Contemporary Mathematics

Unitarizable Minimal Principal Series of Reductive Groups

Dan Barbasch, Dan Ciubotaru, and Alessandra Pantano

To Bill Casselman and Dragan Miličić

Abstract. The aim of this paper is to give an exposition of recent progress on the determination of the unitarizable Langlands quotients of minimal principal series for reductive groups over the real or p-adic fields in characteristic 0.

Contents

1. Introduction

1.1. The aim of this paper is to give an exposition of some recent progress on the computation of the unitary dual of a reductive group over a local field of characteristic 0. The results are for Langlands quotients of minimal principal series, and form the subject of [**Ba1**], [**BP**], and [**BC1**].

Let $\mathbb{F}$ be the real field or a p-adic field of characteristic zero, and let $|\ |$ denote the absolute value, respectively the p-adic norm. When $\mathbb{F}$ is p-adic, we let

$$\mathbb{O} = \{x \in \mathbb{F} : |x| \leq 1\}, \text{ and } \mathcal{P} = \{x \in \mathbb{F} : |x| < 1\}, \tag{1.1}$$

be the ring of integers, respectively the unique prime ideal. Let $\mathbb{O}^\times = \mathbb{O} \setminus \mathcal{P}$ be the set of invertible elements in the ring of integers. We fix an uniformizer $\varpi \in \mathcal{P}$, and then $\mathcal{P} = \varpi\mathbb{O}$. The quotient $\mathbb{O}/\mathcal{P}$ is isomorphic to a finite field $\mathbb{F}_q$ of characteristic p.

Let $G(\mathbb{F})$ be the $\mathbb{F}$-points of a linear connected reductive group G defined over $\mathbb{F}$. Assume that $G(\mathbb{F})$ is split. This means that G has a Cartan subgroup H such that $H(\mathbb{F})$ is isomorphic to a product of $r := \text{rank}(G)$ copies of $\mathbb{F}^\times$. In this paper we will deal almost exclusively with split groups. Since $G(\mathbb{F})$ is split, G also has a Borel subgroup which is defined over $\mathbb{F}$. Choose such a Borel subgroup B, and a split Cartan subgroup $H \subset B$.

The reductive connected algebraic group G is determined by the root datum $(\mathcal{X}, R, \mathcal{Y}, \check{R})$, where $\mathcal{X}$ is the lattice of algebraic characters of H, $\mathcal{Y}$ is the lattice of algebraic cocharacters of H, R is the set of roots of H in G and $\check{R}$ are the coroots. Then $H(\mathbb{F}) = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{F}^{\times}$. The choice of Borel B determines the positive roots R^+ and positive coroots $\check{R}^+$. Also let $\mathfrak{h} = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{C}$ be the complex Lie algebra of $H(\mathbb{F})$.

Let ${}^{\vee}G$ be the *dual* complex connected group corresponding to $(\mathcal{Y}, \check{R}, \mathcal{X}, R)$. Denote by $\check{H} := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}^{\times}$. This is a Cartan subgroup of ${}^{\vee}G$. Its Lie algebra is $\check{\mathfrak{h}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$. The torus $\check{H}$ has a polar decomposition

$$\check{H} = \check{H}_e \cdot \check{H}_h, \text{ where } \check{H}_e = \mathcal{X} \otimes_{\mathbb{Z}} S^1 \text{ and } \check{H}_h = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R}_{>0}. \tag{1.2}$$

A semisimple element $s \in {}^{\vee}G$ is called elliptic (respectively, hyperbolic) if s is conjugate to an element of $\check{H}_e$ (respectively, $\check{H}_h$). Similarly, the Lie algebra $\check{\mathfrak{h}}$ of $\check{H}$ has a decomposition into a real (hyperbolic) part $\check{\mathfrak{h}}_{\mathbb{R}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R}$ and an imaginary (elliptic) part $\check{\mathfrak{h}}_{i\mathbb{R}} = \mathcal{X} \otimes_{\mathbb{Z}} i\mathbb{R}$.

We fix a maximal compact subgroup K in $G(\mathbb{F})$ in the real case, and $K = G(\mathbb{O})$ in the p-adic case. Set ${}^0H = H(\mathbb{F}) \cap K$. In the real case ${}^0H$ is isomorphic to $(\operatorname{rank} G)$-copies of $\mathbb{Z}/2\mathbb{Z}$. The familiar notation for ${}^0H$ in this case is M. In the p-adic case, ${}^0H$ is isomorphic with $(\operatorname{rank} G)$-copies of $\mathbb{O}^{\times}$.

For a character χ of $H(\mathbb{F})$, the *minimal principal series* $X(\chi)$ is defined to be

$$X(\chi) = \operatorname{Ind}_{B(\mathbb{F})}^{G(\mathbb{F})}(\chi \otimes 1), \tag{1.3}$$

where Ind means unitary induction. Each such principal series has finite composition series, and in particular a canonical completely reducible subquotient denoted $L(\chi)$. This paper is concerned with the problem of when the constituents of $L(\chi)$ are unitary.

A character $\chi\colon H(\mathbb{F}) \to \mathbb{C}$ is called *spherical* (or *unramified*) if its restriction to ${}^0H$ is trivial, *i.e.* ${}^0\chi := \chi|_{{}^0H} = triv$. It is called *nonspherical* (or *ramified*) otherwise.

There are two main themes in this paper: the first is to describe what is known about the unitarizability of the unramified subquotient $L(\chi)$, and the second is to relate the unitarizability of the Langlands subquotients of ramified minimal principal series $X(\chi)$ of $G(\mathbb{F})$ with the spherical unitary dual for certain groups $G({}^0\chi)$ attached to ${}^0\chi$ (see definition in 1.9).

1.2. We begin with the case when the character χ of $H(\mathbb{F})$ is unramified.

A representation (π, V) is called *spherical* if the set V^K of vectors fixed by K is nontrivial. By Frobenius reciprocity $X(\chi)^K$ is 1-dimensional when χ is unramified. Thus $X(\chi)$ is spherical, and has a unique irreducible subquotient which is spherical. In the unramified case, this is precisely the canonical subquotient $L(\chi)$ alluded to earlier.

The basic example, going back to [**Bar**] is $G = SL(2, \mathbb{R})$. Then $H(\mathbb{R}) = \mathbb{R}^{\times}$. The unramified character χ can be written as

$$\chi = \chi_\nu : H(\mathbb{R}) = \mathbb{R}^{\times} \to \mathbb{C},\ \chi_\nu(z) = |z|^\nu,\ z \in \mathbb{R}^{\times},\ \text{for some } \nu \in \mathbb{C}. \tag{1.4}$$

When ν is purely imaginary, $X(\chi_\nu)$ is irreducible (and unitary). If ν is real, then $X(\nu)$ is reducible if and only if $\langle \check{\alpha}, \nu \rangle \in 2\mathbb{Z} + 1$. When $\langle \check{\alpha}, \nu \rangle = 1$, respectively -1, then the trivial representation is a quotient, respectively a submodule of $X(\chi)$. The spherical Langlands subquotient $L(\chi_\nu)$ is unitary for

$$\begin{aligned}(1.5)\qquad &\nu \in i\mathbb{R}, \quad \text{and,}\\ &\nu \in \mathbb{R},\ \text{such that}\ -1 \le \langle \check\alpha, \nu\rangle \le 1.\end{aligned}$$

1.3. Now we consider the similar example for $SL(2,\mathbb{F})$, when $\mathbb{F}$ is p-adic. This is again well-known ([**Sa**]), and we refer the reader to chapter 9 of [**Cas2**] for a detailed treatment. The unramified characters of $\mathbb{H}(\mathbb{F}) = \mathbb{F}^\times$ can be described similarly to (1.4), but since now $|\ |$ is the p-adic norm, the actual parameterization is

(1.6)

$$\chi = \chi_\nu : H(\mathbb{F}) = \mathbb{F}^\times \to \mathbb{C},\ \chi_\nu(z) = |z|^\nu,\ z \in \mathbb{F}^\times,\ \text{for some}\ \nu \in \mathbb{C}/(2\pi i/\log q)\mathbb{Z}.$$

When ν is real, $X(\chi_\nu)$ is reducible if and only if $\langle\check\alpha,\nu\rangle = \pm 1$, which is the value of the parameter where the trivial representation is a subquotient. But when ν is purely imaginary, there are two cases one needs to consider. If $\nu \neq \pi i/\log q$, then $X(\chi_\nu)$ is irreducible (and unitary). If $\nu = \pi i/\log q$, then $X(\chi_\nu)$ decomposes into a sum of two irreducible submodules, both of which are unitary. It is worth noting that the Weyl group $W = S_2$ acts by $\nu \to -\nu$, and $\nu = \pi i/\log q$ is the only nontrivial parameter so that χ_ν is fixed by W.

The spherical Langlands subquotient $L(\chi_\nu)$ is unitary for

$$\begin{aligned}(1.7)\qquad &\nu \in i(\mathbb{R}/(2\pi/\log q)\mathbb{Z}),\ \nu \neq \pi i/\log q,\\ &\nu = \pi i/\log q, \quad \textit{and}\\ &\nu \in \mathbb{R},\ \text{such that}\ -1 \le \langle\check\alpha,\nu\rangle \le 1.\end{aligned}$$

What should be noted from equations (1.4), (1.5) and (1.6), (1.7) is that, if ν is assumed real, then the parameter sets for the spherical representations *and* the unitary sets are *the same* for both real and p-adic case. This suggests the following natural questions:

(a) Can the determination of the unitary representations be reduced to the case when the parameter is (in some technical sense) real?
(b) If the parameter is assumed real, can one generalize the identification of spherical unitary duals between the real and p-adic cases to all split groups?

When $\mathbb{F} = \mathbb{R}$, the answer to (a) is the well-known reduction to real infinitesimal character to Levi subgroups for unitary representations (see Chapter XVI in [**Kn**] for example). In the particular case of spherical representations, the reduction has a simple form. Assume $\nu = \Re\nu + i\Im\nu$ with $\Im\nu \neq 0$. $\Im\nu$ defines a proper parabolic subgroup $P_{\Im\nu} = L_{\Im\nu}U_{\Im\nu}$ of G (similarly to the construction in section 2.9). Then

$$(1.8)\qquad L(\chi_\nu) = \operatorname{Ind}_{P_{\Im\nu}}^G(L_{M_{\Im\nu}}(\chi_{\Re\nu})),$$

and since this is unitary induction, $L(\chi_\nu)$ is unitary for G if and only if $L_{M_{\Im\nu}}(\chi_{\Re\nu})$ is unitary for $M_{\Im\nu}$ (a proper Levi).

When $\mathbb{F}$ is p-adic, the situation is more complicated as already seen in the case $\nu = \pi i/\log q$ in $SL(2)$. The answer is positive nevertheless for the representations which appear in unramified principal series, and it is the subject of [**BM2**].

The answer to (b) is known to be positive, at least when G is of classical type ([**Ba1**]). But in order to explain this (and the reduction to "real infinitesimal character" in the p-adic case), we need to introduce the Iwahori-Hecke algebra and its graded versions.

1.4. Let $\mathbb{F}$ be a p-adic field. In this case, the set of spherical irreducible representations are part of a larger class, the Iwahori-spherical representations. Let $K_1 \subset K$ be the subgroup of K of elements congruent to Id modulo $\mathcal{P}$. Then there is an exact sequence

$$1 \longrightarrow K_1 \longrightarrow K \xrightarrow{\pi} K/K_1 \cong G(\mathbb{F}_q) \longrightarrow 1. \tag{1.9}$$

The group $B(\mathbb{F}_q)$ is a Borel subgroup of $G(\mathbb{F}_q)$. Then $\mathcal{I} := \pi^{-1}(B(\mathbb{F}_q))$ is an open compact subgroup of K, called an *Iwahori subgroup*. The *Iwahori-Hecke algebra*, denoted $\mathcal{H}$ is the algebra of locally constant compactly supported $\mathcal{I}$-biinvariant functions under convolution. If (π, V) has $\mathcal{I}$-fixed vectors, then $\mathcal{H}$ acts on $V^{\mathcal{I}}$. Let $C(\mathcal{I}, triv)$ be the category of admissible representations such that each subquotient is generated by its $\mathcal{I}$-fixed vectors. A central theorem in the representation theory of reductive p-adic groups is the following result of Borel and Casselman.

THEOREM 1.1 (Borel-Casselman). *The association $V \mapsto V^{\mathcal{I}}$ is an equivalence of categories between $C(\mathcal{I}, triv)$ and the category of finite dimensional representations of $\mathcal{H}$.*

1.5. The Hecke algebra $\mathcal{H}$ also has a $*$ operation $f^*(g) := \overline{f(g^{-1})}$. So it makes sense to talk about hermitian and unitary $\mathcal{H}$-modules. The next theorem transforms the analytic problem of classifying $\mathcal{I}$-spherical unitary representation of the group $G(\mathbb{F})$ ($\mathbb{F}$ p-adic), which is about infinite dimensional representations, to the algebraic problem of classifying finite dimensional unitary representations of $\mathcal{H}$.

THEOREM 1.2 ([**BM1**],[**BM2**]). *(π, V) is a unitary representation of $G(\mathbb{F})$, $\mathbb{F}$ p-adic, if and only if $V^{\mathcal{I}}$ is a unitary module of the Iwahori-Hecke algebra $\mathcal{H}$.*

The proof of this theorem requires the classification of the irreducible finite dimensional representations of $\mathcal{H}$ from the work of Kazhdan-Lusztig ([**KL**]). We give an outline of the machinery involved, more details are available in later sections. The algebra $\mathcal{H}$ has two well-known descriptions in terms of generators and relations. The original description is in terms of the affine Weyl group ([**IM**]). We will use a second one due to Bernstein, because it is better suited for our purposes (see section 3.1). Schur's lemma for $\mathcal{H}$ holds, so the center of $\mathcal{H}$ acts by scalars on any irreducible finite dimensional representation. By the aforementioned result of Bernstein, the characters of the center of $\mathcal{H}$, which we call *infinitesimal characters* in analogy with the real case, can be identified with semisimple orbits of ${}^{\vee}G$.

Therefore, they are also in 1-1 correspondence with W-orbits of elements of $\check{H}$. Suppose $s \in \check{H}$ denotes an infinitesimal character. Decompose $s = s_e \cdot s_h$, where s_e is elliptic, and s_h hyperbolic.

DEFINITION 1.3. The infinitesimal character s is called *real*, if the W-orbit of s_e has only one element.

[**BM1**] proves theorem 1.2 in the case when the infinitesimal character is real. Then [**BM2**] extends the result to all infinitesimal characters. In the process, [**BM2**] shows that it is sufficient to determine the unitary dual of the graded version of the Iwahori-Hecke algebra defined in [**Lu1**], denoted $\mathbb{H}$, and in addition it is enough to consider modules with real infinitesimal character only.

The affine graded Hecke algebra is the associated graded object to a filtration of ideals of $\mathcal{H}$ defined relative to a (W-orbit) of an elliptic element s_e. It has an explicit description in terms of generators and relations, similar to the Bernstein presentation ([**Lu1**]). For example when $s_e = 1$, as a vector space, $\mathbb{H}$ is generated by a copy of $\mathbb{C}[W]$, the group algebra of W, and a copy of the symmetric algebra $\mathbb{A} = Sym(\mathfrak{h})$, with a nontrivial commutation relation between $\mathbb{C}[W]$ and $\mathbb{A}$. (See 3.3 for the precise definition.) The center of $\mathbb{H}$ is $\mathbb{A}^W$, so the central ("infinitesimal") characters are parameterized by W-orbits in $\check{\mathfrak{h}}$. We call an infinitesimal character for $\mathbb{H}$ *real* if it is an orbit of elements in $\check{\mathfrak{h}}_{\mathbb{R}}$.

When $s_e \neq 1$, we refer the reader to [**BM2**], and we will only give the details (which are representative) for the case of $SL(2,\mathbb{F})$. [**BM2**] proves that the unitarity of a $\mathcal{H}$-module with infinitesimal character $s = s_e s_h$ is equivalent with the unitarity of the corresponding $\mathbb{H}_{s_e}$-module with (real) infinitesimal character $\log s_h$, where $\mathbb{H}_{s_e}$ is the algebra obtain by "grading" at $W \cdot s_e$.

