1}) \tag{5.19}$$

for α_i a simple root, $\delta_{i-1} \in \widehat{M}$ (in the W-orbit of δ) and $\nu_{i-1} \in \mathfrak{a}^*$.

In view of this result, we only need to understand the operator $a_\mu(s_\alpha,\delta,\nu)$ for α simple. The computation of $a_\mu(s_\alpha,\delta,\nu)$ can largely be reduced to a similar computation for the rank-one group MG^α, where G^α is the $SL(2)$ or $Mp(2)$ subgroup of G attached to the root α. (See Section 5.3 for a description of G^α for $Mp(4)$.)

We recall the construction of the group G^α. For every (simple) root α we can define a Lie algebra homomorphism

$$\phi_\alpha\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_0 \tag{5.20}$$

as in (4.3.5) of [**16**]. The image of ϕ_α is a subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The corresponding connected subgroup of G is denoted by G^α.

Exponentiating ϕ_α, we obtain a group homomorphism

$$\Phi_\alpha\colon Mp(2,\mathbb{R}) \to G \tag{5.21}$$

with image G^α. The structure of G^α depends on whether the map Φ_α factors through $SL(2,\mathbb{R})$.

If Φ_α factors through $SL(2,\mathbb{R})$, we say that the root α is "not metaplectic". In this case, G^α is isomorphic to $SL(2,\mathbb{R})$, and the maximal compact subgroup $K^\alpha \subset G^\alpha$ is isomorphic to $U(1)$. If Φ_α does not factor through $SL(2,\mathbb{R})$, then we call α "metaplectic". For metaplectic roots, $G^\alpha \simeq Mp(2,\mathbb{R})$ and $K^\alpha \simeq \widetilde{U(1)}$.
We can give a more explicit description of the metaplectic roots: if G is not of type G_2, then the metaplectic roots are exactly the long roots. If G is of type G_2, then all roots are metaplectic.

Let us go back to the task of computing the operator $a_\mu(s_\alpha,\delta,\nu)$. Recall that

$$a_\mu(s_\alpha,\delta,\nu)T(v) = \int_{\bar N\cap(\sigma_\alpha N\sigma_\alpha^{-1})} e^{-(\nu+\rho)\log \underline{a}(\bar n)} T\left(\mu\left(\sigma_\alpha \underline{k}(\bar n)\right)^{-1} v\right) d\bar n$$

for all $T\in \mathrm{Hom}_M(\mu,\delta)$ and all $v\in E_\mu$. Let $G^\alpha = K^\alpha A^\alpha N^\alpha$ be the Iwasawa decomposition of G^α. Note that:

- $\bar N\cap(\sigma_\alpha N\sigma_\alpha^{-1}) = \bar N^\alpha$
- The Iwasawa decompositions of $\bar n\in\bar N^\alpha$ inside G and G^α coincide, because
$$K^\alpha = K\cap G^\alpha \qquad A^\alpha = A\cap G^\alpha \qquad N^\alpha = N\cap G^\alpha.$$
- If α is simple, the restriction of $\rho = \frac{1}{2}\left[\sum_{\alpha\in\Delta^+}\alpha\right]$ to $\mathrm{Lie}(A^\alpha)$ equals ρ^α.

Then we can write

$$a_\mu(s_\alpha,\delta,\nu)T(v) \int_{\bar N^\alpha} e^{-(\nu+\rho^\alpha)\log \underline{a}^\alpha(\bar n)} T\left(\mu\left(\sigma_\alpha \underline{k}^\alpha(\bar n)\right)^{-1} v\right) d\bar n \tag{5.22}$$

for all $T\in\mathrm{Hom}_M(\mu,\delta)$ and $v\in E_\mu$. This formula coincides with the one for the corresponding operator for the group MG^α on the representation $\mu|_{MK^\alpha}$.

It follows that any decomposition of μ into MK^α-invariant subspaces must be preserved by the operator $a_\mu(s_\alpha,\delta,\nu)$.

Let $\mu = \bigoplus_{l\in\mathbb{Z}/2}\varphi_l^\alpha$ be the decomposition of μ as a direct sum of isotypic components of characters of K^α. The group M stabilizes the subspace $(\varphi_l^\alpha+\varphi_{-l}^\alpha)$, for every $l\in\mathbb{N}/2$. Hence the decomposition

$$\mu = \bigoplus_{l\in\mathbb{N}/2}\left(\varphi_l^\alpha+\varphi_{-l}^\alpha\right) \tag{5.23}$$

is stable under the action of MK^α. The corresponding decompositions of $\mathrm{Hom}_M(\mu,\delta)$ and $\mathrm{Hom}_M(\mu,s_\alpha\delta)$ are stable under the action of $a_\mu(s_\alpha,\delta,\nu)$. More precisely,

$$a_\mu(s_\alpha,\delta,\nu)\colon \mathrm{Hom}_M\left(\varphi_l^\alpha+\varphi_{-l}^\alpha,\delta\right)\to\mathrm{Hom}_M\left(\varphi_l^\alpha+\varphi_{-l}^\alpha,s_\alpha\delta\right) \tag{5.24}$$

for all $l\in\mathbb{N}/2$.

The computation of $a_\mu(s_\alpha,\delta,\nu)$ on $\mathrm{Hom}_M\left(\varphi_l\oplus\varphi_{-l},V^\delta\right)$ can be carried out explicitly, by evaluating an integral in $SL(2,\mathbb{R})$ or $Mp(2)$. We will not display the calculation here but just state the result. If

$$T\in\mathrm{Hom}_M(\varphi_l\oplus\varphi_{-l},V^\delta),$$

then $a_\mu(s_\alpha,\delta,\nu)$ maps T to

$$c_l(\alpha,\nu)\,T\circ\mu(\sigma_\alpha^{-1}). \tag{5.25}$$

The constant $c_l(\alpha,\nu)$ depends on l and on the inner product $\langle\nu,\,\alpha^\vee\rangle$, and is equal to:

$$c_l(\alpha,\nu) := \frac{\pi\Gamma(\lambda)}{2^{\lambda-1}\Gamma\left(1+\frac{\lambda-1+l}{2}\right)\Gamma\left(1+\frac{\lambda-1-l}{2}\right)}. \tag{5.26}$$

Here $\lambda = \langle\nu,\check\alpha\rangle$, and Γ denotes the Gamma function. Note that $c_l(\alpha,\nu) = c_{-l}(\alpha,\nu)$.

To simplify the notation, we introduce a normalization.
Choose a fine K-type μ_δ containing δ. The space $\mathrm{Hom}_M(\mu_\delta,\delta)$ is one-dimensional, hence the operator $a_{\mu_\delta}(w,\delta,\nu)$ acts on it by a scalar. We normalize the operator $\mathcal{A}_P(w,\delta,\nu)$ so that this scalar is one.

On each K-type μ we obtain a normalized operator $a'_\mu(s_\alpha, \delta, \nu)$, which acts on $\mathrm{Hom}_M(\varphi_l^\alpha + \varphi_{-l}^\alpha, \delta)$ by

$$a'_\mu(s_\alpha, \delta, \nu)T = c'_l(\alpha, \nu)\, T \circ \mu(\sigma_\alpha^{-1})$$

where

$$c'_l(\alpha,\nu) = \frac{c_l(\alpha,\nu)}{c_{\frac{1}{2}}(\alpha,\nu)} \qquad \text{if } l \text{ belongs to } \mathbb{Z} + \frac{1}{2};$$

$$c'_l(\alpha,\nu) = \frac{c_l(\alpha,\nu)}{c_1(\alpha,\nu)} \qquad \text{if } l \text{ is an odd integer;}$$

$$c'_l(\alpha,\nu) = \frac{c_l(\alpha,\nu)}{c_0(\alpha,\nu)} \qquad \text{if } l \text{ is an even integer.}$$

Using the expression of $c_l(\alpha, \nu)$ in terms of Γ functions, and the property

$$\Gamma(z+n) = z(z+1)(z+2)\cdots(z+n-1)\Gamma(z) \quad \forall\, n > 0$$

of the Γ function, we obtain:

- $c'_{-\frac{1}{2}+2m}(\alpha,\nu) = (-1)^m \frac{(\frac{1}{2}-\lambda)(\frac{5}{2}-\lambda)\cdots(2m-\frac{3}{2}-\lambda)}{(\frac{1}{2}+\lambda)(\frac{5}{2}+\lambda)\cdots(2m-\frac{3}{2}+\lambda)}$
- $c'_{\frac{1}{2}+2m}(\alpha,\nu) = (-1)^m \frac{(\frac{3}{2}-\lambda)(\frac{7}{2}-\lambda)\cdots(2m-\frac{1}{2}-\lambda)}{(\frac{3}{2}+\lambda)(\frac{7}{2}+\lambda)\cdots(2m-\frac{1}{2}+\lambda)}$
- $c'_{2m+1}(\alpha,\nu) = (-1)^m \frac{(2-\lambda)(4-\lambda)\cdots(2m-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2m+\lambda)}$
- $c'_{2m}(\alpha,\nu) = (-1)^m \frac{(1-\lambda)(3-\lambda)\cdots(2m-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2m-1+\lambda)}$

for all $m > 0$, and of course $c'_{\frac{1}{2}}(\alpha,\nu) = c'_1(\alpha,\nu) = c'_0(\alpha,\nu) = 1$. As usual, $\lambda = \langle \nu, \check{\alpha}\rangle$.

¿From now on, all our intertwining operators will be normalized. With abuse of notation, we will replace $a'_\mu(w,\delta,\nu)$ by $a_\mu(w,\delta,\nu)$ and $c'_l(\alpha,\nu)$ by $c_l(\alpha,\nu)$. In particular, we will write:

$$c_l(\alpha,\nu) = \begin{cases} 1 & \text{if } l = 0, 1, \text{or } \frac{1}{2} \\ -\frac{\frac{1}{2}-\lambda}{\frac{1}{2}+\lambda} = -\frac{\frac{1}{2}-\langle \nu,\alpha^\vee\rangle}{\frac{1}{2}+\langle \nu,\alpha^\vee\rangle} & \text{if } l = \frac{3}{2} \\ -\frac{1-\lambda}{1+\lambda} = -\frac{1-\langle \nu,\alpha^\vee\rangle}{1+\langle \nu,\alpha^\vee\rangle} & \text{if } l = 2. \end{cases} \tag{5.27}$$

The action of the operator

$$a_\mu(s_\alpha,\delta,\nu)\colon\ \mathrm{Hom}_M(\mu,\delta) \to \mathrm{Hom}_M(\mu, s_\alpha\delta)$$

is now completely understood. To conclude the subsection, we specify the parity of the indices l appearing in the decomposition

$$\mathrm{Hom}_M(\mu,\delta) = \bigoplus_{l\in\mathbb{N}/2} \mathrm{Hom}_M(\varphi_l^\alpha + \varphi_{-l}^\alpha, \delta).$$

The Lie algebra of K^α is generated by the element

$$Z_\alpha := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Because $K^\alpha = \exp(i\mathbb{R}Z_\alpha)$, we can identify the space ϕ_l^α (defined to be the isotypic component of the character l of K^α inside μ) with the l-generalized eigenspace of $d\mu(iZ_\alpha)$. We note that the element $m_\alpha = \exp(i\pi Z_\alpha)$ has order 2 if α is not metaplectic, and has order 4 otherwise. This condition imposes strong restrictions on the possible eigenvalues of $d\mu(iZ_\alpha)$.
If α is not metaplectic, then $d\mu(iZ_\alpha)$ has integer eigenvalues for every K-type μ. If α is metaplectic, then the eigenvalues of $d\mu(iZ_\alpha)$ are integers if the K-type μ is non-genuine (i.e. if $\mu(-I)$ is trivial), and half-integers if μ is genuine. Therefore:

- if μ is genuine and α is metaplectic, then every index l appearing in the decomposition
$$\mathrm{Hom}_M(\mu,\delta) = \bigoplus_{l\in\mathbb{N}/2} \mathrm{Hom}_M(\varphi_l^\alpha + \varphi_{-l}^\alpha, \delta)$$
belongs to $\mathbb{Z}+\frac{1}{2}$;
- if μ is not genuine, or if μ is genuine and α is not metaplectic, then every index l belongs to $\mathbb{Z}$. Precisely, l is an odd integer if $\delta(m_\alpha) = -I$ and it is an even integer otherwise.

Finally, we observe that the action of Z_α on E_μ extends to an action of Z_α^2 on the space $\mathrm{Hom}_M(\mu,\delta)$, because Z_α^2 commutes with M. The action is given by:
$$T \mapsto T \circ (d\mu(Z_\alpha))^2 .$$
So we can think of
$$\mathrm{Hom}_M(\mu,\delta) = \bigoplus_{l\in\mathbb{N}/2} \mathrm{Hom}_M(\varphi_l^\alpha + \varphi_{-l}^\alpha, \delta)$$
as the decomposition of $\mathrm{Hom}_M(\mu,\delta)$ into generalized eigenspaces for Z_α^2.

5.2.2. *Unitarity of Langlands quotients.* We restrict our attention to representations with real infinitesimal character, so we assume ν to be real.
Suppose that ν is strictly dominant for the positive root system determined by N. Then the principal series $I_P(\delta,\nu)$ has a unique irreducible Langlands quotient $\bar{X}_P(\delta,\nu)$. The purpose of this section is to discuss the unitarity of $\bar{X}_P(\delta,\nu)$.

By work of Knapp and Zuckermann, $\bar{X}_P(\delta,\nu)$ is Hermitian if and only if the long Weyl group element w_0 satisfies
$$w_0\delta \simeq \delta \quad \text{and} \quad w_0\nu = -\nu.$$
We will assume that the above conditions are met. Then the normalized operator $\mathcal{A}_P(w_0,\delta,\nu)$ (introduced in the previous section) is defined by a converging integral. Let $\delta(w_0)$ be any intertwining operator displaying $w_0\delta$ as equivalent to δ. The composition
$$L(w_0,\delta,\nu) := \delta(w_0)\mathcal{A}_P(w_0,\delta,\nu)\colon I_P(\delta,\nu) \to I_P(\delta,-\nu) \tag{5.28}$$
is Hermitian, and induces a G-invariant form $\langle\, , \rangle$ on $\mathcal{H}_P(\delta,\nu)$ by
$$\langle f, g\rangle = (L(w_0,\delta,\nu)f, g)_{L^2(K)} \qquad \forall\, f,g \in \mathcal{H}_P(\delta,\nu). \tag{5.29}$$
Here
$$(f,g)_{L^2(K)} = \int_K f(k)\overline{g(k)}dk. \tag{5.30}$$
The form $\langle\, , \rangle$ descends to a nondegenerate G-invariant form on the quotient of $\mathcal{H}_P(\delta,\nu)$ by the kernel of the operator $\mathcal{A}_P(w_0,\delta,\nu)$, which is isomorphic to $\bar{X}_P(\delta,\nu)$.