1.6. We explain the example of $SL(2,\mathbb{F})$, $\mathbb{F}$ p-adic, in this setting, and illustrate the role of the Hecke algebra. The dual complex group is ${}^\vee G = PGL(2,\mathbb{C})$. The algebra $\mathcal{H}$ is generated (in Bernstein's presentation) by T corresponding to the nontrivial element of W, and θ corresponding to the unique coroot, subject to the relations

$$T^2 = (q-1)T + q, \qquad T\theta = \theta^{-1}T + (q-1)(\theta+1). \tag{1.10}$$

The center of $\mathcal{H}$ is generated by $\theta + \theta^{-1}$. Homomorphisms of the center correspond to W-orbits of semisimple elements $s = s_e s_h \in \check{H}$. In the correspondence 1.1, if V is a subquotient of $X(\chi_\nu)$ (as in (1.6)), then the center of $\mathcal{H}$ acts on $V^{\mathcal{I}}$ by $q^\nu + q^{-\nu}$. The infinitesimal character $s \in \check{H}$ is (the W-orbit of) $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, where $a = \nu \log q/\pi$. As noted in section 1.3, the principal series $X(\chi_\nu)$ is irreducible for $\nu \neq 1, \nu \neq \pi i/\log q$. When it is reducible, the constituents are: the trivial module ($triv$), the Steinberg module (St), which is the unique submodule (discrete series) at $\nu = 1$, the two components $\overline{X}_{sph}$ and $\overline{X}_{nonsph}$ at $\nu = \pi i/\log q$. They correspond to the four one-dimensional Hecke modules as follows:

$SL(2)$-mod	$\mathcal{H}$-mod	Action of $Z(\mathcal{H})$	Inf. char. s	s_e
$triv$	$T = q,\ \theta = q$	$q + q^{-1}$	$\begin{pmatrix} \log q/\pi & 0 \\ 0 & \pi/\log q \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
St	$T = -1,\ \theta = q^{-1}$	$q + q^{-1}$		
$\overline{X}_{sph}$	$T = q,\ \theta = -1$	-2	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$\overline{X}_{nonsph}$	$T = -1,\ \theta = -1$	-2		

Relative to s_e, there are three cases as follows.

1.6.1. $s_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The infinitesimal character s is real. The affine graded Hecke algebra is generated by $t,\ \omega$ subject to the relations

$$t^2 = 1, \qquad t\omega + \omega t = 1. \tag{1.11}$$

This case is relevant for the unitarity of the complementary series in $SL(2,\mathbb{F})$.

1.6.2. $s_e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The infinitesimal character s is real. Because $\theta(s_e) = -1$, the affine graded Hecke algebra is the group algebra of the affine group generated by $t,\ \omega$ with relations

$$t^2 = 1, \qquad t\omega + \omega t = 0. \tag{1.12}$$

The finite dimensional representation theory of this algebra is also well known, but quite different from case 1.6.1. In particular, at real infinitesimal character the only unitary representations occur at $s_h = 1$. They correspond to the two modules $\overline{X}_{sph}$ and $\overline{X}_{nonsph}$.

1.6.3. $s_e = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. When $\zeta^2 \neq \pm 1$, the infinitesimal characters are not real, so there is a reduction to a smaller algebra. The affine graded Hecke algebra $\mathbb{H}_{s_e}$ in [**BM2**] is generated by $E_\zeta,\ E_{\zeta^{-1}},\ t,\ \omega$ satisfying the following relations:

(1) $E_\zeta^2 = E_\zeta,\ E_{\zeta^{-1}}^2 = E_{\zeta^{-1}},\ E_\zeta \cdot E_{\zeta^{-1}} = 0,\ E_\zeta + E_{\zeta^{-1}} = 1$, in other words they are projections,
(2) $tE_\zeta = E_{\zeta^{-1}}t, \qquad E_\zeta t = tE_{\zeta^{-1}}$,
(3) $t^2 = 1, \qquad t\omega + \omega t = 1$.

Let $\mathcal{M}_2$ be the algebra of 2×2 matrices with complex coefficients and the usual basis E_{ij}. Let $\mathbb{A}$ be the polynomial algebra generated by ω. Then theorem 3.3 in [**BM2**] states that the map $m \otimes a \mapsto \Psi(m) \cdot a$ from $\mathcal{M}_2 \otimes_{\mathbb{C}} \mathbb{A}$ to $\mathbb{H}_{s_e}$

$$\begin{aligned} &\Psi : \mathcal{M}_2 \longrightarrow \mathbb{H}_{s_e} \\ &E_{11} \mapsto E_\zeta, \quad E_{12} \mapsto E_\zeta t, \quad E_{21} \mapsto tE_\zeta, \quad E_{22} \mapsto tE_\zeta t = E_{\zeta^{-1}} \end{aligned} \tag{1.13}$$

is an algebra isomorphism. Therefore $\mathbb{H}_{s_e}$ is Morita equivalent to $\mathbb{A}$. The equivalence also preserves unitarity. It follows that the only unitary representation with such infinitesimal character is the trivial one. This case corresponds to the unitarity of $X(\chi_\nu)$, ν purely imaginary, but $\nu \neq 0, \pi i/\log q$.

1.7. Now we can describe the role of the graded algebra $\mathbb{H}$ in the determination of the spherical unitary dual of $G(\mathbb{R})$. The $SL(2)$ examples suggest that, in this correspondence, under the appropriate technical assumption ("real infinitesimal character"), there should be a matching of the unitary representations. The technical notion for proving this correspondence is that of *petite K-types* for real groups, which were defined in [**Ba1**] and studied further in [**Ba2**] and [**BP**].

In theorem 1.1, the unramified principal series $X(\chi)$ correspond to the induced Hecke modules $\mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\chi$, where χ can be identified with an element of $\check{\mathfrak{h}}$. The action of $\mathbb{H}$ on this induced module, which we will denote by $X(\chi)$ as well, is by multiplication on the left. Therefore as a $\mathbb{C}[W]$ module it is isomorphic to the left regular representation,

$$\sum_{(\psi, V_\psi) \in \widehat{W}} V_\psi \otimes V_\psi^*. \tag{1.14}$$

A module (π, V) of $\mathbb{H}$ is called spherical if it has nontrivial W-fixed vectors, *i.e.* $V^W \neq (0)$. It is clear from (1.14) that the $\mathbb{H}$-module $X(\chi)$ contains the trivial W-type with multiplicity 1, so it has a unique spherical subquotient, denoted again by $L(\chi)$. In theorem 1.1, spherical modules for $\mathbb{H}$ match spherical modules for $G(\mathbb{F})$.

Assume that χ is real and dominant, and such that the spherical quotient $L(\chi)$ is hermitian. Then $X(\chi)$ has an invariant hermitian form so that the radical is the maximal proper invariant subspace (so that the quotient is $L(\chi)$). Each space V_ψ^* inherits a hermitian form $\mathcal{A}_\psi^{\mathbb{H}}(\chi)$ depending continuously on χ so that $L(\chi)$ is unitarizable if and only if all $\mathcal{A}_\psi^{\mathbb{H}}(\chi)$, $\psi \in \widehat{W}$, are positive semidefinite. The set of *relevant* W-types defined in [**Ba1**], [**Ba2**], [**BC2**] is a minimal set of W-representations with the property that an $L(\chi)$ is unitary if and only if the form $\mathcal{A}_\psi$ is positive definite for all ψ relevant.

Example 1.4. For $G = SL(n)$, the representations σ of the Weyl group $W = S_n$ are parameterized by partitions, and the set of relevant W-types are formed only of the partitions with at most two parts ([**Ba1**]). In $G = E_8$, there are 112 W-types, and only nine are called relevant ([**BC2**]).

Now consider the case $\mathbb{F} = \mathbb{R}$ and χ a real unramified character. Assume that there is $w \in W$ so that $w\chi = \chi^{-1}$. This is the condition χ must satisfy so that $X(\chi)$ admits an invariant hermitian form. Assume further that χ is dominant so that $L(\chi)$ is the quotient by the maximal proper invariant subspace. The maximal proper invariant subspace is also the radical of the hermitian form, so that $L(\chi)$ inherits a nondegenerate hermitian form. Then for every K-type μ, there is a hermitian form $\mathcal{A}_\mu^{\mathbb{R}}(\chi)$ on $\mathrm{Hom}_K[\mu, X(\chi)]$. The module $L(\chi)$ is unitary if and only if $\mathcal{A}_\mu^{\mathbb{R}}(\chi)$ is positive semidefinite for all μ. By Frobenius reciprocity, we can interpret $\mathcal{A}_\mu^{\mathbb{R}}(\chi)$ as a hermitian form on $(V_\mu^{{}^0H})^*$. Moreover $(V_\mu^{{}^0H})^*$ is a representation of W (not necessarily irreducible). We denote it by ψ_μ.

The petite K-types are K-types such that the real operator $\mathcal{A}_\mu^{\mathbb{R}}(\chi)$ coincides with the Hecke operator $\mathcal{A}_{\psi_\mu}^{\mathbb{H}}(\chi)$. The petite spherical K-types for split $G(\mathbb{R})$ are studied in [**Ba1, Ba2**].

Example 1.5. For the spherical principal series of $SL(2,\mathbb{R})$, (where $K = SO(2)$ and $W = \mathbb{Z}/2\mathbb{Z}$) the only petite $SO(2)$-types appearing in $X(\chi)$ are (0), and (± 2), which correspond to the trivial, and the sign W-representations respectively.

The main result is that, for every simple split group $G(\mathbb{R})$, every relevant W-type occurs in $(V_\mu^{{}^0H})^*$ for a petite μ. The consequence is that the set of unitary spherical representations with real infinitesimal character for a real split group $G(\mathbb{R})$ is contained in the set of unitary spherical parameters with real infinitesimal character for the corresponding p-adic group $G(\mathbb{F})$. In fact, for the split classical groups, [**Ba1**] proves that these sets are equal. (This is false for nonsplit groups, see [**Ba3**] for complex groups, [**Ba4**] and [**BC3**] for unitary groups.)

1.8. The description of the spherical unitary dual (for real infinitesimal character) when $G(\mathbb{F})$ is a simple split group has a particularly nice form. The details are in section 5, where we also explain the main theorem.

View the spherical parameter χ as an element of $\check{\mathfrak{h}}_{\mathbb{R}}$. Then, motivated by the results in [**KL**], one can attach to χ a nilpotent ${}^\vee G$-orbit $\check{\mathcal{O}}(\chi)$ in $\check{\mathfrak{g}}$ as follows. Note first that because $\chi \in \check{\mathfrak{g}}$ is semisimple, the centralizer ${}^\vee G(\chi)$ is connected. In fact, it is a Levi subgroup of ${}^\vee G$. Define

$$\check{\mathfrak{g}}_1 = \{X \in \mathfrak{g} : [\chi, X] = X\}. \tag{1.15}$$

This vector space consists of nilpotent elements of $\check{\mathfrak{g}}$. The group ${}^\vee G(\chi)$ acts on $\check{\mathfrak{g}}_1$ via the adjoint representation with finitely many orbits. Therefore there exists a unique orbit which is open (dense) and we define $\check{\mathcal{O}}(\chi)$ to be its ${}^\vee G$-saturation.

For every nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$, we fix a Lie triple $\{\check{e},\check{h},\check{f}\}$ (see [**CM**]), and we denote by $\mathfrak{z}(\check{\mathcal{O}})$ the centralizer in $\check{\mathfrak{g}}$ of this Lie triple. Then, in particular, any parameter χ for which $\check{\mathcal{O}}(\chi)=\check{\mathcal{O}}$ can be written (up to W-conjugacy) as

$$\chi=\check{h}/2+\nu,\quad \nu\in\mathfrak{z}(\check{\mathcal{O}}). \tag{1.16}$$

In fact, according to [**BM1**], $\check{\mathcal{O}}$ is the unique maximal nilpotent orbit for which χ can be written in this way.

Definition 1.6. We define the *complementary series attached to* $\check{\mathcal{O}}$, denoted by $CS_{\check{\mathfrak{g}}}(\check{\mathcal{O}})$, to be the set of spherical parameters χ such that $L(\chi)$ is unitary and $\check{\mathcal{O}}(\chi)=\check{\mathcal{O}}$. Clearly, the spherical unitary dual of $G(\mathbb{F})$ is the disjoint union of all $CS_{\check{\mathfrak{g}}}(\check{\mathcal{O}})$.

When $\check{\mathcal{O}}=0$, *i.e.* the trivial nilpotent orbit in $\check{\mathfrak{g}}$, the set of parameters χ such that $\check{\mathcal{O}}(\chi)=0$ corresponds to those characters χ for which $X(\chi)$ is irreducible. By the results of [**Vo4**] in the real case, [**BM4**] in the adjoint p-adic case, these are precisely the spherical principal series which are generic, *i.e.* admit Whittaker models. So $CS_{\check{\mathfrak{g}}}(0)$ consists of the unitary generic spherical parameters for $G(\mathbb{F})$.

The reducibility of the spherical principal series $X(\chi)$ is well-known. $X(\chi)$ is reducible if and only if

$$\begin{aligned}&\langle\check{\alpha},\chi\rangle\in 2\mathbb{Z}+1,\quad \text{in the real case,}\\ &\langle\check{\alpha},\chi\rangle=1,\quad \text{in the } p\text{-adic case,}\end{aligned} \tag{1.17}$$

for all $\alpha\in R$. Assuming χ is dominant, $CS_{\check{\mathfrak{g}}}(0)$ is necessarily a subset of the complement in the dominant Weyl chamber of $\check{\mathfrak{h}}_{\mathbb{R}}$ of the arrangement of hyperplanes given by (1.17). The region of $\check{\mathfrak{h}}_{\mathbb{R}}$ in the dominant Weyl chamber on which all coroots $\check{\alpha}$ are strictly less than 1 is called the *fundamental alcove*. Moreover, any region in $\check{\mathfrak{h}}_{\mathbb{R}}$ which is conjugate under the affine Weyl group to the fundamental alcove is called an alcove. Since the spherical principal series $X(\chi)$ is irreducible at $\chi=0$, by unitary induction and a well-known deformation argument, the parameters in the fundamental alcove must be in $CS_{\check{\mathfrak{g}}}(0)$.

We can now list the main results of [**Ba1**] and [**BC1**]. Earlier, for split p-adic groups, the spherical unitary dual in type A was determined in [**Ta**], for types B,C,D in [**BM3**], and for G_2 in [**Mu**]. For split real groups, types A and G_2 are part of [**Vo1**] and [**Vo2**], respectively.

Theorem 1.7 ([**Ba1**],[**BC1**]). *Assume $\mathbb{F}$ is a p-adic field. Recall that $G(\mathbb{F})$ is a simple split group, $\check{\mathcal{O}}$ is a nilpotent orbit in $\check{\mathfrak{g}}$, with a fixed Lie triple $\{\check{e},\check{h},\check{f}\}$ in $\check{\mathfrak{g}}$, whose centralizer is $\mathfrak{z}(\check{\mathcal{O}})$. The spherical complementary series $CS_{\check{\mathfrak{g}}}(\check{\mathcal{O}})$ are defined in definition 1.6.*

(1) *A spherical parameter $\chi=\check{h}/2+\nu$, $\nu\in\mathfrak{z}(\check{\mathcal{O}})$ is in $CS_{\check{\mathfrak{g}}}(\check{\mathcal{O}})$ if and only if ν is in $CS_{\mathfrak{z}(\check{\mathcal{O}})}(0)$. There is one exception to this rule in type F_4, one in type E_7, and six in type E_8 (tabulated in section 6).*
(2) *$CS_{\check{\mathfrak{g}}}(0)$ is the disjoint union of 2^ℓ alcoves in $\mathfrak{h}_{\mathbb{R}}$, where ℓ and the explicit description of the alcoves are listed in section 5.3.*

Theorem 1.8 ([**Ba1**]). *When $\mathbb{F}=\mathbb{R}$, the same description of the spherical unitary dual holds for split classical groups.*

In sections 5.5 and 5.6, we give an explicit description of the spherical unitary parameters for classical groups in terms of the Zelevinski-type strings introduced in [**BM3**] and [**Ba1**]. We then explain, following [**Ba1**], how this allows one to write any given infinitesimal character χ in the form (1.16), and to check if it parameterizes a unitary representation as in the theorem above.

For the split exceptional real groups, the same theorem is expected to hold, but at this point, by the correspondence relevant W-types/petite K-types, we only know that $CS_{\check{\mathfrak{g}}}(\check{\mathcal{O}})$ in the real case is a subset of $CS_{\check{\mathfrak{g}}}(\check{\mathcal{O}})$ in the p-adic ([**Ba2, BC2**]).