Note that the irreducible Hermitian representation $\bar{X}_P(\delta,\nu)$ admits a unique nondegenerate invariant form (up to a constant); hence proving that $\bar{X}_P(\delta,\nu)$ is nonunitary amounts to showing that the form $\langle\,,\rangle$ is indefinite.

We begin by making a convenient choice of $\delta(w_0)$. Let $(\mu_\delta, E_{\mu_\delta})$ be a fine K-type containing δ. Because μ_δ contains δ with multiplicity one, we can canonically identify δ with its unique copy inside μ_δ. Then V^δ is identified with the isotypic component $E_{\mu_\delta}(\delta)$ of δ in E_{μ_δ}. (Recall that V^δ is the representation space for δ.) We define two actions of M on V^δ: one is the restriction of μ_δ to M (identified with δ), the other is by $w_0\cdot\delta$. The operator $\mu_\delta(w_0)$ maps $V^\delta = E_{\mu_\delta}(\delta)$ into itself (because $w_0\cdot\delta\simeq\delta$), and intertwines the two actions. Hence we can choose $\delta(w_0)=\mu_\delta(w_0)$.

This gives:

$$L(w_0,\delta,\nu) = \mu_\delta(w_0)\mathcal{A}_P(w_0,\delta,\nu). \tag{5.31}$$

By Frobenius reciprocity, $L(w_0,\delta,\nu)$ gives rise to an operator

$$l_\mu(w_0,\delta,\nu)\colon \operatorname{Hom}_M(\mu,\delta)\to\operatorname{Hom}_M(\mu,\delta) \tag{5.32}$$

for every K-type μ appearing in the principal series. The operator $l_\mu(w_0,\delta,\nu) := \mu_\delta(w_0)a_\mu(w_0,\delta,\nu)$ carries all the signature information on the K-type μ, and is zero if μ does not appear in the quotient $\bar{X}(\delta,\nu)$.

Next, we give a factorization of $l_\mu(w_0,\delta,\nu)$ as a product of operators corresponding to simple root reflections. Recall that if

$$w_0 = s_{\alpha_r}s_{\alpha_{r-1}}\dots s_{\alpha_1}$$

is a *minimal* decomposition of w_0 as a product of simple reflections, then the operator $a_\mu(w_0,\delta,\nu)$ factors as

$$a_\mu(w_0,\delta,\nu)=\prod_{j=1}^{r} a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})$$

with $x_0=1$, $\delta_0=\delta=x_0\cdot\delta$, $\nu_0=\nu=x_0\cdot\nu$, and

$$\delta_j=\underbrace{s_{\alpha_j}s_{\alpha_{j-1}}\dots s_{\alpha_1}}_{x_j}\cdot\delta = x_j\cdot\delta \qquad \nu_j=\underbrace{s_{\alpha_j}s_{\alpha_{j-1}}\dots s_{\alpha_1}}_{x_j}\cdot\nu = x_j\cdot\nu$$

for $j\geq 1$. So we can write:

$$\begin{aligned} l_\mu(w_0,\delta,\nu) = \mu_\delta(w_0)a_\mu(w_0,\delta,\nu) = \mu_\delta(w_0)\left[\prod_{j=1}^{r} a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})\right] = \\ = \mu_\delta(x_r)\left[\prod_{j=1}^{r} a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})\right]\mu_\delta(x_0)^{-1} = \\ = \prod_{j=1}^{r}\left[\mu_\delta(x_j)a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})\mu_\delta(x_{j-1})^{-1}\right] = \\ = \prod_{j=1}^{r}\left[\mu_\delta(\sigma_{\alpha_j})a_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})\right] = \prod_{j=1}^{r} l_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1}) \end{aligned}$$

The operator $a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})$ carries $\operatorname{Hom}_M(\mu,\delta_{j-1})$ into $\operatorname{Hom}_M(\mu,\delta_j)$. For all $k=1\dots r$, the M-type $\delta_k=x_k\delta$ is the representation

$$\delta_k(m)v=\delta(\sigma_k^{-1}m\sigma_k)v$$

of M on the space $V^\delta = E_{\mu_\delta}(\delta)$. Here σ_k is a representative in K for the root reflection x_k. To obtain a more natural realization of the representation δ_k, we will replace δ_k by its unique copy inside μ_δ.

The fine K-type μ_δ contains every M-type in the Weyl group orbit of δ with multiplicity one. In particular, μ_δ contains a unique representation ρ_k isomorphic to δ_k. The representation space for ρ_k is the isotypic component of δ_k in E_{μ_δ}, denoted by $E_{\mu_\delta}(\delta_k)$; the action of M on $E_{\mu_\delta}(\delta_k)$ is given by (the restriction of) μ_δ. The map $\mu_\delta(x_k)$ carries $V^\delta = E_{\mu_\delta}(\delta)$ into $E_{\mu_\delta}(x_k\delta) = E_{\mu_\delta}(\delta_k)$, and intertwines the δ_k-action of M on V^δ with the ρ_k-action of M on $E_{\mu_\delta}(\delta_k)$. Applying $\mu_\delta(x_k)$ to the range, we obtain an isomorphism

$$\mu_\delta(x_k)\colon \operatorname{Hom}_M(\mu,\delta_k) \to \operatorname{Hom}_M(\mu,\rho_k), \tag{5.33}$$

for all $k = 1 \dots n$. Let us return to our factorization. Recall that

$$l_\mu(w_0,\delta,\nu) = \prod_{j=1}^{r} \left[\mu_\delta(x_j) a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1}) \mu_\delta(x_{j-1})^{-1}\right].$$

For all $j = 1 \dots r$, there is a commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_M(\mu,\delta_{j-1}) \arrow[r, "a_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})"'] & \operatorname{Hom}_M(\mu,\delta_j) \arrow[d, "\mu_\delta(x_j)"] \\
\operatorname{Hom}_M(\mu,\rho_{j-1}) \arrow[u, "\mu_\delta(x_{j-1})^{-1}"] \arrow[r, "\mu_\delta(\sigma_{\alpha_j})a_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})"] & \operatorname{Hom}_M(\mu,\rho_j)
\end{tikzcd}$$

hence we can write

$$l_\mu(w_0,\delta,\nu) = \prod_{j=1}^{r} \left[\mu_\delta(\sigma_{\alpha_j}) a_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})\right] = \prod_{j=1}^{r} l_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1}). \tag{5.34}$$

For all $j = 1 \dots r$, we set $l_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1}) := \mu_\delta(\sigma_{\alpha_j}) a_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})$, and we regard $l_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})$ as an operator acting on $\operatorname{Hom}_M(E_\mu, E_{\mu_\delta})$.

The operators $l_\mu(w_0,\delta_k,\nu)$ coming from characters δ_k in the Weyl group orbit of δ admit a similar decomposition. Notice that

- We can combine the various operators $l_\mu(w_0,\delta_k,\nu)$ to get an operator

$$l_\mu(w_0,\nu)\colon \operatorname{Hom}_M\left(E_\mu, E_{\mu_\delta}\right) \to \operatorname{Hom}_M\left(E_\mu, E_{\mu_\delta}\right). \tag{5.35}$$

- The operator $l_\mu(w_0,\nu)$ factors as a product of operators corresponding to simple reflections:

$$l_\mu(w_0,\nu) = \prod_{j=1}^{r} l_\mu(s_{\alpha_j},\nu_{j-1}). \tag{5.36}$$

Each factor can be interpreted as an operator acting on $\operatorname{Hom}_M(E_\mu, E_{\mu_\delta})$, and is easy to compute: write

$$\operatorname{Hom}_M(E_\mu, E_{\mu_\delta}) = \bigoplus_{l\in\mathbb{N}/2} E(-l^2)$$

for the decomposition of $\mathrm{Hom}_M(E_\mu, E_{\mu_\delta})$ into generalized eigenspaces for the action of $Z^2_{\alpha_j}$ (by $T \mapsto T \circ \mathrm{d}\mu(Z_{\alpha_j})^2$). The factor $l_\mu(s_{\alpha_j}, \nu_{j-1})$ acts on $E(-l^2)$ by

$$T \mapsto c_l\, \mu_\delta(\sigma_{\alpha_j}) T \mu(\sigma_{\alpha_j}^{-1}),$$

where $c_l = c_l(\alpha_j, \nu_{j-1})$ are the scalars introduced in the previous section.

The operator $l_\mu(w_0, \nu)$ carries all the signature information for the Hermitian form $\langle , \rangle$ on the μ-isotypic subspace of the principal series representation $I_P(\delta_k, \nu)$, for every M-type δ_k occurring in μ_δ which is fixed by w_0.

We will compute the operator $l_\mu(w_0, \nu)$ for a particular K-type μ of $Mp(4)$, and show that it is not positive semidefinite for one choice of δ. This will prove that the nondegenerate G-invariant Hermitian form on $\overline{X}(\delta, \nu)$ is indefinite, hence the representation is nonunitary.

5.3. The Groups G^α for $Mp(4)$. Let

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm e_1 \pm e_2,\ \pm 2e_1,\ \pm 2e_2\} \tag{5.37}$$

be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Recall that for each root $\alpha \in \Delta$ we can define a map

$$\phi_\alpha \colon \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}_0 \tag{5.38}$$

as in (4.3.5) of [**16**]. The image is a subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Let G_α be the corresponding connected subgroup of $Mp(4)$. Then G_α is isomorphic to either $SL(2, \mathbb{R})$ or $Mp(2)$ (since G_α is the identity component of the inverse image under pr of the corresponding subgroup of $Sp(4, \mathbb{R})$). The map ϕ_α exponentiates to a map

$$\Phi_\alpha \colon Mp(2) \to G \tag{5.39}$$

with image G_α. We define

$$Z_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}_0 \tag{5.40}$$

$$\sigma_\alpha = \exp\left(\frac{\pi}{2} Z_\alpha\right) \in K \quad \text{and} \tag{5.41}$$

$$m_\alpha = \exp(\pi Z_\alpha) = \sigma_\alpha^2 \in M. \tag{5.42}$$

Note that σ_α is a representative in $Mp(4)$ for the root reflection s_α; its projection is a representative in $Sp(4)$ for s_α:

$$pr(\sigma_\alpha) = \Phi_{\alpha,L} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with $\Phi_{\alpha,L}$ the corresponding map for the linear groups.

We now write down the maps ϕ_α and the elements Z_α, σ_α and m_α for $Mp(4)$. For $\alpha = 2e_1$,

$$\phi_{2e_1}\begin{pmatrix} u & v \\ w & -u \end{pmatrix} = \left(\begin{array}{cc|cc} u & 0 & v & 0 \\ 0 & 0 & 0 & 0 \\ \hline w & 0 & -u & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{5.43}$$

$$Z_{2e_1} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left[\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \frac{i}{2}\right] \tag{5.44}$$

$$\sigma_{2e_1} = \exp\left[\begin{pmatrix} \frac{\pi}{2}i & 0 \\ 0 & 0 \end{pmatrix}, \frac{\pi}{4}i\right] = \left[\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, e^{\frac{\pi}{4}i}\right] \tag{5.45}$$

$$m_{2e_1} = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, i\right] = a. \tag{5.46}$$

For $\alpha = 2e_2$,

$$\phi_{2e_2}\begin{pmatrix} u & v \\ w & -u \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & u & 0 & v \\ \hline 0 & 0 & 0 & 0 \\ 0 & w & 0 & -u \end{array}\right) \tag{5.47}$$

$$Z_{2e_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) = \left[\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \frac{i}{2}\right] \tag{5.48}$$

$$\sigma_{2e_2} = \exp\left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\pi}{2}i \end{pmatrix}, \frac{\pi}{4}i\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, e^{\frac{\pi}{4}i}\right] \tag{5.49}$$

$$m_{2e_2} = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, i\right] = b. \tag{5.50}$$

For $\alpha = e_1 - e_2$,

$$\phi_{e_1-e_2}\begin{pmatrix} u & v \\ w & -u \end{pmatrix} = \left(\begin{array}{cc|cc} u & v & 0 & 0 \\ w & -u & 0 & 0 \\ \hline 0 & 0 & -u & -w \\ 0 & 0 & -v & u \end{array}\right) \tag{5.51}$$

$$Z_{e_1-e_2} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right) = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right] \tag{5.52}$$

$$\sigma_{e_1-e_2} = \exp\left[\begin{pmatrix} 0 & \frac{\pi}{2} \\ -\frac{\pi}{2} & 0 \end{pmatrix}, 0\right] = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right] \tag{5.53}$$

$$m_{e_1-e_2} = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1\right] = y. \tag{5.54}$$

Finally, for $\alpha = e_1 + e_2$,

$$\phi_{e_1+e_2}\begin{pmatrix} u & v \\ w & -u \end{pmatrix} = \left(\begin{array}{cc|cc} u & 0 & 0 & v \\ 0 & u & v & 0 \\ \hline 0 & w & -u & 0 \\ w & 0 & 0 & -u \end{array}\right) \tag{5.55}$$

$$Z_{e_1+e_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right) = \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 0\right] \tag{5.56}$$

$$\sigma_{e_1+e_2} = \exp\left[\begin{pmatrix} 0 & \frac{\pi}{2}i \\ \frac{\pi}{2}i & 0 \end{pmatrix}, 0\right] = \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 1\right] \tag{5.57}$$

$$m_{e_1+e_2} = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1\right] = y. \tag{5.58}$$

For any root α, there is a surjective map

$$pr_\alpha \colon G_\alpha \to SL(2, \mathbb{R}), \tag{5.59}$$

which is either injective or two-to-one, depending on whether G_α is linear or not. The element m_α belongs to the inverse image of $m = diag(-1, -1)$ under this map, so $pr_\alpha(m_\alpha^2) = 1$. Note that m_α has order two if pr_α is injective, and has order four otherwise. Therefore, the order of m_α determines whether G_α is isomorphic to $SL(2)$ or $Mp(2)$. We find:

$$G_{2e_1} \simeq G_{2e_2} \simeq Mp(2) \tag{5.60}$$

and

$$G_{e_1-e_2} \simeq G_{e_1+e_2} \simeq SL(2). \tag{5.61}$$

We choose

$$\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{2e_1,\, 2e_2,\, e_1 \pm e_2\} \tag{5.62}$$

so that the simple roots are

$$\alpha = e_1 - e_2 \quad \text{and} \quad \beta = 2e_2. \tag{5.63}$$

Then

$$w_0 = s_\alpha s_\beta s_\alpha s_\beta \tag{5.64}$$

is a minimal decomposition of the long Weyl group element w_0 as a product of simple reflections.