1.8.1. The pictures below show the set of spherical unitary parameters for $G(\mathbb{R}) = SO(3,2)$ ($\mathbb{H} = \mathbb{H}(C_2)$) and $G(\mathbb{R}) = Sp(4,\mathbb{R})$ ($\mathbb{H} = \mathbb{H}(B_2)$) respectively. In both cases, the set is partitioned into a disjoint union of complementary series attached to nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$; the orbits are listed in the adjoint tables. Note that the complementary series attached to the trivial nilpotent orbit ($\check{\mathcal{O}} = (1,\ldots,1)$) is the fundamental alcove. The complementary series attached to the regular nilpotent orbit consists of only one point, the trivial representation.

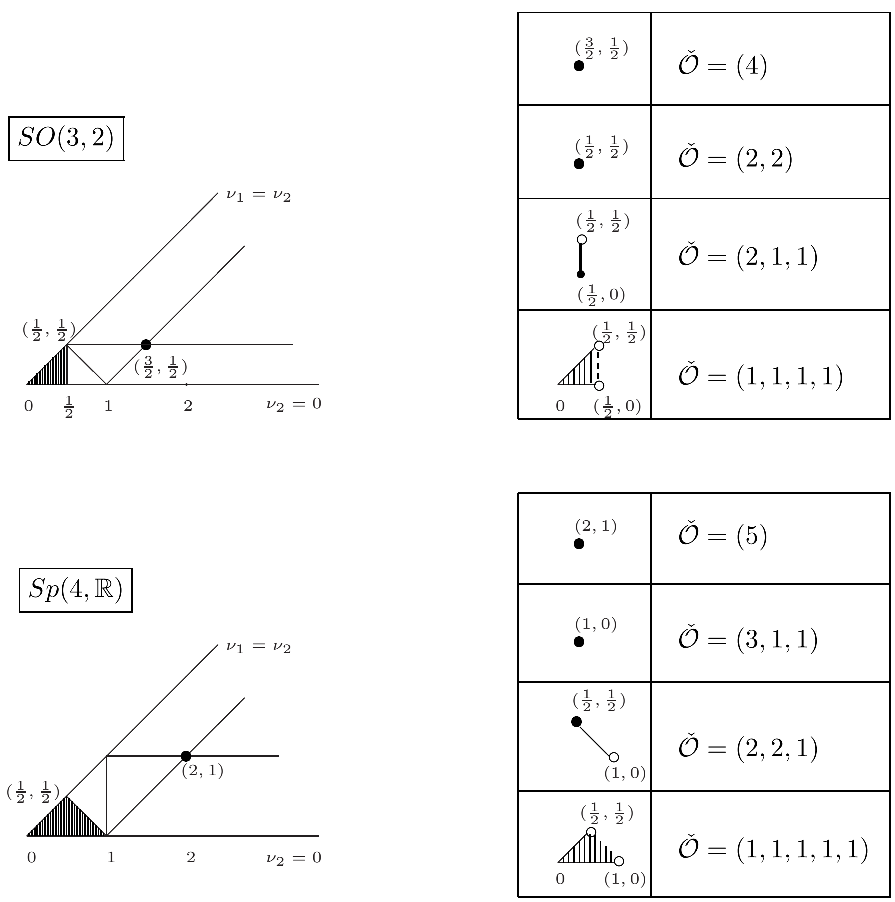


FIGURE 1. Spherical unitary parameters for $SO(3,2)$ and $Sp(4,\mathbb{R})$

Note that $Sp(4,\mathbb{R}) \cong Spin(3,2)$, and indeed the pictures are the same, up to the change of coordinates $(\nu_1,\nu_2) \mapsto (\frac{\nu_1+\nu_2}{2}, \frac{\nu_1-\nu_2}{2})$ (from the $Sp(4,\mathbb{R})$ parameters to the $SO(3,2)$ parameters).

1.9. Now we consider the ramified principal series. If χ is a ramified character of $H(\mathbb{F})$, recall that ${}^0\chi$ is the restriction to the compact part ${}^0H$. To ${}^0\chi$, one associates the homomorphisms

$$\hat{\chi} : \mathbb{O}^\times \to \check{H}, \qquad \text{when } \mathbb{F} \text{ is } p\text{-adic,}$$
$$\hat{\chi} : \mathbb{Z}/2\mathbb{Z} = \{1,-1\} \to \check{H}, \qquad \text{when } \mathbb{F} = \mathbb{R}, \tag{1.18}$$

defined by the property that

$$\lambda \circ \hat{\chi}(z) = {}^0\chi \circ \lambda(z),\ z \in \mathbb{O}^\times \text{ (or } z \in \mathbb{Z}/2\mathbb{Z} \subset \mathbb{R}^\times),\ \text{for all } \lambda \in \mathcal{Y}, \tag{1.19}$$

where in the left hand side λ is regarded as a character of $\check{H}$ and in right hand side as a cocharacter of $H(\mathbb{F})$. In the real case, the image of $\hat{\chi}$ can clearly be identified with a single semisimple (elliptic) element of order 2 in $\check{H}$.

Define

$$^\vee G(\hat{\chi}) = \text{ the centralizer of the image of } \hat{\chi} \text{ in } {}^\vee G. \tag{1.20}$$

In the p-adic case, it is proved in [**Ro**] that there exists an elliptic semisimple element s of ${}^\vee G$ such that the the identity component of ${}^\vee G(\hat{\chi})$ equals the identity component of the centralizer of s in ${}^\vee G$. Moreover, the component group $\check{R}_{{}^0\chi}$ of ${}^\vee G(\hat{\chi})$ is abelian. In the real case this is a classical result (Knapp-Stein): the group ${}^\vee G(\hat{\chi})$ is the one defined by the *good roots* with respect to ${}^0\chi$, while $\check{R}_{{}^0\chi}$ is the dual of the R-group. (For a definition of good (co)roots, see section 2.3 if $\mathbb{F}$ is real, and section 3.2 if $\mathbb{F}$ is p-adic.)

Let $G({}^0\chi)$ be the connected split subgroup of $G(\mathbb{F})$ whose complex dual is the identity component of ${}^\vee G(\hat{\chi})$. The methods described in this paper give a way to compare the unitary Langlands quotients of the ramified principal series $X(\chi) = \mathrm{Ind}_{B(\mathbb{F})}^{G(\mathbb{F})}(\chi \otimes 1)$ with the spherical unitary dual of $G({}^0\chi)$.

Remark. One may consider the extension $G({}^0\chi)'$ of $G({}^0\chi)$ by the (abelian) R-group $R_{{}^0\chi} \cong \check{R}_{{}^0\chi}$, and say that a representation (π, V) of $G({}^0\chi)'$ is *quasi-spherical* if the restriction of π to $G({}^0\chi)$ is spherical. A refinement would be to explain the unitarizable Langlands quotients of the ramified principal series via the quasi-spherical unitary dual of $G({}^0\chi)'$, rather then the spherical unitary dual of $G({}^0\chi)$.

1.10. We give some examples illustrating the construction of $G({}^0\chi)$ when χ is ramified.

1.10.1. The first example is the nonspherical principal series of $SL(2,\mathbb{R})$. Let ${}^0\chi$ be the sign character of $\mathbb{Z}/2\mathbb{Z}$. We can write $\chi = {}^0\chi \otimes |\cdot|^\nu$, and we assume that ν is real. It is well-known that $X(\chi)$ is reducible when $\nu = 0$, and then it splits into a sum of two unitary representations $L(\chi) = X(\chi) = L(\chi)_1 \oplus L(\chi)_{-1}$ (the limits of discrete series). Moreover, $\nu = 0$ is the only value of the parameter ν for which $L(\chi)$ is unitary. In this case, the nontrivial element in the image of $\hat{\chi}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $PGL(2,\mathbb{C})$. The centralizer ${}^\vee G(\hat{\chi})$ has two components: the identity component is the diagonal $\check{H}$, and the other is formed by the matrices with 0 on the diagonal. Then $G({}^0\chi)$ is the diagonal torus $H(\mathbb{R}) = \mathbb{R}^\times$, and the only unramified real unitary character is the trivial one. This can be formulates as: $L(\chi)$ is unitary if and only if χ is an unramified unitary parameter of $G({}^0\chi)$. One refines this statement by introducing $G({}^0\chi)'$. The R-group is $\mathbb{Z}/2\mathbb{Z}$, and $G({}^0\chi)' = H(\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z} = O(1,1)$, where the nontrivial $\mathbb{Z}/2\mathbb{Z}$ element acts by flipping the diagonal entries. There are two quasi-spherical unitary representations of $G({}^0\chi)'$ corresponding to the trivial, respectively the sign representations of $Z/2\mathbb{Z}$.

1.10.2. An example of a nontrivial character ${}^0\chi$ for p-adic $SL(2,\mathbb{F})$, p odd, is as follows. First set

$$sgn : \mathbb{F}_q^\times \to \{\pm 1\}, \quad sgn(a) = \begin{cases} 1, & a \in (\mathbb{F}_q^\times)^2, \\ -1, & a \notin (\mathbb{F}_q^\times)^2 \end{cases} \tag{1.21}$$

Then ${}^0H \cong \mathbb{O}^\times$, and the analog of (1.9) is

$$1 \longrightarrow 1 + \mathcal{P} \longrightarrow \mathbb{O}^\times \longrightarrow \mathbb{F}_q^\times \longrightarrow 1. \tag{1.22}$$

The character ${}^0\chi$ we consider is the pull back of sgn from $\mathbb{F}_q$ to $\mathbb{O}^\times$.

Later in the paper, in section 3.2, we will define a correspondence between the minimal principal series for $G(\mathbb{R})$ and certain minimal principal series of p-adic $G(\mathbb{F})$, by sending the sign character of $\mathbb{Z}/2\mathbb{Z}$ to a fixed nontrivial quadratic character ${}^0\chi$ of $\mathbb{O}^\times$.

1.10.3. When G has connected center, then ${}^\vee G$ has simply-connected derived subgroup, and ${}^\vee G(\hat{\chi})$ is connected by a well known theorem of Steinberg. Using the results of [**Ro**] and [**KL**], the Langlands classification for the ramified principal series in the p-adic case was obtained in this case in [**Re**]. When G has connected center, $G({}^0\chi)' = G({}^0\chi)$ is the (endoscopic) group of $G(\mathbb{F})$ we consider, and the intent is to match the unitary Langlands quotients of the ramified principal series $\mathrm{Ind}_{B(\mathbb{F})}^{G(\mathbb{F})}(\chi \otimes 1)$ with the spherical unitary dual of $G({}^0\chi)$.

For example, when $G = SO(2n+1)$, the groups which appear as $G({}^0\chi)$ in our cases are precisely the elliptic (in the sense of not being contained in any Levi subgroup) endoscopic split groups $SO(2(n-m)+1,\mathbb{F}) \times SO(2m+1,\mathbb{F})$.

1.10.4. On the other hand, when $G = Sp(2n)$, the groups which appear are the split groups $G({}^0\chi) = Sp(2n-2m,\mathbb{F}) \times SO(2m,\mathbb{F})$, and $G({}^0\chi)' = Sp(2n-2m,\mathbb{F}) \times O(2m,\mathbb{F})$. When $\mathbb{F} = \mathbb{R}$, the case of nonspherical minimal principal series for $Sp(2n,\mathbb{R})$ is presented in more detail in sections 2.6 and 4.7.1.

For example, consider $G = Sp(4)$. If $\mathbb{F} = \mathbb{R}$, then ${}^0H = (\mathbb{Z}/2\mathbb{Z})^2$, so there are four characters ${}^0\chi$ of ${}^0H$: ${}^0\chi = triv \otimes triv$, $triv \otimes \chi_0$, $\chi_0 \otimes triv$, and $\chi_0 \otimes \chi_0$, where $\chi_0 = sgn$. Since $triv \otimes \chi_0$ and $\chi_0 \otimes triv$ are conjugate, there are really only three cases. When $\mathbb{F}$ is p-adic, ${}^0H = (\mathbb{O}^\times)^2$. We fix a quadratic, nontrivial character χ_0 of $\mathbb{O}^\times$, and only consider the similar three ${}^0\chi$'s: $triv \otimes triv$, $triv \otimes \chi_0$, and $\chi_0 \otimes \chi_0$ respectively. Then we have:

${}^0\chi$	$G({}^0\chi)$	$G({}^0\chi)'$
$triv \otimes triv$	$Sp(4,\mathbb{F})$	$Sp(4,\mathbb{F})$
$triv \otimes \chi_0$	$Sp(2,\mathbb{F}) \times SO(2,\mathbb{F})$	$Sp(2,\mathbb{F}) \times O(2,\mathbb{F})$
$\chi_0 \otimes \chi_0$	$SO(4,\mathbb{F})$	$O(4,\mathbb{F})$

The first case is the unramified one. The notation in the table may be confusing when $\mathbb{F} = \mathbb{R}$: we mean the split orthogonal groups, in classical notation, $SO(n,n)$ and $O(n,n)$.

1.11. Similarly to the unramified case, we define $\mathbb{H}({}^0\chi)$ to be the graded Hecke algebra for ${}^\vee G({}^0\chi)$, and we denote by $\mathbb{H}'({}^0\chi)$ the extension of $\mathbb{H}({}^0\chi)$ by the dual R-group. (These extensions of the graded Hecke algebra are defined in section 3.7).

Notice that if ${}^0\chi$ is trivial, *i.e.* if the principal series $X(\chi)$ is spherical, then the R-group is trivial. In this case $\mathbb{H}'({}^0\chi) = \mathbb{H}({}^0\chi)$ is the (usual) Hecke algebra $\mathbb{H}$ attached to the coroot system of $G(\mathbb{F})$. Also, when G has connected center, or more generally whenever the R-group of ${}^0\chi$ is trivial, $\mathbb{H}'({}^0\chi) = \mathbb{H}({}^0\chi)$.

In the p-adic case, at least when G has connected center, the generalization of theorem 1.2 and of the [**BM2**] translation of unitarity to the graded Hecke algebra appears to hold. We will not consider this problem in the present paper, but we hope to pursue it in future work.

The generalization of the notion of petite K-types in [**BP**] provides a similar matching of intertwining operators for the ramified principal series of the real split group on one hand, with intertwining operators for the Hecke algebra $\mathbb{H}({}^0\chi)$ on the other. For example, for the nonspherical principal series of $SL(2,\mathbb{R})$, the only petite $SO(2)$-types are the two *fine* K-types (section 2.3), (± 1). The difficulty of defining and computing the petite K-types for nonspherical principal series in general lies in the fact that one needs to capture at the same time the specifics of both the spherical and nonspherical principal series of $SL(2,\mathbb{R})$. This is realized in [**BP**], and we postpone the technical details until section 4. Instead we present the example of $Sp(4,\mathbb{R})$.

1.12. The example of $\mathbf{Sp(4,\mathbb{R})}$. In this example, we use the more customary notation δ for ${}^0\chi$, and $\hat{\delta}$ for $\hat{\chi}$ (defined in 1.9). There are four minimal principal series $X(\delta,\nu)$ ($M = {}^0H = (\mathbb{Z}/2\mathbb{Z})^2$): $\delta_0 = triv \otimes triv$, $\delta_1^+ = triv \otimes sgn$, $\delta_1^- = sgn \otimes triv$, and $\delta_2 = sgn \otimes sgn$. The maximal compact group is $K = U(2)$, whose representations are parameterized by pairs of integers (a,b), $a \geq b$. The ramified character χ is written $\chi = \delta \otimes \nu$, where $\nu = (\nu_1,\nu_2)$, $\nu_1,\nu_2 \in \mathbb{R}$. We assume that $\nu_1 \geq \nu_2 \geq 0$, *i.e.* that the parameter is (weakly) dominant.

As in the spherical case, it is well known when $X(\delta,\nu)$ admits an invariant hermitian form. Since the parameter ν is assumed real, the condition is that there exists $w \in W$ such that

$$w \cdot \delta \cong \delta, \text{ and } w \cdot \nu = -\nu. \tag{1.23}$$

In the $Sp(4,\mathbb{R})$ cases, all quotients $L(\delta,\nu)$ are hermitian.

We denote the Weyl groups of $G(\delta)$ and $G(\delta)'$, by W_δ^0 and W_δ respectively. For every K-type μ, similarly to the spherical case, one has an operator $\mathcal{A}_\mu^{\mathbb{R}}(\delta,\nu)$ on the space

$$\mathrm{Hom}_M[\mu|_M : \delta], \tag{1.24}$$

whose signature we would like to compute. This space carries a representation of W_δ (and W_δ^0), which we denote as before by ψ_μ. The conditions in section 4 used to define petite K-types are such that this operator is the same as the "p-adic" operator on ψ_μ for the Hecke algebra $\mathbb{H}(\delta)$ (or more precisely, the extended version $\mathbb{H}'(\delta)$).

The tables with examples of petite K-types μ, their operators $\mathcal{A}_\mu^{\mathbb{R}}(\nu_1,\nu_2)$, and the set of unitary Langlands subquotients of minimal principal series are next. In the tables below, by "mult." we mean the dimension of the space (1.24).

1.12.1. $\delta_0 = triv \otimes triv$. This is the spherical principal series. In this case ${}^\vee G(\hat{\delta}_0) = SO(5, \mathbb{C})$. We may assume $\nu_1 \geq \nu_2 \geq 0$. Then $W_\delta = W^0_\delta = W = W(B_2)$. The representations of $W(B_2)$ are parameterized by pairs of partitions of total sum 2. There are 4 one dimensional representations, and one two-dimensional, labeled $(1) \times (1)$.

μ	mult.	$\psi_\mu \in \widehat{W}$	$\mathcal{A}^{\mathbb{R}}_\mu(\nu)$
$(0,0)$	1	$(2)\times(0)$	1
$(1,-1)$	1	$(1,1)\times(0)$	$\frac{(1-(\nu_1+\nu_2))(1-(\nu_1-\nu_2))}{(1+(\nu_1+\nu_2))(1+(\nu_1-\nu_2))}$
$(2,2)$	1	$(0)\times(2)$	$\frac{(1-\nu_1)(1-\nu_2)}{(1+\nu_1)(1+\nu_2)}$
$(2,0)$	2	$(1)\times(1)$	m

where $m = \frac{1}{c(\nu_1,\nu_2)}\begin{pmatrix} (1+\nu_2)[(1+\nu_1)+(1-\nu_1)(\nu_1^2-\nu_2^2)] & 2\nu_1(1-\nu_2^2) \\ 2\nu_1(1-\nu_2^2) & (1-\nu_2)[(1-\nu_1)+(1+\nu_1)(\nu_1^2-\nu_2^2)] \end{pmatrix}$, and $c(\nu_1,\nu_2) = (1+\nu_1)(1+\nu_2)(1+(\nu_1-\nu_2))(1+(\nu_1+\nu_2))$.

The spherical quotient $L(\nu)$ is unitary if and only if $\nu_1+\nu_2 \leq 1$, or $(\nu_1,\nu_2) = (2,1)$.

1.12.2. $\delta_1^+ = triv \otimes sgn$ *and* $\delta_1^- = sgn \otimes triv$. In this case, ${}^\vee G(\hat{\delta}_1^+) \cong S[O(3,\mathbb{C}) \times O(2,\mathbb{C})]$, and ${}^\vee G(\hat{\delta}_1^-) \cong S[O(2,\mathbb{C}) \times O(3,\mathbb{C})]$, $W^0_{\delta_1^\pm} = W(A_1)$ and $W_{\delta_1^\pm} = W^0_{\delta_1^\pm} \times W(A_1)$.

The two principal series are $X(\delta_1^+,(\nu_1,\nu_2))$ and $X(\delta_1^-,(\nu_1,\nu_2))$ with $\nu_1 \geq \nu_2 \geq 0$. If $\nu_1 = \nu_2$, they have the same Langlands quotient. There is a difference however: in the case of δ_1^+, the parameter ν_1 corresponds to the $SL(2)$ in $G(\delta_1^+)$, and ν_2 to the $SO(1,1)$, while in the case of δ_1^-, ν_1 corresponds to $SO(1,1)$ and ν_2 to $SL(2)$ in $G(\delta_1^-)$.

1) We consider first δ_1^+ and $\nu_1 \geq \nu_2 \geq 0$. We need to distinguish between two cases:

(i) $\nu_2 > 0$. The Langlands quotient $L(\delta_1^+,\nu)$ is irreducible. The operators are as in the following table.

μ	mult.	$\psi_\mu \in \widehat{W^0_\delta}$	$\psi_\mu \in \widehat{W_\delta}$	$\mathcal{A}^{\mathbb{R}}_\mu(\nu)$
$(1,0)$	1	$triv$	$triv \times triv$	$+1$
$(0,-1)$	1	$triv$	$triv \times sgn$	-1
$(2,1)$	1	sgn	$sgn \times triv$	$\frac{1-\nu_1}{1+\nu_1}$
$(-1,-2)$	1	sgn	$sgn \times sgn$	$-\frac{1-\nu_1}{1+\nu_1}$

Thus $L(\nu)$ is **not unitary** for $\nu_2 > 0$.

(ii) $\nu_2 = 0$. Then $X(\delta_1^+,(\nu_1,0))$ is a direct sum of two modules, $X_{(1,0)}(\delta_1^+,(\nu_1,0))$ and $X_{(0,-1)}(\delta_1^+,(\nu_1,0))$. The Langlands quotient is a direct sum of two irreducible modules, $L_{(1,0)}(\delta_1^+,(\nu_1,0))$ and $L_{(0,-1)}(\delta_1^+,(\nu_1,0))$, distinguished by the fact that the former contains the K-type $(1,0)$ (and $(2,1)$), and the latter contains $(0,-1)$ (and $(-1,-2)$). For $\nu_1 \geq 0$, $L_{(1,0)}$ is the unique irreducible quotient of $X_{(1,0)}$, and $L_{(0,-1)}$ is the unique irreducible quotient of $X_{(0,-1)}$. The intertwining operator is as in the table above. Thus the Langlands quotients are **not unitary** for $1 < \nu_1$. On the other hand, $X_{(1,0)} = L_{(1,0)}$ and $X_{(0,-1)} = L_{(0,-1)}$ for $0 \leq \nu_1 < 1$. At $\nu_1 = 0$, $X(\delta_1^+,(0,0))$ is unitarily induced from a unitary character δ_1^+, and equal to the direct sum of $L_{(1,0)}$ and $L_{(0,-1)}$. Thus $L_{(1,0)}(0,0)$ and $L_{(0,-1)}(0,0)$ are both unitary. It follows that $L_{(1,0)}$ and $L_{(0,-1)}$ are unitary for $0 \leq \nu_1 \leq 1$ by the continuity of the hermitian form in the parameter ν_1.

In this case, $G(\delta_1^+) = SL(2,\mathbb{R}) \times SO(1,1)$. A spherical parameter for this group, (ν_1, ν_2) with ν_1 on the $SL(2,\mathbb{R})$ and ν_2 on the $SO(1,1)$, is hermitian if and only if $\nu_2 = 0$. The representations $L(\delta_1^+, (\nu_1, 0))$ are unitary if and only if the corresponding spherical $L(\nu_1)$ on $SL(2,\mathbb{R})$ is unitary.

2) Now consider δ_1^-, and $\nu_1 \geq \nu_2 \geq 0$. The intertwining operators are as in the table for δ_1^+ above. But in this case, the Langlands quotient $L(\delta_1^-, (\nu_1, \nu_2))$ is irreducible if and only if $\nu_1 > 0$, which implies that unless $(\nu_1, \nu_2) = (0, 0)$, the Langlands quotient is not unitary. At $(0, 0)$, there are two Langlands quotients, same as for δ_1^+, and they are both unitary.

The group $G(\delta_1^-)$ is $SO(1,1) \otimes SL(2,\mathbb{R})$, with ν_1 corresponding to $SO(1,1)$ and ν_2 to $SL(2,\mathbb{R})$.

The conclusion is that a hermitian parameter for $\delta_1^\pm$ *with* $\nu_1 \geq \nu_2 \geq 0$ is unitary if and only if the corresponding parameter for $G(\delta_1^\pm)$ is unitary.

1.12.3. $\delta_2 = sgn \otimes sgn$. In this case, ${}^\vee G(\hat{\delta}_2) = S[O(4,\mathbb{C}) \times O(1,\mathbb{C})]$, and $W_\delta^0 = W(A_1) \times W(A_1)$, $W_\delta = W_\delta^0 \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong W(B_2)$. We may assume $\nu_1 \geq \nu_2 \geq 0$. There are two cases:

(i) $\nu_2 > 0$. The Langlands quotient is irreducible, and the operators are as in the table

μ	mult.	$\psi_\mu \in \widehat{W_\delta^0}$	$\psi_\mu \in \widehat{W_\delta}$	$\mathcal{A}_\mu^{\mathbb{R}}(\nu)$
$(1,1)$	1	$triv \otimes triv$	$(2) \times (0)$	1
$(-1,-1)$	1	$triv \otimes triv$	$(0) \times (2)$	1
$(2,0)$	1	$sgn \otimes sgn$	$(1,1) \times (0)$	$\frac{(1-(\nu_1+\nu_2)(1-(\nu_1-\nu_2)}{(1+(\nu_1+\nu_2)(1+(\nu_1-\nu_2)}$
$(0,-2)$	1	$sgn \otimes sgn$	$(0) \times (1,1)$	$\frac{(1-(\nu_1+\nu_2)(1-(\nu_1-\nu_2)}{(1+(\nu_1+\nu_2)(1+(\nu_1-\nu_2)}$
$(1,-1)$	2	$triv \otimes sgn$ $+sgn \otimes triv$	$(1) \times (1)$	$\begin{pmatrix} \frac{1-(\nu_1-\nu_2)}{1+(\nu_1-\nu_2)} & 0 \\ 0 & \frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)} \end{pmatrix}$

The Langlands quotient is unitary if and only if $0 < \nu_2 \leq 1 - \nu_1$.

(ii) $\nu_2 = 0$. In this case $X(\delta_2, (\nu_1, 0))$ is the direct sum of modules $X_{(1,1)}(\nu_1, 0)$ and $X_{(-1,-1)}(\nu_1, 0)$. The Langlands subquotient is the direct sum of two irreducible modules $L_{(1,1)}(\nu_1, 0)$ and $L_{(-1,-1)}(\nu_1, 0)$, characterized by the fact that the former contains the K-type $(1,1)$ (and $(2,0)$, and a copy of $(1,-1)$), and the latter contains $(-1,-1)$ (and $(0,-2)$, and a copy of $(1,-1)$). For $\nu_1 \geq 0$, $L_{(1,1)}(\nu_1, 0)$ is the unique irreducible quotient of $X_{(1,1)}$, and $L_{(-1,-1)}(\nu_1, 0)$ the unique irreducible subquotient of $X_{(-1,-1)}$. The intertwining operator is obtained by taking $\nu_2 \to 0$ in the table above. This means in particular that the operator on $(1,-1)$ is $\frac{1-\nu_1}{1+\nu_1} Id$. Thus both $L_{(1,1)}(\nu_1, 0)$ and $L_{(-1,-1)}(\nu_1, 0)$ are **not unitary** for $1 < \nu_1$. At $\nu_1 = 0$, the module $X(\delta_2, (0,0))$ is unitarily induced from the unitary character δ_2, so both $L_{(1,1)}(0,0)$ and $L_{(-1,-1)}(0,0)$ are unitary. Since both $X_{(1,1)}$ and $X_{(-1,-1)}$ are irreducible for $0 \leq \nu_1 < 1$, the unitarity of $L_{(1,1)}$ and $L_{(-1,-1)}$ for $0 \leq \nu_1 \leq 1$ follows from the continuity of the hermitian form in the parameter ν_1.

In this case, $G(\delta) = SO(2,2)$. The unitary representations $L(\delta_2,(\nu_1,\nu_2))$ match the spherical unitary dual of this group, which in the coordinates used in this subsection, consists of parameters (ν_1,ν_2) satisfying $0 \le \nu_2 \le 1-\nu_1$.

1.12.4. As already alluded to in the cases above, every principal series $X(\delta_i,\nu)$ contains certain special K-types, called *fine* (see section 2.3). They are as follows:

δ_0	$(0,0)$
$\delta_1^{\pm}$	$(1,0),\ (0,-1)$
δ_2	$(1,1),\ (-1,-1)$

In conclusion, the matching for unitary nonspherical principal series of $Sp(4,\mathbb{R})$ can be formulated as follows.

Proposition 1.9. *Assume that $\nu = (\nu_1,\nu_2)$ is weakly dominant, and (δ,ν) is hermitian. Fix a fine K-type μ and let $L_\mu(\delta,(\nu_1,\nu_2))$ denote the irreducible Langlands quotient which contains the fine K-type μ.*

Then $L_\mu(\delta,(\nu_1,\nu_2))$ is unitary if and only if (ν_1,ν_2) parameterizes a unitary spherical representation for the split group $G(\delta)$.

Remarks.

(1) The spherical unitary dual for $G(\delta)$ can be read from the theorems in 1.6.

(2) Even though δ_1^+ and δ_1^- are conjugate, the corresponding unitary duals are not the same. This should not be surprising in view of the Langlands classification of admissible representations. The fact that in the classification, we must assume (ν_1,ν_2) is dominant implies that $(\delta_1^+,(\nu_1,\nu_2))$ is conjugate to $(\delta_1^-,(\nu_1',\nu_2'))$ if and only if $\nu_1 = \nu_2 = \nu_1' = \nu_2'$.

(3) The correspondence $(\delta,(\nu_1,\nu_2)) \mapsto (\nu_1,\nu_2)$ from parameters of G to spherical parameters of $G(\delta)$ does not necessarily take hermitian parameters to hermitian parameters. Again in the δ_1^- example, any parameter (ν_1,ν_2) is hermitian for G. For the factor $SO(1,1) = \mathbb{R}^\times$, the hermitian dual of a real character $z \mapsto |z|^{\nu_1}$ is $z \mapsto |z|^{-\nu_1}$, so the only hermitian character is the unitary one, *i.e.* for $\nu_1 = 0$. One of the roles of the group $G(\delta)'$ is to fix this problem, as well as to take into account that the Langlands quotient is not always irreducible. See also the remark in 1.9. For this example, $G(\delta_1^-)' = O(1,1) \times SL(2,\mathbb{R})$. The maximal compact group of $O(1,1)$ is $O(1)\times O(1)$, and for $SO(1,1)$ it is $S[O(1)\times O(1)]$. We describe the quasi-spherical representations of $O(1,1) = \mathbb{R}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$. Explicitly we realize $O(1,1)$ as the subgroup preserving the form $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The quasispherical representations have restrictions to $O(1) \times O(1)$ which consist of *triv* $\otimes$ *triv* and *det* $\otimes$ *det*. The analogues of the $X(\delta,\nu)$ are the two dimensional representations κ_{ν_1}, given by the formulas

(1.25)

$$\kappa_{\nu_1} : \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mapsto \begin{pmatrix} |z|^{\nu_1} & 0 \\ 0 & |z|^{-\nu_1} \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & |z|^{\nu_1} \\ |z|^{-\nu_1} & 0 \end{pmatrix}, \ z \in \mathbb{R}^\times.$$

The restriction of κ_{ν_1} to $SO(1,1)$ is the sum of the characters $|\ |^{\nu_1}$ and $|\ |^{-\nu_1}$, and the restriction to $O(1) \times O(1)$ is the sum of *triv* $\otimes$ *triv* and *det* $\otimes$ *det*. The representation κ_{ν_1} is irreducible for $\nu_1 \neq 0$, and decomposes into the sum of the *trivial* and the *determinant* representation for $\nu_1 = 0$. So the κ_{ν_1} for $\nu_1 \neq 0$ and the *trivial* and *determinant* representations for $\nu_1 = 0$ can be viewed as the analogues of the $L_\mu(\delta, \nu)$. We can make this precise by pairing the two fine K-types of $X(\delta_1^-, \nu)$, $(1, 0)$ and $(0, -1)$ with $triv \otimes triv$ and $det \otimes det$ of $O(1) \times O(1)$ respectively. Furthermore every κ_{ν_1} is hermitian, but not unitary if $\nu_1 > 0$. Thus we have set up a correspondence between parameters $L_\mu(\delta_1^-, (\nu_1, \nu_2))$ for G with parameters $L_{\phi(\mu)}((\nu_1, \nu_2))$ which is 1-1 onto parameters for $G(\delta_1^-)'$ satisfying $\nu_1 \geq \nu_2$ which matches hermitian parameters with hermitian parameters, and unitary parameters with unitary parameters. The only drawback is that there is a choice in the matching of fine K-types which is not canonical.

A similar result can be stated for δ_1^+ and δ_2. We summarize this discussion in a proposition.