5.4. The Operator on the K-Type $\mu = \left(\frac{3}{2}, \frac{1}{2}\right)$. We will compute the operator $l(w_0, \nu)$ for the K-type $\mu = \left(\frac{3}{2}, \frac{1}{2}\right)$. First, we need to understand the structure of μ_δ and μ, and their restriction to M.

5.4.1. *The K-Type μ_δ.* The K-type μ_δ has highest weight

$$\left(\frac{1}{2}, -\frac{1}{2}\right) = (1, 0) + \left(-\frac{1}{2}, -\frac{1}{2}\right)$$

and is isomorphic to $\mathbb{C}^2 \otimes \det^{-\frac{1}{2}}$. Here $\mathbb{C}^2$ denotes the standard representation of $U(2)$. So an element $(g, z) \in \widetilde{U}(2)$ acts on a vector $v \in E_{\mu_\delta} \simeq \mathbb{C}^2$ by

$$\mu_\delta\,(g, z)\, v := z^{-1}\, gv.$$

We choose the basis $\{b_1, b_2\}$ of E_{μ_δ}, with $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$\mu_\delta(a) = \mu_\delta\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, i\right] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \text{ and} \tag{5.65}$$

$$\mu_\delta(b) = \mu_\delta\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, i\right] = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{5.66}$$

Because the elements a and b generate M, the subspaces $\langle b_1\rangle$ and $\langle b_2\rangle$ are clearly stable under the action of M. It is easy to see that

$$\langle b_1\rangle \simeq \delta_1 \quad \text{and} \quad \langle b_2\rangle \simeq \delta_2.$$

The matrices of $\mu_\delta(\sigma_\alpha)$ and $\mu_\delta(\sigma_\beta)$ with respect to the basis $\{b_1, b_2\}$ are as follows.

$$\mu_\delta(\sigma_\alpha) = \mu_\delta\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.67}$$

$$\mu_\delta(\sigma_\beta) = \mu_\delta\left[\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, e^{\frac{\pi i}{4}}\right] = \begin{pmatrix} e^{-\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{\pi i}{4}} \end{pmatrix}. \tag{5.68}$$

5.4.2. *The K-type μ.* The K-type μ has highest weight

$$\left(\frac{3}{2}, \frac{1}{2}\right) = (1,0) + \left(\frac{1}{2}, \frac{1}{2}\right)$$

and is isomorphic to $\mathbb{C}^2 \otimes \det^{\frac{1}{2}}$. So an element $(g,z) \in \widetilde{U}(2)$ acts on a vector $v \in E_\mu \simeq \mathbb{C}^2$ by

$$\mu(g,z)\, v := z\, gv.$$

The differentiated action is

$$\mathrm{d}\mu(Z,z)\, v = (Z + zI)\, v,$$

with I the identity matrix.

We choose the basis $\{f_1, f_2\}$ of E_μ, with $f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$\mu(a) = \mu\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, i\right] = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \text{ and} \tag{5.69}$$

$$\mu(b) = \mu\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, i\right] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{5.70}$$

The subspaces $\langle f_1\rangle$ and $\langle f_2\rangle$ are M-invariant, and

$$\langle f_1\rangle \simeq \delta_2 \quad \text{and} \quad \langle f_2\rangle \simeq \delta_1.$$

We give the matrices of $\mu\left(\sigma_\alpha^{-1}\right)$, $\mathrm{d}\mu(Z_\alpha)$, $\mathrm{d}\mu(Z_\alpha)^2$ and $\mu\left(\sigma_\beta^{-1}\right)$, $\mathrm{d}\mu(Z_\beta)$ and $\mathrm{d}\mu(Z_\beta)^2$ with respect to the basis $\{f_1, f_2\}$.

$$\mu\left(\sigma_\alpha^{-1}\right) = \mu\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.71}$$

$$\mathrm{d}\mu(Z_\alpha) = d\mu\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.72}$$

$$\mathrm{d}\mu(Z_\alpha)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.73}$$

and

$$\mu\left(\sigma_\beta^{-1}\right)=\mu\left[\left(\begin{array}{cc}1 & 0\\ 0 & -i\end{array}\right), e^{-\frac{\pi i}{4}}\right]=\left(\begin{array}{cc}e^{-\frac{\pi i}{4}} & 0\\ 0 & e^{-\frac{3\pi i}{4}}\end{array}\right) \tag{5.74}$$

$$\mathrm{d}\mu\left(Z_\beta\right)=d\mu\left[\left(\begin{array}{cc}0 & 0\\ 0 & i\end{array}\right), \frac{i}{2}\right]=\left(\begin{array}{cc}\frac{i}{2} & 0\\ o & \frac{3i}{2}\end{array}\right) \tag{5.75}$$

$$\mathrm{d}\mu\left(Z_\beta\right)^2=\left(\begin{array}{cc}-\frac{1}{4} & 0\\ 0 & -\frac{9}{4}\end{array}\right). \tag{5.76}$$

5.4.3. *The Operator $l_\mu(w_0,\nu)$ on* $\mathrm{Hom}_M(E_\mu, E_{\mu_\delta})$. We choose the basis

$$\begin{array}{ll} T_1(f_1)=0 & T_2(f_1)=b_2\\ T_1(f_2)=b_1 & T_2(f_2)=0 \end{array}$$

of the space $E := \mathrm{Hom}_M\left(V^\mu, V^{\mu_\delta}\right)$.
Note that $T_1 \in \mathrm{Hom}_M\left(V^\mu, E_{\mu_\delta}(\delta_1)\right)$, and $T_2 \in \mathrm{Hom}_M\left(V^\mu, E_{\mu_\delta}(\delta_2)\right)$.

We compute the operator

$$l_\mu(s_\gamma,\lambda)\colon E \to E$$

associated to each simple reflection γ. Recall the recipe:

- Decompose E into eigenspaces for the action of $\mathrm{d}\mu(Z_\gamma)^2$: $E=\bigoplus_{l\in\mathbb{N}/2} E(-l^2)$, with
$$E(-l^2) := \{T\in E\colon T\circ \mathrm{d}\mu(Z_\gamma)^2=(-l^2)T\}$$
- The operator $l_\mu(s_\gamma,\lambda)$ acts on an element $T\in E(-l^2)$ by
$$l_\mu(s_\gamma,\lambda)T=c_l(\gamma,\lambda)\,\mu_\delta(\sigma_\gamma)T\mu(\sigma_\gamma)^{-1},$$
with
$$c_l(\gamma,\lambda)=\begin{cases}1 & \text{if } l=0,1,\text{or } \frac{1}{2}\\ -\frac{\frac{1}{2}-\langle\lambda,\gamma^\vee\rangle}{\frac{1}{2}+\langle\lambda,\gamma^\vee\rangle} & \text{if } l=\frac{3}{2}\\ -\frac{1-\langle\lambda,\gamma^\vee\rangle}{1+\langle\lambda,\gamma^\vee\rangle} & \text{if } l=2\end{cases} \tag{5.77}$$
(for our purpose we do not need the constants c_l for other values of l).
For brevity of notation, we write: $\mu_\delta(\sigma_\gamma)T\mu(\sigma_\gamma)^{-1} := \psi_\mu(s_\gamma)T$.

We start by computing the operator $l_\mu(s_\alpha,\lambda)$ (for $\alpha=e_1-e_2$).
Recall that $\mathrm{d}\mu(Z_\alpha^2)=-I$, so for all $T\in E$ and all $v\in E_\mu$, we have

$$T\left(\mathrm{d}\mu(Z_\alpha)^2 v\right)=-T(v).$$

Hence $E=E_{-1}$ and $l_\mu(s_\alpha,\lambda)\equiv\psi_\mu\left(s_\alpha\right)$.
For all $v=a_1f_1+a_2f_2\in E_\mu$, we compute:

$$\psi_\mu\left(s_\alpha\right)T_1\left(v\right)=\mu_\delta(\sigma_\alpha)T_1\left(\mu\left(\sigma_\alpha^{-1}\right)v\right)=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)T_1\left(\left(\begin{array}{cc}0 & -1\\ 1 & 0\end{array}\right)\begin{pmatrix}a_1\\ a_2\end{pmatrix}\right)=$$
$$=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)T_1\begin{pmatrix}-a_2\\ a_1\end{pmatrix}=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)\begin{pmatrix}a_1\\ 0\end{pmatrix}=\begin{pmatrix}0\\ -a_1\end{pmatrix}=-T_2(v).$$

Similarly,

$$\psi_\mu\left(s_\alpha\right)T_2\left(v\right)=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)T_2\left(\left(\begin{array}{cc}0 & -1\\ 1 & 0\end{array}\right)\left(\begin{array}{c}a_1\\ a_2\end{array}\right)\right)=$$
$$=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)T_2\begin{pmatrix}-a_2\\ a_1\end{pmatrix}=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)\begin{pmatrix}0\\ -a_2\end{pmatrix}=\begin{pmatrix}-a_2\\ 0\end{pmatrix}=-T_1(v).$$

Then

$$l_\mu(s_\alpha,\lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{5.78}$$

Now we do the corresponding computations for $\beta = 2e_2$.
For all $j = 1,2$ and all $v \in E_\mu$, we have:

$$T_j\left(\mathrm{d}\mu(Z_\beta)^2 v\right) = T_j\left(\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{9}{4} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = T_j\begin{pmatrix} -\frac{1}{4}a_1 \\ \\ -\frac{9}{4}a_2 \end{pmatrix} = \begin{cases} -\frac{9}{4}T_j & \text{if } j=1 \\ \\ -\frac{1}{4}T_j & \text{if } j=2 \end{cases}$$

so $E \simeq E\left(-\frac{1}{4}\right) \oplus E\left(-\frac{9}{4}\right)$, with

$$E\left(-\frac{1}{4}\right) = \langle T_2\rangle \quad \text{and} \quad E\left(-\frac{9}{4}\right) = \langle T_1\rangle.$$

It follows that

$$l_\mu(s_\beta,\lambda)T_j = \begin{cases} c_{3/2}(\beta,\lambda)\,\psi_\mu(s_\beta)T_j & \text{if } j=1 \\ \\ \psi_\mu(s_\beta)T_j & \text{if } j=2. \end{cases}$$

Recall that $c_{3/2}(\beta,\lambda) = -\frac{\frac{1}{2}-\langle\lambda,\beta^\vee\rangle}{\frac{1}{2}+\langle\lambda,\beta^\vee\rangle}$. We now compute $\psi_\mu(s_\beta)T_j$, for $j = 1,2$.

$$\begin{aligned} \psi_\mu(s_\beta)\,T_j(v) &= \mu_\delta(\sigma_\beta)T_j\left(\mu\left(\sigma_\beta^{-1}\right)v\right) \\ &= \begin{pmatrix} e^{-\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{\pi i}{4}} \end{pmatrix} T_j\left(\begin{pmatrix} e^{-\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{3\pi i}{4}} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} e^{-\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{\pi i}{4}} \end{pmatrix} T_j\begin{pmatrix} e^{-\frac{\pi i}{4}}a_1 \\ e^{-\frac{3\pi i}{4}}a_2 \end{pmatrix} \\ &= \begin{cases} -T_j(v) & \text{if } j=1 \\ +T_j(v) & \text{if } j=2. \end{cases} \end{aligned}$$

Hence we get:

$$l_\mu(s_\beta,\lambda)T_j = \begin{cases} +\frac{\frac{1}{2}-\langle\lambda,\beta^\vee\rangle}{\frac{1}{2}+\langle\lambda,\beta^\vee\rangle}\,T_j & \text{if } j=1 \\ \\ T_j & \text{if } j=2. \end{cases}$$

Equivalently,

$$l_\mu(s_\beta,\lambda) = \begin{pmatrix} \frac{\frac{1}{2}-\langle\lambda,\beta^\vee\rangle}{\frac{1}{2}+\langle\lambda,\beta^\vee\rangle} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.79}$$

We are now ready to compute the full intertwining operator:

$$l_\mu(w_0,\nu) = l_\mu(s_\alpha, s_\beta s_\alpha s_\beta\nu)\,l_\mu(s_\beta, s_\alpha s_\beta\nu)\,l_\mu(s_\alpha, s_\beta\nu)\,l_\mu(s_\beta,\nu)\,.$$

Write $\nu = (\nu_1,\nu_2)$, with $\nu_1 \geq \nu_2 \geq 0$. Then equations (5.78) and (5.79) give:

- $l_\mu(s_\alpha, s_\beta s_\alpha s_\beta\nu) = l_\mu(s_\alpha, s_\beta\nu) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

- $l_\mu(s_\beta,\nu) = \begin{pmatrix} \frac{\frac{1}{2}-\langle \nu,\beta^\vee\rangle}{\frac{1}{2}+\langle \nu,\beta^\vee\rangle} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\frac{1}{2}-\nu_2}{\frac{1}{2}+\nu_2} & 0 \\ 0 & 1 \end{pmatrix}$

- $l_\mu(s_\beta, s_\alpha s_\beta\nu) = \begin{pmatrix} \frac{\frac{1}{2}-\langle s_\alpha s_\beta\nu,\beta^\vee\rangle}{\frac{1}{2}+\langle s_\alpha s_\beta\nu,\beta^\vee\rangle} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\frac{1}{2}-\nu_1}{\frac{1}{2}+\nu_1} & 0 \\ 0 & 1 \end{pmatrix}$

and we obtain:

$$l_\mu(w_0,\nu) = \begin{pmatrix} \frac{\frac{1}{2}-\nu_2}{\frac{1}{2}+\nu_2} & 0 \\ 0 & \frac{\frac{1}{2}-\nu_1}{\frac{1}{2}+\nu_1} \end{pmatrix}. \tag{5.80}$$