Proposition 1.10. *Let ϕ denote the (non-canonical) correspondence for fine K-types. Assume that ν is weakly dominant for G. Let $L_{\phi(\mu)}(G(\delta)', \nu)$ be the unique irreducible quasi-spherical representation of $G(\delta)'$ which contains $\phi(\mu)$. Then:*

1. *$L_\mu(\delta, \nu)$ is hermitian if and only if $L_{\phi(\mu)}(G(\delta)', \nu)$ is hermitian.*
2. *$L_\mu(\delta, (\nu_1, \nu_2))$ is unitary if and only if $L_{\phi(\mu)}(G(\delta)', \nu)$ is unitary.*

The two propositions 1.9 are indicative of the general case. The calculation of nonspherical petite K-types in [**BP**], given here in section 4, implies the "only if" statement of the proposition for all nonspherical principal series of classical split simple groups. For exceptional groups there is at least one minimal principal series for $G = F_4$, the one labeled δ_3 in table 1 in section 2.7, for which the unitary set is larger than the spherical unitary dual for the corresponding $G(\delta_3)$.

1.13. We give an outline of the paper. Section 2 presents basic facts and examples about minimal principal series of quasisplit real groups, intertwining operators, fine types, and R-groups. Section 3 is concerned with minimal principal series of split p-adic groups and affine Hecke algebras. Section 4 presents the idea and the construction of petite K-types, and the relation between intertwining operators for minimal principal series of real split groups and intertwining operators for (extended) graded Hecke algebras. In section 5, we record the main elements involved in the determination of the spherical unitary dual for split real and p-adic classical groups and for split p-adic exceptional groups. Section 6 has lists of parameters for unitary spherical representations of split groups.

Some of the results on which this exposition is based were presented in June 2006, at the Snowbird conference on "Representations of real Lie groups", in honor of B. Casselman and D. Miličić. We would like to thank the organizers, particularly P. Trapa, for the invitations to attend the conference and for their efforts in creating a successful meeting. This research was supported by the NSF grants DMS-0300172 and FRG-0554278.

2. Minimal principal series for real groups

In this section, and in section 4, we will use the classical notation from the theory of reductive Lie groups. For example, if G is a real group, the Lie algebra of G is denoted by $\mathfrak{g}_0$ and the complexification by $\mathfrak{g}$, and a minimal parabolic subgroup P has the decomposition $P = MAN$. The set of inequivalent irreducible representations of a group H is denoted by $\widehat{H}$.

2.1. Minimal Principal Series. The definitions and basic properties of minimal principal series can be found for example in chapter 4 of [**Vo3**].

Let G denote a quasisplit real linear reductive group in the sense of [**Vo3**]. The main case of interest in this paper is when G is the real points of a linear connected reductive group *split* over $\mathbb{R}$. Let K be a maximal compact subgroup corresponding to the Cartan involution θ.

Let $P = MAN$ be a minimal parabolic subgroup of G (a Borel subgroup), and $H = MA$ be the Cartan subgroup. The group M is abelian, because G is quasisplit. (If G is split, then M is a finite abelian 2-group.) The Iwasawa decomposition is $G = KAN$. One identifies $\mathfrak{a}^*(\cong \check{\mathfrak{a}})$ with $\widehat{A}$ via

$$\nu \mapsto e^\nu, \ (e^\nu)(a) = \exp(\nu(\log a)), \ \nu \in \mathfrak{a}^*, a \in A. \tag{2.1}$$

For every irreducible representation (δ, V^δ) of M, and every element ν of $\check{\mathfrak{a}}$, we denote the *minimal principal series* by

$$X(\delta, \nu) := \mathrm{Ind}_{MAN}^G(\delta \otimes e^\nu \otimes 1). \tag{2.2}$$

(Ind denotes normalized induction.) The K-structure of $X(\delta, \nu)$ is easy to describe:

(1) The restriction of $X(\delta, \nu)$ to K is

$$X(\delta, \nu)|_K = \mathrm{Ind}_M^K(\delta). \tag{2.3}$$

(2) For every K-type (μ, E_μ), one has Frobenius reciprocity

$$\mathrm{Hom}_K\left(\mu, X(\delta, \nu)\right) = \mathrm{Hom}_M\left(\mu|_M, \delta\right). \tag{2.4}$$

Definition 2.1. A representation (π, V) of G is called *spherical* if π has fixed vectors under K. The minimal principal series $X(\nu) := X(triv, \nu)$ is called the *spherical principal series*.

2.2. Denote by

$$\begin{aligned} &\Delta(\mathfrak{g}, \mathfrak{h}) && \text{the set of roots of } \mathfrak{h} \text{ in } \mathfrak{g}, \\ &\Delta^+(\mathfrak{g}, \mathfrak{h}) && \text{the set of roots in } \Delta(\mathfrak{g}, \mathfrak{h}) \text{ whose root spaces lie in } \mathfrak{n}, \\ &\Delta && \text{the set of } \textit{restricted roots} \text{ of } \mathfrak{a} \text{ in } \mathfrak{g}, \\ &\bar{\Delta} && \text{the } \textit{reduced roots} \text{ in } \Delta, \\ &\Delta^+ \subset \Delta && \text{the restricted roots whose root spaces lie in } \mathfrak{n}_0. \end{aligned} \tag{2.5}$$

If G is split, then $\bar{\Delta} = \Delta$. The Weyl group of Δ is

$$W = N_G(A)/A = N_K(A)/M. \tag{2.6}$$

A root in $\Delta(\mathfrak{g}, \mathfrak{h})$ is called *real* if $\theta\alpha = -\alpha$. It is called complex otherwise (recall that G is assumed quasisplit). A root in Δ is called real if it is a restriction of a real root in $\Delta(\mathfrak{g}, \mathfrak{h})$), or otherwise it is called complex. When G is split, all roots in Δ are real.

For every $\alpha \in \Delta$, define

$$\begin{aligned} &\mathfrak{a}_0^\alpha = \text{the kernel of } \alpha \text{ on } \mathfrak{a}_0, \text{ with} \\ &A^\alpha = \text{the corresponding connected subgroup of } G, \\ &M^\alpha A^\alpha = Z_G(A^\alpha), \text{ where the Lie algebra of } M^\alpha \text{ is} \\ &\mathfrak{m}_0^\alpha = \text{the algebra spanned by } \mathfrak{m}_0 \text{ and } \{E_{c\alpha} \text{ root vectors}: \ c \geq 1\}, \\ &K^\alpha = K \cap M^\alpha. \end{aligned} \tag{2.7}$$

When the root $\alpha \in \bar{\Delta}$ is real, this gives rise an $SL(2)$-subgroup

$$G^\alpha = K^\alpha A^\alpha N^\alpha \cong SL(2,\mathbb{R}) \tag{2.8}$$

as follows. Let

$$\phi_\alpha \colon sl(2,\mathbb{R}) \to \mathfrak{g}_0 \tag{2.9}$$

be a root homomorphism such that $H_\alpha = \phi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \mathfrak{a}_0$, and $E_\alpha := \phi\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ is an α-root vector. Then G^α is the connected subgroup of G with Lie algebra $\phi(sl(2,\mathbb{R}))$.

We will need the following notation later:

$$\begin{aligned} &Z_\alpha := E_\alpha + \theta(E_\alpha) \in \mathfrak{k}_0^\alpha \\ &\sigma_\alpha := \exp\left(\frac{\pi}{2} Z_\alpha\right) \in K^\alpha \text{ is a representative of } s_\alpha \text{ in } N_K(A) \\ &m_\alpha := \sigma_\alpha^2 \in M \text{ an element of order 2.} \end{aligned} \tag{2.10}$$

When G is split semisimple, the elements m_α generate M.

2.3. Fine *K*-types. We will only recall the properties of *fine M-types* and *fine K-types* that we need later (for a general definition see chapter 4 of [**Vo3**]).

The number of fine M-types is directly related to the disconnectedness of M. The two extreme cases are as follows.

(1) If G is a split semisimple group, then any M-type is fine. In this case, a K-type is fine if and only if

$$\text{the eigenvalues of } \mu(iZ_\alpha) \text{ are in } \{0, \pm 1\} \text{ for all (real) } \alpha. \tag{2.11}$$

(2) if G is a complex semisimple group, then only the trivial M-type and the trivial K-type are fine.

If μ is a fine K-type containing the fine M-type δ, then μ is *minimal* (in the sense of [**Vo3**]) in the principal series $X(\delta,\nu)$.

The number of fine K-types containing a given fine M-type is related to the *R-group* of δ, which we now recall.

Definition 2.2. Define Δ_δ to be the subset of roots α in $\bar{\Delta}$ such that either α is complex, or, if α is real, then $\delta(m_\alpha) = 1$. One calls Δ_δ the *good roots with respect to δ*. Let ${}^\vee\Delta = \{\check{\alpha} : \alpha \in \Delta\}$ be the set of *good coroots*.

If δ is fine, then Δ_δ is a root system. We denote the corresponding Weyl group by $W_\delta^0 = W(\Delta_\delta)$. We also set

$$W_\delta = \{w \in W \colon w \cdot \delta \simeq \delta\} \tag{2.12}$$

(the stabilizer of δ in W). Let $\Delta_\delta^+ = \Delta^+ \cap \Delta_\delta$, and similarly define ${}^\vee\Delta_\delta^+$. Then

(1) W_δ^0 is a normal subgroup of W_δ, and

$$W_\delta = W_\delta^0 \rtimes R_\delta^c$$

with $R_\delta^c = \{w \in W \colon w({}^\vee\Delta_\delta^+) = {}^\vee\Delta_\delta^+\}$. Note that R_δ^c is contained in the Weyl group generated by the reflections through the roots perpendicular to $\rho(\Delta_\delta)$. Since these roots are strongly orthogonal, R_δ^c is an abelian 2-group.

(2) The quotient

$$R_\delta = W_\delta / W_\delta^0 \tag{2.13}$$

is a finite abelian 2-group (isomorphic to R_δ^c). We call R_δ the *R-group* of δ.

(3) If G is connected semisimple (and has a complexification), then we can identify W_δ with the stabilizer of the good co-roots for δ:

$$W_\delta = \{w \in W \colon w({}^\vee\Delta_\delta) = {}^\vee\Delta_\delta\}.$$

THEOREM 2.3 ([**Vo3**], 4.3.16). *Let δ be a fine M-type. Set*

$$A(\delta) = \{\mu \text{ fine } K\text{-type such that } \operatorname{Hom}_M(\mu|_M, \delta) \neq \{0\}\}. \tag{2.14}$$

Then

(1) $A(\delta) \neq \emptyset$.

(2) *If $\mu \in A(\delta)$, then $\mu|_M = \bigoplus_{\delta' \in W\cdot\delta} \delta'$. In particular, μ appears with multiplicity one in $X(\delta, \nu)$.*

(3) *There is a natural simply transitive action of $\widehat{R_\delta}$ on $A(\delta)$.*

($\widehat{R_\delta}$ is a group, because R_δ is abelian.) For the action of $\widehat{R_\delta}$ on $\widehat{K}$, and in particular on $A(\delta)$, we refer the reader to 4.3.47 in [**Vo3**]. Notice that the cardinality of R_δ equals the number of fine K-types containing δ (as well as the number of fine K-types contained in $X(\delta, \nu)$).

Examples. (a) If $G = SL(2, \mathbb{R})$, then $K = SO(2)$ and $M = \mathbb{Z}/2\mathbb{Z}$. There are two M-types *triv* and *sgn*, and they are both fine.
If δ is trivial, then $\Delta_\delta = \Delta$, $R_\delta = \{1\}$ and $X(\delta, \nu)$ contains a unique fine $SO(2)$-type ($\mu = (0)$). If δ is *sign*, then $\Delta_\delta = \emptyset$, $R_\delta = \mathbb{Z}/2\mathbb{Z}$ and $X(\delta, \nu)$ contains two fine $SO(2)$-types, namely $(+1)$ and (-1).

(b) If $G = SL(2, \mathbb{C})$, then $K = SU(2)$, and $M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong U(1)$. So $\widehat{M} \cong \mathbb{Z}$, but (by definition) only the trivial M-type is fine. If $\delta = triv$, then $\Delta_\delta = \Delta$ and $\#R_\delta = \#A(\delta) = 1$.

(c) If $G = SU(2,1)$, then $K = S(U(2) \times U(1))$, and $M = U(1)$. Again $\widehat{M} \cong \mathbb{Z}$, but (by definition) only the trivial M-type is fine. If $\delta = triv$, then $\#R_\delta = \#A(\delta) = 1$.

2.4. Subquotients of Principal Series. Let δ be a fine M-type and let $A(\delta)$ be the set of fine K-types in $X(\delta, \nu)$ from theorem 2.3. Since every $\mu \in A(\delta)$ appears in $X(\delta, \nu)$ with multiplicity one, there is a unique irreducible subquotient of $X(\delta, \nu)$ which contains μ. We set

$$L(\delta, \nu)(\mu) = \text{ the unique irreducible subquotient of } X(\delta, \nu) \text{ containing } \mu. \tag{2.15}$$

In general, $L(\delta, \nu)(\mu)$ may contain other fine K-types (other than μ).

Definition 2.4. Define

$$W(\nu) = \{w \in W : w\nu = \nu\},$$
$$W_\delta(\nu) = W(\nu) \cap W_\delta, \quad W_\delta^0(\nu) = W_\delta^0 \cap W(\nu),$$
$$R_\delta(\nu) = W_\delta(\nu)/W_\delta^0(\nu) \subset R_\delta,$$
$$R_\delta^\perp(\nu) = \{\chi \in \widehat{R_\delta} \colon \chi(r) = 1, \text{ for all } r \in R_\delta(\nu)\}. \tag{2.16}$$

Theorem 2.5 ([**Vo3**]). *Assume $\mu \in A(\delta)$.*

1. *Two subquotients $L(\delta,\nu)(\mu)$ and $L(\delta,\nu')(\mu)$ are equivalent if and only if there exists $w \in W_\delta$ such that $\nu' = w\nu$.*
2. *Every irreducible $(\mathfrak{g}, K)$-module containing the K-type μ is equivalent to a subquotient $L(\delta,\nu)(\mu)$ for some $\nu \in \mathfrak{a}^*$.*
3. *Let $\mu' \in A(\delta)$ be another fine K-type. Then $L(\delta,\nu)(\mu) = L(\delta,\nu)(\mu')$ if and only if μ and μ' are conjugate by an element of $R_\delta^\perp(\nu)$.*

2.5. A parameter $\nu \in \mathfrak{a}^*$ is called *weakly dominant* (respectively, *strictly dominant*) if

$$\langle \Re(\nu), \check{\alpha}\rangle \geq 0 \text{ (respectively, } \langle \Re(\nu), \check{\alpha}\rangle > 0), \text{ for all } \alpha \in \Delta^+. \tag{2.17}$$

An important consequence of theorem 2.5 is:

Corollary 2.6. *If $\Re(\nu)$ is strictly dominant, then there exists a unique irreducible subquotient which contains all fine K-types in $A(\delta)$.*

On the other hand, when ν is purely imaginary ($\Re(\nu) = 0$), every fine K-type in $A(\delta)$ parameterizes one constituent of $X(\delta,\nu)$ (a result obtained first by Knapp-Stein):

Theorem 2.7 (Knapp-Stein). *If $\nu \in \mathfrak{a}^*$ is purely imaginary, then*

$$X(\delta,\nu) = \bigoplus_{i=1}^{|R_\delta(\nu)|} X^i(\delta,\nu), \tag{2.18}$$

where $X^i(\delta,\nu)$ are irreducible and inequivalent subrepresentations. Moreover, each $X^i(\delta,\nu)$ contains some fine K-type $\mu \in A(\delta)$.

The trivial K-type is the only fine K-type in $A(triv)$, so every spherical principal series $X(\nu) = X(triv,\nu)$ has a unique irreducible spherical subquotient $L(\nu) = L(triv,\nu)(0)$. In particular, the spherical principal series $X(\nu)$ is irreducible for all ν purely imaginary. The original proof of this result is due to Kostant.

2.6. An example: $Sp(2n,\mathbb{R})$. Let G be the split group $Sp(2n,\mathbb{R})$. Then $K \simeq U(n)$ and $M \simeq \mathbb{Z}_2^n$. Let

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j : i,\, j = 1\dots n,\, i < j\} \cup \{2\epsilon_l : l = 1\dots n\} \tag{2.19}$$

be the set of positive roots. Note that every root is real. The Weyl group $W = \mathcal{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ acts as the group of all permutations and sign changes on $\{\epsilon_1,\, \epsilon_2 \dots \epsilon_n\}$.

The abelian group M has 2^n irreducible inequivalent representations, all of the form

$$\delta_S \colon \operatorname{diag}(\lambda_1,\, \lambda_2, \dots,\, \lambda_n,\, \lambda_1,\, \lambda_2, \dots,\, \lambda_n) \mapsto \prod_{j \in S} \lambda_j \tag{2.20}$$

for some $S \subset \{1, \cdots, n\}$. Because G is split and semisimple, every M-type is fine. The Weyl group partitions $\widehat{M}$ into $(n+1)$ conjugacy classes; we choose representatives:

$$\delta_0 = \delta_{\emptyset} \quad \text{and} \tag{2.21}$$

$$\delta_p = \delta_{\{n-p+1, n-p+2, \dots, n\}} \quad \text{for } 1 \le p \le n. \tag{2.22}$$

We compute the number of irreducible subquotients of $X(\delta_i, \nu)$, for all $i = 0 \dots p$.

δ_0 is the trivial representation of M, so $\Delta_{\delta_\emptyset} = \Delta$, $W_{\delta_\emptyset} = W^0_{\delta_\emptyset} = W$ and $R_{\delta_\emptyset} = \{1\}$. Since there is a unique fine K-type containing δ_0, the principal series $X(\delta_0, \nu)$ has a unique subquotient $L(\nu) = L(\delta_0, \nu)(0)$.

Now assume $1 \le p \le n$. To identify the good roots for δ_p, we need to evaluate δ_p on the elements m_α's. For every (positive) root α, write

$$m_\alpha = \exp\left(2\pi i \|\alpha\|^{-2} H_\alpha\right) = \operatorname{diag}(d_1,\, d_2, \dots, d_n,\, d_1,\, d_2, \dots, d_n).$$

Notice that the entries of m_α are 1, with the exception of $d_i = d_j = -1$ if $\alpha = \epsilon_i \pm \epsilon_j$, and $d_k = -1$ if $\alpha = 2\epsilon_k$. Hence we find:

$$\delta_p(m_{\epsilon_i \pm \epsilon_j}) = \begin{cases} +1, & \text{if either } 1 \le i < j \le n-p \text{ or } n-p+1 \le i < j \le n \\ -1, & \text{otherwise} \end{cases} \tag{2.23}$$

and

$$\delta_p(m_{2\epsilon_k}) = \begin{cases} +1, & \text{if } 1 \le k \le n-p \\ -1, & \text{otherwise.} \end{cases} \tag{2.24}$$

The good roots for δ_p are

$$\Delta_{\delta_p} = \{\pm\epsilon_i \pm \epsilon_j\}_{1 \le i < j \le n-p} \cup \{\pm 2\epsilon_k\}_{1 \le k \le n-p} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{n-p+1 \le i < j \le n}. \tag{2.25}$$

For brevity of notation, set $C_0 = D_0 = D_1 = \emptyset$ and $C_1 = A_1$. Then

$$W^0_{\delta_p} = C_{n-p} \times D_p \quad \forall\, p = 1 \dots n. \tag{2.26}$$

Next, we compute the stabilizer of δ_p. Recall that W_{δ_p} can be identified with the subgroup of Weyl group elements that preserve the good coroots for δ_p. Every permutation and sign change on the sets $\{\epsilon_1 \dots \epsilon_{n-p}\}$ and $\{\epsilon_{n-p+1} \dots \epsilon_n\}$ has this property. Hence

$$W_{\delta_p} = C_{n-p} \times C_p \qquad \text{for all } p = 1, \cdots, n. \tag{2.27}$$

Note that, for all $p = 1 \dots n$, $W^0_{\delta_p}$ is a normal subgroup of W_{δ_p} of index 2.
Because $R_{\delta_p} \simeq \mathbb{Z}/2\mathbb{Z}$, δ_p is contained into two distinct fine K-types (notably $\Lambda^p(\mathbb{C}^n)$ and its dual), which we denote by $\mu^+_{\delta_p}$ and $\mu^-_{\delta_p}$.

Finally, we give the $R_{\delta_p}(\nu)$-group. Write $\nu = (a_1, a_2, \dots, a_n)$ with

$$a_1 \ge a_2 \ge \cdots \ge a_{n-p} \ge a_{n-p+1} \ge \cdots \ge a_n \ge 0.$$

Then

$$R_{\delta_p}(\nu) = \begin{cases} \{1\} & \text{if } 0 \notin \{a_{n-p+1}, \dots, a_n\} \Leftrightarrow a_n \neq 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 \in \{a_{n-p+1}, \dots, a_n\} \Leftrightarrow a_n = 0. \end{cases} \tag{2.28}$$

We conclude that

- If the last entry of ν is nonzero, then the principal series $X(\delta_p, \nu)$ has a unique irreducible subquotient $L(\delta, \nu)(\mu^+_{\delta_p}) = L(\delta, \nu)(\mu^-_{\delta_p})$.

- If the last entry of ν is zero, then the principal series $X(\delta_p, \nu)$ has two distinct subquotients $L(\delta, \nu)(\mu^+_{\delta_p}) \neq L(\delta, \nu)(\mu^-_{\delta_p})$.

2.7. Lists of fine *K*-types for split groups. For the convenience of the reader, we record a list of examples of fine M-types, fine K-types, R-groups and sets of good roots for split simple linear groups. For brevity, we do not include the trivial M-type, and we only give one fine M-type for each orbit of W in $\widehat{M}$. We list, instead, all the fine K-types containing a given fine M-type.

Table 1: Table of fine types

Group	Fine M-type δ	Fine K-types	R_δ-group	Δ_δ
$SL(2n+1,\mathbb{R})$	$\delta_p,\ 1 \le p \le n$	$\Lambda^p(\mathbb{C}^{2n+1})$	1	$A_{p-1} + A_{2n-p}$
$SL(2n,\mathbb{R})$	$\delta_p,\ 1 \le p < n$	$\Lambda^p(\mathbb{C}^{2n})$	1	$A_{p-1} + A_{2n-p-1}$
	δ_n	$\Lambda^n(\mathbb{C}^{2n})_+$ $\Lambda^n(\mathbb{C}^{2n})_-$	$\mathbb{Z}/2\mathbb{Z}$	$A_{n-1} + A_{n-1}$
$Sp(2n,\mathbb{R})$	$\delta_p,\ 1 \le p \le n$	$\Lambda^p(\mathbb{C}^n)$, $\Lambda^p(\mathbb{C}^n)^*$	$\mathbb{Z}/2\mathbb{Z}$	$C_{n-p} + D_p$
$SO(n+1,n)$	$\delta_p,\ 1 \le p \le n$	$1 \otimes \Lambda^p(\mathbb{C}^n)$	1	$B_{n-p} + B_p$
$SO(2n+2,2n+1)_0$	$\delta_p,\ 1 \le p \le n$	$1 \otimes \Lambda^p(\mathbb{C}^{2n+1})$	1	$B_{2n+1-p} + B_p$
$SO(2n+1,2n)_0$	$\delta_p,\ 1 \le p < n$	$1 \otimes \Lambda^p(\mathbb{C}^{2n})$	1	$B_{2n-p} + B_p$
	δ_n	$1 \otimes \Lambda^n(\mathbb{C}^{2n})_+$ $1 \otimes \Lambda^n(\mathbb{C}^{2n})_-$	$\mathbb{Z}/2\mathbb{Z}$	$B_n + B_n$
$SO(2n+1,2n+1)_0$	$\delta_p,\ 1 \le p \le n$	$1 \otimes \Lambda^p(\mathbb{C}^{2n+1})$ $\Lambda^p(\mathbb{C}^{2n+1}) \otimes 1$	$\mathbb{Z}/2\mathbb{Z}$	$D_{2n+1-p} + D_p$
$SO(2n,2n)_0$	$\delta_p,\ 1 \le p < n$	$1 \otimes \Lambda^p(\mathbb{C}^{2n})$ $\Lambda^p(\mathbb{C}^{2n}) \otimes 1$	$\mathbb{Z}/2\mathbb{Z}$	$D_{2n-p} + D_p$
	δ_n	$1 \otimes \Lambda^p(\mathbb{C}^{2n})_+$ $\Lambda^p(\mathbb{C}^{2n})_+ \otimes 1$ $1 \otimes \Lambda^p(\mathbb{C}^{2n})_-$ $\Lambda^p(\mathbb{C}^{2n})_- \otimes 1$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$D_n + D_n$
G_2	δ_3	$V_3 \otimes \mathbb{C}$	1	$A_1 + \widetilde{A}_1$
F_4	δ_3 δ_{12}	$(2\|0,0,0)$ $(1\|1,0,0)$	1 1	C_4 $B_3 + A_1$
E_6	δ_{27} δ_{36}	ω_2 $2\omega_1$	1 1	D_5 $A_5 + A_1$
E_7	δ_{28} δ_{36} δ_{63}	$\omega_2,\ \omega_6$ $2\omega_1,\ 2\omega_7$ $\omega_1 + \omega_7$	$\mathbb{Z}/2\mathbb{Z}$ $\mathbb{Z}/2\mathbb{Z}$ 1	E_6 A_7 $D_6 + A_1$
E_8	δ_{120} δ_{135}	ω_2 $2\omega_1$	1 1	$E_7 + A_1$ D_8

We explain the notation in the table.

(1) If G is a split simple linear group of classical type, the group M consists of diagonal matrices. We have denoted by δ_k the character of M that maps an element m of M into the product of the last k diagonal entries of m.

(2) If G is a split simple linear group of exceptional type, δ_k denotes the character of M that has a k-dimensional orbit under the action of the Weyl group.

The group K is a maximal compact subgroup of G, as follows:

G	K
$SL(m,\mathbb{R})$	$SO(m)$
$Sp(2n,\mathbb{R})$	$U(n)$
$SO(p,q)$	$S(O(p)\times O(q))$
$SO(p,q)_0$	$SO(p)\times SO(q)$

G	K
G_2	$SU(2)\times SU(2)/\{\pm I\}$
F_4	$Sp(1)\times Sp(3)/\{\pm I\}$
E_6	$Sp(4)/\{\pm I\}$
E_7	$SU(8)/\{\pm I\}$
E_8	$Spin(16)/\{I,w\}$

The notation for the maximal compact subgroup when $G=E_8$ means that K is a quotient of $Spin(16)$ by a central $\mathbb{Z}/2\mathbb{Z}$, not equal to $SO(16)$. For all n, we denote by $\mathbb{C}^n$ the standard representation of $SO(n)$, $O(n)$, $SU(n)$ and $U(n)$. We write V_3 for the three-dimensional irreducible representation of $SU(2)$ (on the space of homogeneous polynomials of degree 2 in 2 variables). The fine K-types of E_6, E_7 and E_8 are described in terms of fundamental weights; the ones of F_4 are described in terms of standard (Bourbaki) coordinates.

Remark 2.8. For the purpose of computing intertwining operators on principal series, we need to fix *a priori* a fine K-type μ_δ containing δ. If the cardinality of $A(\delta)$ is not one, the choice of μ_δ is not canonical; we choose:

G	δ	μ_δ
$SL(2n,\mathbb{R})$	δ_n	$\Lambda^n(\mathbb{C}^{2n})_+$
$SO(2n+1,2n)_0$	δ_n	$1\otimes\Lambda^n(\mathbb{C}^{2n})_+$
$SO(2n+1,2n+1)_0$	$\delta_p\ (1\le p\le n)$	$1\otimes\Lambda^p(\mathbb{C}^{2n+1})$
$SO(2n,2n)_0$	$\delta_p\ (1\le p\le n)$	$1\otimes\Lambda^p(\mathbb{C}^{2n})$
$SO(2n,2n)_0$	δ_n	$1\otimes\Lambda^n(\mathbb{C}^{2n})_+$
E_7	δ_{28}	w_2
E_7	δ_{36}	$2w_1$

This choice of μ_δ will be used to define the full intertwining operator in section 2.8, the W_δ-representation in section 4.2, and the bijection between $\widehat{R_\delta}$ and $A(\delta)$ in sections 2.9 and 4.2.

2.8. Intertwining operators. We recall some basic facts about intertwining operators for minimal principal series of real groups (see chapter 7 of [**Kn**] for more details).

Fix an M-type δ and a character ν of A, and let $X(\delta,\nu)$ be the minimal principal series induced from $P=MAN$. Denote by $\bar{P}=MA\bar{N}$ the opposite parabolic. For every $w\in W$, one defines a *formal intertwining operator*

$$A(w,\delta,\nu)\colon X(\delta,\nu)\longrightarrow X(w\delta,w\nu),\qquad (2.29)$$
$$(A(w,\delta,\nu)F)(g)=\int_{\bar{N}\cap(wN)}F(gw\bar{n})\,d\bar{n},\quad g\in G.$$

Formally, the integral defines an intertwining operator, but it may not converge for general ν.

2.8.1. The intertwining operator $A(w,\delta,\nu)$ can be decomposed as follows. (This is the *Gindikin-Karpelevič decomposition.*)

PROPOSITION 2.9. *Assume that $w = w_1 w_2$, with $\ell(w) = \ell(w_1)\ell(w_2)$. Then*

$$A(w,\delta,\nu) = A(w_1, w_2\delta, w_2\nu) \circ A(w_2,\delta,\nu). \tag{2.30}$$

COROLLARY 2.10. *If $w = s_1 \cdot s_2 \cdot \dots \cdot s_m$ is a minimal decomposition of w as a product of simple reflections, then $A(w,\delta,\nu)$ factors as*

$$A(w,\delta,\nu) = A(s_1, w_1\delta, w_1\nu) \cdot A(s_2, w_2\delta, w_2\nu) \cdot \dots \cdot A(s_m, w_m\delta, w_m\nu), \tag{2.31}$$

where $w_k = s_{k+1} \dots s_m$, for all $1 \le k \le m$.

If

$$\langle \Re(\nu), \beta \rangle > 0, \text{ for every root } \beta \in \Delta^+ \text{ such that } w\beta \notin \Delta^+, \tag{2.32}$$

then the integral in (2.29) is actually convergent. This is proved using the decomposition (2.31) and an investigation of the rank one cases (we will make this more precise below).

2.8.2. For every K-type (μ, E_μ), the intertwining operator $A(w,\delta,\nu)$ induces an operator

$$A_\mu(w,\delta,\nu)\colon \operatorname{Hom}_K(\mu, X(\delta,\nu)) \longrightarrow \operatorname{Hom}_K(\mu, X(w\delta, w\nu)). \tag{2.33}$$

By Frobenius reciprocity (2.4), this can be regarded as an operator

$$A_\mu(w,\delta,\nu)\colon \operatorname{Hom}_M(\mu,\delta) \longrightarrow \operatorname{Hom}_M(\mu, w\delta). \tag{2.34}$$

Via (2.30), $A_\mu(w,\delta,\nu)$ also acquires a decomposition into factors corresponding to simple reflections. Let $A_\mu(s_\alpha,\delta,\nu)$ be such a simple reflection factor.

REMARK 2.11. Observe that the operator $A_\mu(s_\alpha,\delta,\nu)$ for G agrees with the operator $A_{\mu|_{MK^\alpha}}(s_\alpha,\delta,\nu|_{\mathfrak{a}^\alpha})$ for the real rank one group MG^α.

More precisely, if G is split,

$$\mu = \bigoplus_{m\in\mathbb{Z}} \mu^\alpha_m \tag{2.35}$$

is the decomposition of the K-type μ into isotypic components of $K^\alpha = SO(2)$ $(\widehat{K^\alpha} = \mathbb{Z})$, then the decomposition

$$\operatorname{Hom}_M(\mu,\delta) = \bigoplus_{m\in\mathbb{N}} \operatorname{Hom}_M(\mu^\alpha_m + \mu^\alpha_{-m}, \delta) \tag{2.36}$$

is preserved by $A_\mu(s_\alpha,\delta,\nu)$. Moreover, the restriction of $A_\mu(s_\alpha,\delta,\nu)$ to $\operatorname{Hom}_M(\mu^\alpha_m + \mu^\alpha_{-m})$ coincides with the operator $A_{\mu^\alpha_m+\mu^\alpha_{-m}}(s_\alpha,\delta,\nu|_{\mathfrak{a}^\alpha})$ for MG^α.

2.9. Reducibility. We discuss the reducibility of the principal series $X(\delta,\nu)$.

Assume that $\Re(\nu)$ is weakly dominant with respect to Δ^+. The first instance in which $X(\delta,\nu)$ becomes reducible is when the R-group $R_\delta(\nu)$ is nontrivial.