For all $i = 1, 2$, let $\bar{X}(\delta_i,\nu)$ be the irreducible constituent of the principal series $I_P(\delta_i,\nu)$ containing the lowest K-type $\mu_\delta = \left(\frac{1}{2}, -\frac{1}{2}\right)$. The operator $l_\mu(w_0,\nu)$ carries the signature information on the K-type $\mu = \left(\frac{3}{2}, \frac{1}{2}\right)$ for *both* $\bar{X}(\delta_1,\nu)$ and $\bar{X}(\delta_2,\nu)$, and can be interpreted as the direct sum of the operators

$$l_\mu(w_0,\delta_1,\nu)\colon \operatorname{Hom}_M(\nu,\delta_1) \to \operatorname{Hom}_M(\nu,\delta_1)$$

and

$$l_\mu(w_0,\delta_2,\nu)\colon \operatorname{Hom}_M(\nu,\delta_2) \to \operatorname{Hom}_M(\nu,\delta_2).$$

Recall that $l_{\mu_\delta}(w_0,\delta_i,\nu) = 1$ for both $i = 1, 2$, because of our normalization. We can sometimes use the operator $l_\mu(w_0,\nu)$ to detect the nonunitarity of $\bar{X}(\delta_i,\nu)$:

REMARK 5.1. If the i^{th}-entry of $l_\mu(w_0,\nu)$ is negative, then the nondegenerate Hermitian form on $\bar{X}(\delta_i,\nu)$ is indefinite, and the representation is nonunitary.
If the i^{th}-entry of $l_\mu(w_0,\nu)$ is zero, then the K-type μ does not appear in $\bar{X}(\delta_i,\nu)$.
If the i^{th}-entry of $l_\mu(w_0,\nu)$ is positive, then we cannot draw any conclusion regarding the unitarity of $\bar{X}(\delta_i,\nu)$.

We are interested in the Mystery representation, that has infinitesimal character $\nu = \left(\frac{3}{2}, \frac{1}{2}\right)$. In this case

$$l_\mu(w_0,\nu) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

hence the previous remark implies that $\overline{X}(\delta_2,\nu)$ is nonunitary.

THEOREM 5.2. *The Mystery representation of $Mp(4)$ is nonunitary.*

Consequently, we obtain the following result:

THEOREM 5.3. *Let X be an irreducible admissible representation of $Mp(4)$. Then X is unitary and ω-regular if and only if X is either an $A_{\mathfrak{q}}(\Omega)$ or a Meta-$A_{\mathfrak{q}}(\lambda)$ representation.*

6. Appendix

6.1. Lie Groups and Representations. Consider figures which have continuous sets of symmetries, such as the circle or the sphere. For any small number x, the rotation $k_x = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$ moves the points on the circle by an angle x. We get a one-dimensional set S^1 of rotations that leave the circle looking the same. Similarly, we can draw an imaginary axis of rotation from any point on the sphere to its center. That gives us two dimensions of choices for the axis of rotation plus one dimension for the angle of rotation around the chosen axis; so the rotations of a sphere form a three-dimensional set S^2.

A Lie Group is a continuous group of symmetries. More precisely, it is a C^∞ manifold with a smooth group structure. The sets S^1 and S^2 of rotations of the circle and the sphere are examples of one- and three-dimensional Lie groups. If V is an n-dimensional vector space over a field $\mathbb{F}$, then the set $Aut\,(V) \simeq GL\,(n, \mathbb{F})$ is a Lie group of dimension n^2.

The group S^1 acts on a one-dimensional space by scalars: $k_t u = \lambda\,(t)\,u$, where $\lambda\,(t)$ is a complex number. S^1 also acts on the plane by multiplication:

$$k_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix}, \forall\, u, v \in \mathbb{C}$$

and on functions (e.g. L^2 functions) on the unit circle by left translation:

$$k_t f \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) = f \left(k_t^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \right), \forall\, u, v \in \mathbb{R}\ s.t.\ u^2 + v^2 = 1.$$

The latter is an infinite dimensional vector space.

Representation Theory studies all possible ways in which a Lie group acts on vector spaces. We call all these possibilities the representations of the group. More precisely, a representation of a group G is a continuous homomorphism

$$\phi \colon G \to Aut\,(V) \tag{6.1}$$

for some vector space V. In case all group elements act by the identity operator, we call the representation trivial.

The representations of the rotation group S^1 are related to Fourier series: all the basic (irreducible) representations are one-dimensional, of periodic type. The action is

$$\phi_n\,(k_t)\,v = e^{int} v, \forall\, v \in \mathbb{C}.$$

Applications to science abound. Examples include areas where we use infinite series of the form $f\,(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}$.

A representation of G on V is called irreducible if the vector space V does not have proper, closed subspaces which are themselves left invariant by the group. Irreducible representations are the building blocks of bigger representations.

Remark 6.1. Irreducible representations of compact groups are finite dimensional. Irreducible representations of abelian groups are one-dimensional.

Example 6.2. All irreducible representations of the diagonal group

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0,\, a \in \mathbb{R} \right\} \tag{6.2}$$

are given by

$$\phi_r \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} v = a^r v$$

for $r \in \mathbb{C}$.

Definition 6.3. A representation $\phi \colon G \to Aut\,(V)$ of G on a vector space V is unitary if V is a Hilbert space and all operators $\phi\,(g)$ are unitary operators on V (i.e. preserve the inner product on V).

Example 6.4. Square integrable functions on the circle $L^2\left(S^1\right)$ form a unitary representation of the rotation group S^1.

EXAMPLE 6.5. All irreducible (hence finite dimensional) representations of a compact group are unitary.

EXAMPLE 6.6. A representation ϕ_r of the diagonal group A is unitary if r is purely imaginary.

6.2. The group $SL(2, \mathbf{R})$. Consider the group $G = SL(2, \mathbb{R})$ of 2×2 real matrices with determinant one. In order to construct a large family of representations of $SL(2, \mathbb{R})$, we consider the following subgroups:

$$M = \{\pm I\} \tag{6.3}$$

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \tag{6.4}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} : t \in \mathbb{R} \right\} \equiv S^1$$

REMARK 6.7. The irreducible representations of M are $\{\delta_+, \delta_- : \delta_\pm(-I) = \pm 1\}$. Those of N are also one-dimensional. In our case, N will act trivially, i. e., by fixing all vectors.

6.2.1. *The Lie algebra of $SL(2, \mathbb{R})$.* The set

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\} = \mathfrak{sl}(2, \mathbb{C}) \tag{6.5}$$

is a linear vector space, called the complexified Lie algebra of $SL(2, \mathbb{R})$. The real Lie algebra $\mathfrak{g}_0$, with real entries, is the tangent space at the identity of the Lie group $SL(2, \mathbb{R})$. We will use similar notation for other Lie algebras, e.g. $\mathfrak{k}$ and $\mathfrak{k}_0$ are the complexified and the real Lie algebra of K.
The space $\mathfrak{g}$ has a (bilinear) bracket operation $[x, y] = xy - yx$ and basis elements

$$H = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; X = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \; Y = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

They satisfy:

$$\begin{array}{ll} [x, x] = 0 & [H, X] = 2X \\ [x, [y, z]] = [[x, y], z] + [y, [x, z]] & [H, Y] = -2Y \\ & [X, Y] = H. \end{array}$$

Note that if we define $\mathfrak{g}_\lambda = \{Z \in \mathfrak{g} \colon [H, Z] = \lambda Z\}$, then $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_2 + \mathfrak{g}_{-2}$.