Partition the restricted roots according to their inner product with $\Re(\nu)$:

$$\Delta = \Delta_L \sqcup \Delta_U^+ \sqcup \Delta_U^-$$

with

$$\begin{aligned}
\Delta_L &= \{\alpha \in \Delta\colon \langle \Re(\nu), \alpha\rangle = 0\},\\
\Delta_U^+ &= \{\alpha \in \Delta\colon \langle \Re(\nu), \alpha\rangle > 0\},\\
\Delta_U^- &= \{\alpha \in \Delta\colon \langle \Re(\nu), \alpha\rangle < 0\}.
\end{aligned}$$

The set Δ_L is a root system, and $\Delta^+ = \Delta_L^+ \sqcup \Delta_U^+$.

Denote by L the centralizer of $\Re(\nu)$ in G. This is a Levi subgroup containing the Cartan MA, with Lie algebra $\mathfrak{l}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \left(\bigoplus_{\alpha\in\Delta_L} \mathfrak{g}_\alpha\right)$. Define the parabolic subgroup $Q = LU$ of G with Lie algebra $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \left(\bigoplus_{\alpha\in\Delta_L} \mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha\in\Delta_U^+} \mathfrak{g}_\alpha\right)$.

The following lemma is *induction by stages*.

LEMMA 2.12. $X(\delta,\nu) = \mathrm{Ind}_P^G(\delta\otimes\nu) = \mathrm{Ind}_{LU}^G\left(\mathrm{Ind}_P^L(\delta\otimes\nu)\otimes 1\right)$.

Note that ν is imaginary for L, so in parenthesis we have unitary induction. By the results presented in section 2.4, the unitarily induced module $\mathrm{Ind}_P^L(\delta\otimes\nu)$ decomposes as the direct sum of $\#R_\delta(\nu)$ irreducible inequivalent representations of L. Let

$$X(\delta,\nu) = \bigoplus_{r\in R_\delta(\nu)} X^r(\delta,\nu) \tag{2.37}$$

be the corresponding decomposition of $X(\delta,\nu)$.

Recall that $\widehat{R_\delta}$ acts on the set of fine K-types $A(\delta)$ simply transitively (theorem 2.3). Implicit in (2.37) is the fact that we fixed a particular fine K-type μ_δ. This gives a bijection between R_δ and $A(\delta)$: $r\in R_\delta \mapsto \mu_{\delta,r}$, where $\mu_{\delta,0} = \mu_\delta$.

The fine K-types occurring in $X^r(\delta,\nu)$ form an orbit under the action of $R_\delta^\perp(\nu)$ on $A_\delta(\nu)$.

Let μ be a K-type which occurs in $X(\delta,\nu)$. The space

$$\mathrm{Hom}_K\left(\mu, X(\delta,\nu)\right) = \mathrm{Hom}_M(\mu,\delta) \tag{2.38}$$

carries a representation of W_δ. Identify $\widehat{R_\delta}$ with R_δ^c, so that $W_\delta = W_\delta^0 \rtimes \widehat{R_\delta}$, and regard $\mathrm{Hom}_K\left(\mu, X(\delta,\nu)\right)$ as a $\widehat{R_\delta}$-module (by restriction).
Because $R_\delta(\nu) \subset R_\delta$ and R_δ is abelian, we can regard $\widehat{R_\delta(\nu)}$ as a subset of $\widehat{R_\delta}$, and restrict $\mathrm{Hom}_K\left(\mu, X(\delta,\nu)\right)$ to $\widehat{R_\delta(\nu)}$. The $\widehat{R_\delta(\nu)}$-module structure on $\mathrm{Hom}_K\left(\mu, X(\delta,\nu)\right)$ is compatible with the action of $R_\delta(\nu)^\perp$ on $A(\delta)$.

PROPOSITION 2.13. *Let r be an element of $R_\delta(\nu)$, and let μ be a K-type containing δ. Identify $R_\delta(\nu)$ with its double dual. Then:*

$$\mathrm{Hom}_K\left(\mu, X^r(\delta,\nu)\right) = \mathrm{Hom}_{\widehat{R_\delta(\nu)}}\left(r, \mathrm{Hom}_K\left(\mu, X(\delta,\nu)\right)\right). \tag{2.39}$$

2.10. A second way in which $X(\delta,\nu)$ becomes reducible is when the operator $A(w_0,\delta,\nu)$ has a nontrivial kernel. Let $w_0 = s_1\cdot s_2\cdot\dots\cdot s_m$ be a minimal decomposition of w_0, and let

$$A(w_0,\delta,\nu) = \prod_{i=1}^m A(s_i, w_i\delta, w_i\nu)$$

be the factorization of $A(w_0,\delta,\nu)$ (as in section 2.8). Then $A(w_0,\delta,\nu)$ has a nontrivial kernel if and only if one of its factors has.

PROPOSITION 2.14 ([**Vo3**, 4.2.25]). *The long intertwining operator $A(w_0,\delta,\nu)$ has a nontrivial kernel if and only if there exists a simple root $\alpha\in\Delta$ such that $Re\langle\check\alpha,\nu\rangle \neq 0$ and $\mathrm{Ind}_P^{M^\alpha A^\alpha}(\delta,\nu)$ is reducible.*

This proposition reduces the question of finding the kernel of the long intertwining operator to a similar question for rank one groups, where the answer is known. If G is split, the only rank one root subgroup that appears is $SL(2,\mathbb{R})$, and the answer is particularly simple. (The operators for $SL(2,\mathbb{R})$ are given in section 2.13.)

Corollary 2.15. *Assume that G is split. Then the operator $A(w_0,\delta,\nu)$ has a kernel if and only if there exists a simple (real) root $\alpha\in\Delta$ such that*

$$\begin{aligned}&\langle\check{\alpha},\nu\rangle=k, \text{ for some integer } k\neq 0, \text{ and}\\ &\delta(m_\alpha)=(-1)^{k+1}.\end{aligned} \tag{2.40}$$

The parity condition means that $\langle\check{\alpha},\nu\rangle$ should be an odd integer if $\alpha\in\Delta_\delta$, and an even integer otherwise.

For split groups, reducibility can only occur in the two instances described above. We summarize this in the next statement.

Theorem 2.16. *Let $X(\delta,\nu)$ be a minimal principal series for a real split group. Then $X(\delta,\nu)$ is reducible if and only if*

(i) *the R-group $R_\delta(\nu)$ is non-trivial and/or*
(ii) *there is a simple root α such that the inner product $\langle\check{\alpha},\nu\rangle$ is a non-zero integer k, and*
$$\delta(m_\alpha)=(-1)^{k+1}.$$

Example. If $G=SL(2,\mathbb{R})$, then

$$X(triv,\nu) \text{ is reducible at } \langle\check{\alpha},\nu\rangle\in\{\pm1,\pm3,\pm5,\dots\}, \tag{2.41}$$

while

$$X(sgn,\nu) \text{ is reducible at } \langle\check{\alpha},\nu\rangle\in\{0,\pm2,\pm4,\dots\}. \tag{2.42}$$

The point of reducibility $\langle\check{\alpha},\nu\rangle=0$ for $X(sgn,\nu)$ comes from the R-group (since $R_{sgn}(0)=\mathbb{Z}/2\mathbb{Z}$.).

2.11. Langlands quotient. We look at the connection between intertwining operators and Langlands classification. Let $w_0\in W$ be the long Weyl group element and

$$A(w_0,\delta,\nu)\colon X(\delta,\nu)\longrightarrow X(w_0\cdot\delta,w_0\cdot\nu) \tag{2.43}$$

be the long intertwining operator. We choose a fine K-type $\mu_\delta\in A(\delta)$. Since μ_δ has multiplicity one in $X(\delta,\nu)$, the operator $A_{\mu_\delta}(w_0,\delta,\nu)$ is a scalar function of ν. We normalize $A(w_0,\delta,\nu)$ by this scalar, and denote the resulting operators by

$$A'(w_0,\delta,\nu) \text{ and } A'_\mu(w_0,\delta,\nu). \tag{2.44}$$

So $A'_{\mu_\delta}(w_0,\delta,\nu)=1$.

Theorem 2.17 (Langlands, Miličić). *Assume $\Re(\nu)$ is weakly dominant with respect to Δ^+.*

(1) *The operator $A'(w_0,\delta,\nu)$ has no poles.*
(2) *The (closure of the) image of $A'(w_0,\delta,\nu)$ is the Langlands quotient $L(\delta,\nu)$. This is the unique largest completely reducible quotient of $X(\delta,\nu)$. When $\Re(\nu)$ is strictly dominant, $L(\delta,\nu)$ is irreducible.*

The connection between this classification and fine K-types can be formulated as follows.

COROLLARY 2.18. *Assume $\Re(\nu)$ is weakly dominant with respect to the roots in Δ^+. Every irreducible summand $L^i(\delta,\nu)$ of the Langlands quotient $L(\delta,\nu)$ is of the form $L(\delta,\nu)(\mu)$ for some fine K-type $\mu \in A(\delta)$.*

If $\Re(\nu)$ is strictly dominant, then $L(\delta,\nu)|_K$ contains all the K-types in $A(\delta)$ (each with multiplicity one).

2.12. Hermitian forms. Assume for this section that $\Re(\nu)$ is strictly dominant with respect to the roots in Δ^+. We use the results in section 2.5 to define Hermitian forms and investigate the unitarity of the (irreducible) Langlands quotient $L(\delta,\nu)$. The following result is based on the fact that the Hermitian dual of $L(\delta,\nu)$ is $L(\delta,-\overline{\nu})$.

THEOREM 2.19 (Knapp-Zuckerman). *Let $X(\delta,\nu)$ be a minimal principal series induced from P. Assume that $\Re(\nu)$ is strictly dominant with respect to Δ^+ and let $L(\delta,\nu)$ be the irreducible Langlands quotient of $X(\delta,\nu)$. Then*

(1) *$L(\delta,\nu)$ admits a nondegenerate invariant Hermitian form if and only if*

$$w_0 \cdot \delta \cong \delta \text{ and } w_0 \cdot \nu = -\bar{\nu}. \tag{2.45}$$

(2) *If $L(\delta,\nu)$ is Hermitian, choose an isomorphism $\tau\colon w_0{\cdot}\delta \to \delta$, and construct the operator*

$$\mathcal{A}(w_0,\delta,\nu) = \tau \circ A'(w_0,\delta,\nu). \tag{2.46}$$

Let $\langle\ ,\ \rangle_h$ denote the pairing between the module $L(\delta,\nu)$ and its Hermitian dual. Then the Hermitian form on $L(\delta,\nu)$ is given by

$$\langle x,y\rangle := \langle x,\mathcal{A}(w_0,\delta,\nu)y\rangle_h \quad \text{for all } x,y \in L(\delta,\nu). \tag{2.47}$$

(Note that there is an implicit choice of the normalization, dictated by $A'(w_0,\delta,\nu)$.)

Fix a fine K-type μ_δ containing δ. We can identify the representation space V^δ of δ with the isotypic component of δ in μ_δ, and define

$$\tau = \mu_\delta(w_0). \tag{2.48}$$

REMARK 2.20. If $\Re(\nu)$ is dominant and the Langlands quotient $L(\delta,\nu)$ is Hermitian, then $L(\delta,\nu)$ is unitary (or unitarizable) if and only if the invariant Hermitian form in (2.47) is positive definite.

Because $L(\delta,\nu)$ is the quotient of $X(\delta,\nu)$ modulo the Kernel of the operator $A'(w,\delta,\nu)$, the form on $L(\delta,\nu)$ is positive definite if and only if the operator $A'(w,\delta,\nu)$ is positive semidefinite on $X(\delta,\nu)$.

Restricting the attention to the various K-types in $X(\delta,\nu)$, one obtains the following criterion of unitarity.

COROLLARY 2.21. *If $\Re(\nu)$ is dominant and the Langlands quotient $L(\delta,\nu)$ is Hermitian, then $L(\delta,\nu)$ is unitary if and only if the Hermitian operators*

$$\mathcal{A}_\mu(w,\delta,\nu) \tag{2.49}$$

are positive semidefinite for all K-types μ appearing in $X(\delta,\nu)$.

REMARK 2.22. If $\Re(\nu)$ is weakly dominant, the Hermitian condition in (2.45) can be replaced by:

$$w \cdot Q = \bar{Q},\ w \cdot \delta \cong \delta \text{ and } w \cdot \nu = -\bar{\nu}, \tag{2.50}$$

where Q is the parabolic subgroup of G defined by ν, as in section 2.9. Note that the Langlands subquotient $L(\delta,\nu)$ might be reducible. Equation (2.47) defines a non degenerate invariant Hermitian form on every irreducible constituent of $L(\delta,\nu)$, *i.e.* on every irreducible subquotient of $X(\delta,\nu)$. Note that the form on $L(\delta,\nu)(\pi)$ is normalized so that it takes the value $\varrho_\pi[w]=\pm 1$ on the fine K-type π (see section 4.2). Every operator $\mathcal{A}_\mu(w,\delta,\nu)$ has a block corresponding to $L(\delta,\nu)(\pi)$. If $\varrho_\pi[w]=1$ (respectively $\varrho_\pi[w]=-1$), the subquotient $L(\delta,\nu)(\pi)$ is unitary if and only if this block is positive semidefinite (respectively negative semidefinite) for all K-types μ.

For practical purposes the unitarity criterion stated above is used mainly to prove that a given Hermitian $L(\delta,\nu)$ is *not* unitary, unitarity being proven by other methods.

The idea of petite K-types is to give a small set of K-types on which the operator $\mathcal{A}_\mu(w,\delta,\nu)$ is computable, and these computations are sufficient for ruling out all nonunitary $L(\delta,\nu)$. Before looking at the general case in section 4, we will present some examples of real rank one cases, where this idea will already become apparent.

2.13. Real rank one. In view of theorem 2.5 and remark 2.11, it is of particular importance to have the formulas for the intertwining operators for real rank one groups. These are known (see [**JW**] and the references therein). We give three examples. Assume that ν is real and weakly dominant for the spherical cases below, and strictly dominant for the nonspherical one.

(1) $G=SL(2,\mathbb{R})$.

(a) $\delta=triv$, normalized on μ_0:

$$\mathcal{A}_{\mu_{2k}}(s_\alpha,triv,\nu)=\prod_{j=1}^{|k|}\frac{(2j-1)-\langle\nu,\check\alpha\rangle}{(2j-1)+\langle\nu,\check\alpha\rangle}; \tag{2.51}$$

(b) $\delta=sgn$, normalized on μ_1:

$$\mathcal{A}_{\mu_{2k+1}}(s_\alpha,sgn,\nu)=sgn(k)\prod_{j=1}^{|k+\frac12|-\frac12}\frac{2j-\langle\nu,\check\alpha\rangle}{2j+\langle\nu,\check\alpha\rangle}. \tag{2.52}$$

For all $k\in\mathbb{Z}$, μ_k denotes the character $e^{i\theta}\mapsto e^{ki\theta}$ of $K=SO(2)\cong U(1)$.

(2) $G=SL(2,\mathbb{C})$ and $\delta=triv$, normalized on μ_0:

$$\mathcal{A}_{\mu_k}(s_\alpha,triv,\nu)=\prod_{j=1}^{k}\frac{j-\langle\nu,\check\alpha\rangle}{j+\langle\nu,\check\alpha\rangle}. \tag{2.53}$$

For all $k\in\mathbb{N}$, μ_k denotes the $(k+1)$-dimensional representation of $SU(2)$.

From corollary 2.21 and the calculations above, one concludes that:

- If $G=SL(2,\mathbb{R})$ and $\delta=triv$, the Langlands quotient $L(triv,\nu)$ is unitary if and only if $0\le\langle\check\alpha,\nu\rangle\le 1$. The spherical petite K-types for $G=SL(2,\mathbb{R})$ are μ_0 and μ_2. They have the property that $L(triv,\nu)$ is unitary if and only if the operators $\mathcal{A}_\mu(s_\alpha,triv,\nu)$ are positive semidefinite for $\mu\in\{\mu_0,\mu_2\}$.

- If $G = SL(2,\mathbb{R})$ and $\delta = sgn$, the Langlands quotient $L(sgn,\nu)$ is never unitary for $\nu > 0$. The nonspherical petite K-types for $G = SL(2,\mathbb{R})$ are μ_1 and μ_{-1}. They have the property that $L(sgn,\nu)$ is unitary if and only if the operators $\mathcal{A}_\mu(s_\alpha, sgn,\nu)$ are positive semidefinite for $\mu \in \{\mu_1, \mu_{-1}\}$.
- If $G = SL(2,\mathbb{C})$ and $\delta = triv$, the Langlands quotient $L(triv,\nu)$ is unitary if and only if $0 \leq \langle\check{\alpha},\nu\rangle \leq 1$. The spherical petite K-types for $G = SL(2,\mathbb{C})$ are μ_0 and μ_1. They have the property that $L(triv,\nu)$ is unitary if and only if the operators $\mathcal{A}_\mu(s_\alpha, triv,\nu)$ are positive semidefinite for $\mu \in \{\mu_0, \mu_1\}$.