DEFINITION 6.8. A Lie algebra representation is a homomorphism $\pi \colon \mathfrak{g} \to End(W)$ for some vector space W, preserving the bracket operations:

$$\pi\left([x, y]_{\mathfrak{g}}\right) = [\pi(x), \pi(y)]_{End(W)} = \pi(x)\pi(y) - \pi(y)\pi(x) \quad \forall x, y \in \mathfrak{g}.$$

If W is finite dimensional, then $\pi(x) : W \to W$ is a matrix for all $x \in \mathfrak{g}$.

6.2.2. *Harish-Chandra modules of $G = SL(2, \mathbb{R})$.* If $\phi\colon G \to Aut\,(V)$ is a representation of G, we can restrict the action of the group on V to any subgroup. In particular, we can restrict the representation to the maximal compact group $K = S^1$, since we know a bit more about its representations. This way we hope to be able to say something about the representation of G.

A representation ϕ of G is called admissible if, when restricted the maximal compact group K, each irreducible representation μ of K occurs with finite multiplicity. We call the sum of those copies of μ the isotypic component of μ. If ϕ is admissible, then we can find a dense subspace $W \subset V$ where ϕ is differentiable. The action of $\mathfrak{g}$ on W is defined by

$$\pi\,(Z)\,w = \frac{d}{dt}\phi\,(\exp tZ)\,|_{t=0}w \tag{6.6}$$

for all Z in $\mathfrak{g}$ and w in W.
Suppose that (ϕ, V) is an admissible representation of G. The restriction of ϕ to the compact subgroup K is a Hilbert space direct sum of isotypic components of irreducible representations of K:

$$V = \widehat{\bigoplus_{n\in\mathbb{Z}\colon V(n)\neq 0}} V(n)$$

(recall that the irreducible representations $\{\phi_n\}$ of $K = S^1$ are parameterized by integers). Set

$$W = \bigoplus_{n\in\mathbb{Z}\colon V(n)\neq 0} V(n)$$

(the algebraic direct sum of the $V(n)$'s). Then W is a dense subspace of V and ϕ is differentiable on W. Hence we have an action π of $\mathfrak{g}$ on W, as in (6.6).

Remark 6.9. For all $w \in V(n) \subset W$

$$\begin{aligned}\pi\,(iH)\,w &= \frac{d}{dt}\phi_n\left(\exp t\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)_{t=0} w \\ &= \frac{d}{dt}\left(e^{int}\right)_{t=0} w = inw.\end{aligned}$$

Hence we can identify $V(n)$ with the eigenspace for H of eigenvalue n.

The representation W is called a $(\mathfrak{g}$,$K)$ module, or a Harish-Chandra module, of G. Let H, X, and Y be as above.

Proposition 11. For any Harish-Chandra module W of $G = SL(2, \mathbb{R})$, the action of $\mathfrak{g}$ on W satisfies:

- $H \cdot V(n) \subseteq V(n)$ diagonalizable operator
- $X \cdot V(n) \subseteq V(n+2)$ "raising" operator
- $Y \cdot V(n) \subseteq V(n-2)$ "lowering" operator.

Definition 6.10. Let W be a Harish-Chandra module, and let

$$n_0 := \min_{n\in\mathbb{Z}} \{|n| \colon V(n) \neq 0\}.$$

We say that $V(n_0)$ (or $V(-n_0)$) is the lowest K-type in W. It is the lowest irreducible representation of K occurring in $W|_K$.

It turns out that we can parameterize irreducible ($\mathfrak{g}$,K) modules by their lowest K-types. First assume $n_0 \geq 2$. If W is irreducible, then either $V(-n_0) = 0$ (and $V(n_0) \neq 0$) or $V(n_0) = 0$ (and $V(-n_0) \neq 0$). In the first case, W is generated using X from the representation $V(n_0)$ of K:

$$\begin{aligned} W &\simeq V(n_0) + V(n_0+2) + V(n_0+4) + \dots \\ &= D(n_0). \end{aligned}$$

Here $D(n_0)$ is the discrete series representation with lowest K-type n_0.
Similarly, if $V(n_0) = 0$, then we can generate W from $V(-n_0)$ using Y:

$$\begin{aligned} W &\simeq V(-n_0) + V(-n_0-2) + V(-n_0-4) + \dots \\ &= D(-n_0). \end{aligned}$$

Now consider the case $n_0 = 0$ or 1. Any representation of G with lowest K-type 0 or 1 can be constructed from a representation of a parabolic subgroup $P = MAN$, as follows. Define

$$I_P(\delta,\nu) = \left\{ f\colon G \to V_{\delta,\nu}\colon f \in L^2(K),\ f(gman) = \delta\left(m^{-1}\right) e^{-(\nu-1)\log a} f(g) \right\}$$

where $V_{\delta,\nu}$ is the one-dimensional representation $\delta \otimes \nu \otimes trivial$ of P (with $\delta = \delta_\pm \in \hat{M}$, $\nu \in \mathbb{C} \simeq \hat{A}$). The group G acts on $I_P(\delta,\nu)$ by left translation:

$$\phi(x) f(g) = f(x^{-1}g) \tag{6.7}$$

for all x, $g \in G$ and $f \in I_P(\delta,\nu)$. The representation $I_P(\delta,\nu)$ is called the induced representation (or principal series) with parameters δ and ν .

Remark 6.11. The representation $I_P(\delta,\nu)$ is not necessarily irreducible; e.g., $I_P(\delta_-,0)$ is the sum of two irreducible representations, which we call $D(\pm 1)$. Moreover, every discrete series representation is a submodule of some $I_P(\delta,\nu)$.

The properties of the principal series and the discrete representations of $G = SL(2,\mathbb{R})$ are described in Table 1.

Table 1

(6.8)

Representation	Irreducible	Unitary
$D(n)\,,\ \lvert n\rvert \geq 1$	*yes*	*yes*
$I_P(\delta_\pm, iv) \simeq I_P(\delta_\pm, -iv)$, $v \neq 0$, *real*	*yes*	*yes*
$I_P(\delta_-, 0)$	*no*	*yes*
$I_P(\delta_+, 0)$	*yes*	*yes*
$I_P(\delta_+, u),\ 0 < u < 1$	*yes*	*yes*
$I_P(\delta_\varepsilon, m)$, $m \in \mathbb{Z}$, $\varepsilon = (-1)^m$	*yes*	*no*
$I_P(\delta_\varepsilon, m)$, $m \in \mathbb{Z}$, $\varepsilon = (-1)^{m+1}$	*no*	*no*
$I_P(\delta_\pm, u+iv)$, $u, v \neq 0$	*yes*	*no*

It turns out that these two constructions can be applied to any real reductive Lie group to give an exhaustive classification of its admissible representations.

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