3. Graded Hecke algebra and p-adic groups

3.1. Split p-adic groups. As in the introduction, denote by

$$\begin{aligned} &\mathbb{F} && \text{a } p\text{-adic field of characteristic } 0,\\ &\mathbb{O} && \text{the ring of integers in } \mathbb{F},\\ &\mathcal{P} && \text{the unique prime ideal in } \mathbb{O},\\ &\mathbb{F}_q && \text{the residue field with } q \text{ elements.} \end{aligned} \tag{3.1}$$

Let G denote the $\mathbb{F}$-points of a connected reductive linear algebraic group with root datum $(\mathcal{X}, R, \mathcal{Y}, \check{R})$, split over $\mathbb{F}$, and let B be the Borel subgroup, $H \cong (\mathbb{F}^\times)^{\operatorname{rank} G}$ the Cartan subgroup, $K = \mathbb{G}(\mathbb{O})$ the maximal compact subgroup, ${}^0H = H \cap K \cong (\mathbb{O}^\times)^{\operatorname{rank} G}$ the compact part of H as in section 1.1. We also fix an Iwahori subgroup $\mathcal{I} \subset K$ of G as defined in section 1.3.

Let $\chi\colon H \to \mathbb{C}^\times$ be a character. Recall that χ is unramified if $\chi|_{{}^0H} = 1$, and otherwise χ is called ramified. The minimal principal series $X(\chi)$ is defined by induction, similarly to the one for real groups. It has a finite composition series and, if χ is unramified, it contains a unique irreducible (K-)spherical subquotient. Every irreducible spherical module appears in this way. However, unlike the case of real groups, not every irreducible G-representation can be realized as a subquotient of a minimal (ramified or unramified) principal series.

A remarkable feature of the representation theory of p-adic groups is the that it is often controlled by affine Hecke algebras. This approach originated with the seminal work of Iwahori-Matsumoto ([**IM**]), Borel ([**Bo**]), and Casselman ([**Cas1**]).

Definition 3.1. The *Iwahori-Hecke algebra* $\mathcal{H}(G//\mathcal{I})$ is the algebra of $\mathcal{I}$-biinvariant complex locally constant functions on G with the convolution. Define the category $\mathcal{C}(\mathcal{I}, triv)$ of (admissible) representations of G which are generated by their Iwahori fixed vectors.

Iwahori-Matsumoto have determined the structure of $\mathcal{H}(G//\mathcal{I})$ by a set of generators in correspondence with the affine Weyl groups. We will instead the set of generators and relations introduced first by Bernstein. The Iwahori-Hecke algebra is a specialization of the *affine Hecke algebra* $\mathcal{H}$ which we define now. Let z be an indeterminate (which can then be specialized to $q^{1/2}$). Then $\mathcal{H}$ is an algebra over $\mathbb{C}[z, z^{-1}]$ generated by $\{T_w\}_{w\in W}$ and $\{\theta_x\}_{x\in\mathcal{Y}}$, subject to the relations

$$\begin{aligned} &T_wT_{w'} = T_{ww'}(l(w) + l(w') = l(ww')),\\ &\theta_x\theta_y = \theta_{x+y},\\ &T_s^2 = (z^2-1)T_s + z^2,\\ &\theta_xT_s = T_s\theta_{sx} + (z^2-1)\frac{\theta_x - \theta_{sx}}{1-\theta_\alpha}, \end{aligned} \tag{3.2}$$

where α denotes a simple root, and s is the corresponding simple reflection.

THEOREM 3.2. (1) *(Borel) There exists an equivalence of categories*

$$\mathcal{C}(\mathcal{I}, triv) \longrightarrow \mathcal{H}(G//\mathcal{I})\text{-modules}, \quad V \mapsto V^{\mathcal{I}}. \tag{3.3}$$

(2) *(Casselman) Every irreducible subquotient of the unramified principal series $X(\chi)$ is in $\mathcal{C}(\mathcal{I}, triv)$, and every irreducible object in $\mathcal{C}(\mathcal{I}, triv)$ has this form.*

As explained in section 1.3, this approach has proven successful towards the determination of the unitary dual as well. By [**BM1, BM2**], one reduces the determination of the unitary representations in $\mathcal{C}(\mathcal{I}, 1)$ to that of the unitary dual of $\mathcal{H}(G//\mathcal{I})$, and furthermore to the similar problem for the graded Hecke algebra (section 3.3) introduced in [**Lu1**].

3.2. The Borel-Casselman correspondence was vastly generalized in [**HM**], and in the theory of types ([**BK**] and the references therein). Hecke algebra isomorphisms as in part theorem 3.2.(1) are constructed for many categories of representations of p-adic groups. The Iwahori subgroup $\mathcal{I}$ is replaced by an arbitrary compact open subgroup J, and the trivial character $triv$ of $\mathcal{I}$ is replaced by a character ρ of J. Then one defines the category $\mathcal{C}(J, \rho)$ of representations of G which contain ρ in their restriction to J, and are generated by the ρ-isotypic component. The Iwahori-Hecke algebra is replaced by the algebra $\mathcal{H}(G//J, \rho)$ of (locally constant) complex functions f on G, such that $f(j_1gj_2) = \rho(j_1)f(g)\rho(j_2)$, for all $g \in G, j_1, j_2 \in J$.

In relation to section 2, of particular importance to us will be the case of ramified minimal principal series. The following definitions should be compared with those for real groups in section 2.

Let Δ denote the root system determined by $\mathbb{H}$ and $\mathbb{G}$, Δ^+ the positive roots given by $\mathbb{B}$. Let W be the (finite) Weyl group of Δ. Fix a character χ and let ${}^0\chi = \chi|_{{}^0H} \colon (\mathbb{O}^\times)^{\operatorname{rank} G} \to \mathbb{C}^\times$ be its restriction to ${}^0H$. Define:

$$\begin{aligned} &W_{{}^0\chi} = \text{ stabilizer of } {}^0\chi \text{ in } W\\ &\Delta_{{}^0\chi} = \{\alpha \in \Delta \colon ({}^0\chi) \circ (\check\alpha|_{\mathbb{O}^\times}) = 1\}\\ &\Delta^+_{{}^0\chi} = \Delta_{{}^0\chi} \cap \Delta^+\\ &W^0_{{}^0\chi} = W(\Delta_{{}^0\chi})\\ &R^c_{{}^0\chi} = \{w \in W_{{}^0\chi} \colon w\Delta^+_{{}^0\chi} = \Delta^+_{{}^0\chi}\}. \end{aligned} \tag{3.4}$$

We call $\Delta_{{}^0\chi}$ the *good roots* for ${}^0\chi$. It is easy to see that $W_{{}^0\chi} = W^0_{{}^0\chi} \rtimes R_{{}^0\chi}$.

In [**Ro**], one associates to ${}^0\chi$ a particular compact open subgroup $J_{{}^0\chi}$ of G and a character ρ of $J_{{}^0\chi}$. The pair $(J_{{}^0\chi}, \rho)$ satisfies (among other properties) $J_{{}^0\chi} \cap H = {}^0H$ and $\rho|_{{}^0H} = {}^0\chi$.

THEOREM 3.3 ([**Ro**]). *Let $G_{{}^0\chi}$ denote the split p-adic group determined by $\Delta_{{}^0\chi}$ and $\mathcal{H}(G_{{}^0\chi}//I_{{}^0\chi})$ the corresponding Iwahori-Hecke algebra.*

(1) *There exists a family of $*$-preserving algebra isomorphisms $\mathcal{H}(G//J_{{}^0\chi},\rho) \cong \mathcal{H}(G_{{}^0\chi}//I_{{}^0\chi},1) \rtimes R_{{}^0\chi}$. (Since $R_{{}^0\chi}$ acts on the root datum of $G_{{}^0\chi}$, we can form this extended Hecke algebra.)*
(2) *The group $R_{{}^0\chi}$ is abelian.*
(3) *An irreducible representation of G is in $\mathcal{C}(J_{{}^0\chi},\rho)$ if and only if it is an irreducible subquotient of the ramified principal series $X(\chi)$.*

In conclusion, we see that the representation theory of the unramified principal series is controlled by certain extended Iwahori-Hecke algebras.

One can make a connection with table 1 for split real groups. Recall that, in the real case, every fine M-type $\delta\colon (\mathbb{Z}/2\mathbb{Z})^{\operatorname{rank} G} \to \mathbb{C}^\times$ can be viewed as a character of $(\mathbb{Z}/2\mathbb{Z})^{\operatorname{rank} G}$, or in other words as a $(\operatorname{rank} G)$-tuple of characters of $\mathbb{Z}/2\mathbb{Z}$: $\delta = (\delta_1, \dots, \delta_{\operatorname{rank} G})$, where δ_j is either the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$. Similarly, every character ${}^0\chi\colon {}^0H \to \mathbb{C}^\times$ can be viewed as a $(\operatorname{rank} G)$-tuple of characters of $\mathbb{O}^\times$: ${}^0\chi = ({}^0\chi_1, \dots, {}^0\chi_{\operatorname{rank} G})$, where ${}^0\chi_j\colon \mathbb{O}^\times \to \mathbb{C}^\times$.

Fix a nontrivial quadratic character ${}^0\chi_0$ of $\mathbb{O}^\times$. Then define the correspondence

$$\delta \longrightarrow {}^0\chi_\delta, \ \text{ with } ({}^0\chi_\delta)_j = \begin{cases} 1 & \text{if } \delta_j = triv, \\ {}^0\chi_0 & \text{if } \delta_j = sgn. \end{cases} \tag{3.5}$$

Then all the data associated with δ in the real case is identical with the data associated to ${}^0\chi_\delta$ in the p-adic case. In principle, by an extension of the results in [**BM1, BM2**], one expects that the unitary representations in the ramified principal series in $\mathcal{C}(J_{{}^0\chi},\rho)$ should correspond to the unitary dual of the extended algebra $\mathcal{H}(G_{{}^0\chi}//I_{{}^0\chi},1) \rtimes R_{{}^0\chi}$. The (possibly surprising) observation is that the unitary representations in the nonspherical minimal principal series of the real group are closely related to the unitary dual of the same Hecke algebra. This will become apparent in section 5.

3.3. Definitions. We recall some basic results on graded Hecke algebras. The *affine graded Hecke algebra* $\mathbb{H}$ was introduced in [**Lu1**]. We will only need to consider a special case of the definition. The generators of $\mathbb{H}$ are the elements $\{t_{s_\alpha}\colon \alpha \in \Pi\}$ and $\{\omega\colon \omega \in \mathfrak{h}\}$. Here Π denotes the set of simple roots. As a $\mathbb{C}$-vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}, \tag{3.6}$$

where

$$\mathbb{A} = Sym(\mathfrak{h}). \tag{3.7}$$

The following commutation relations hold:

$$\omega t_{s_\alpha} = t_{s_\alpha} s_\alpha(\omega) + \langle \omega, \alpha \rangle, \quad \alpha \in \Pi,\ \omega \in \mathfrak{h}. \tag{3.8}$$

The center $Z(\mathbb{H})$ of $\mathbb{H}$ consists of the W-invariants in $\mathbb{A}$ ([**Lu1**]):

$$Z(\mathbb{H}) = \mathbb{A}^W. \tag{3.9}$$

On any irreducible $\mathbb{H}$-module, which is necessarily finite dimensional, $Z(\mathbb{H})$ acts by a *central character*. Therefore, the central characters are parameterized by W-conjugacy classes in $\mathfrak{h}$. We will only consider *real* central characters, *i.e.* W-conjugacy classes of hyperbolic semisimple elements in $\check{\mathfrak{h}} \cong \mathfrak{h}^*$.

We say that a module V of $\mathbb{H}$ is *spherical* if the restriction of V to $\mathbb{C}[W]$ contains the trivial W-representation.

Let $\chi \in \check{\mathfrak{h}}$ be a hyperbolic semisimple element. Denote by $\mathbb{C}_\chi$ the corresponding character of $\mathbb{A}$. Then one can form the *spherical principal series* module

$$X(\chi) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\chi. \tag{3.10}$$

It is immediate from the definition that

$$X(\chi) \cong \mathbb{C}[W] \quad \text{as } W\text{-modules}, \tag{3.11}$$

hence $X(\chi)$ contains the trivial W-representation with multiplicity one. Therefore, $X(\chi)$ has a unique spherical subquotient, $L(\chi)$. The following result is well-known, it is the Hecke algebra equivalent of a classical result for groups.

Theorem 3.4. *Let $X(\chi)$ be the spherical principal series defined in (3.6).*

(1) *If χ is dominant, respectively antidominant, then $X(\chi)$ has a unique irreducible quotient, respectively submodule, which is $L(\chi)$.*
(2) *Every irreducible spherical $\mathbb{H}$-module is isomorphic to $L(\chi)$, for some $\chi \in \check{\mathfrak{h}}$.*

3.4. Operators. We would like to consider Hermitian and unitary modules for $\mathbb{H}$. For this, we need a $*$-operation on $\mathbb{H}$. This is given by [**BM3**]:

$$t_w^* = t_{w^{-1}}, \quad w \in W, \qquad \omega^* = -\overline{\omega} + \sum_{\alpha \in \Delta^+} \langle \omega, \alpha \rangle t_{s_\alpha}, \quad \omega \in \mathfrak{h}. \tag{3.12}$$

For every $\alpha \in \Pi$, set

$$r_\alpha = t_{s_\alpha} \check{\alpha} - 1 \tag{3.13}$$

(an element of $\mathbb{H}$). Then, if $w \in W$ and $w = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \dots \cdot s_{\alpha_k}$ is a minimal decomposition of w in W, one can define

$$r_w = r_{\alpha_1} \cdot r_{\alpha_2} \cdot \, \dots \, \cdot r_{\alpha_k}. \tag{3.14}$$

Note that r_w is independent of the choice of the reduced decomposition (*cf.* [**BM3**]), and therefore is well-defined. When $w = w_0$ (the long Weyl group element), we use r_{w_0} to define the *long intertwining operator*:

$$A(w_0, \chi) \colon X(\chi) \longrightarrow X(w_0\chi), \quad t_w \otimes \mathbb{1}_\chi \mapsto t_w r_{w_0} \otimes \mathbb{1}_{w_0\chi}, \ w \in W. \tag{3.15}$$

We obtain the same formula if we replace the r_{w_0} by the element $r_{w_0}(\chi) \in \mathbb{C}[W]$ defined as follows. If

$$w_0 = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \, \dots \, \cdot s_{\alpha_l}$$

is a minimal decomposition of w_0 in W (with $\ell = \#\Delta^+$), then

$$r_{w_0}(\chi) = r'_{\alpha_1} \cdot r'_{\alpha_2} \cdot \, \dots \, \cdot r'_{\alpha_\ell} \tag{3.16}$$

with

$$r'_{\alpha_j} = -\langle \check{\alpha}_j, (s_{\alpha_{j+1}} s_{\alpha_{j+2}} \dots s_{\alpha_\ell})\chi \rangle \, t_{s_{\alpha_j}} - 1. \tag{3.17}$$

The following discussion is again the Hecke algebra analogue of a classical result for groups, we refer the reader for example to [**BM3**] for more details.

Lemma 3.5. *Let $A(w_0, \chi)$ be the operator defined in (3.15). Then $A(w_0, \chi)$ is an intertwining operator, and is polynomial (hence entire) in χ.*

Let (ψ, V_ψ) be a W-type. The operator $A(w_0,\chi)$ induces operators on Hom spaces

$$A_\psi(w_0,\chi)\colon \operatorname{Hom}_W(V_\psi, X(\chi)) \longrightarrow \operatorname{Hom}_W(V_\psi, X(w_0\chi)). \tag{3.18}$$

Furthermore, by Frobenius reciprocity and (3.11), this induces an operator

$$A_\psi(w_0,\chi)\colon V_\psi^* \longrightarrow V_\psi^*. \tag{3.19}$$

The operator $A_{triv}(w_0,\chi)$ is the scalar

$$N(\chi) = \prod_{\alpha\in\Delta^+} (\langle\check\alpha,\chi\rangle + 1). \tag{3.20}$$

We normalize the operators by this scalar and call them $\mathcal{A}(w_0,\chi)$, respectively $\mathcal{A}_\psi(w_0,\chi)$. Then $\mathcal{A}_{triv}(w_0,\chi) = 1$.

Proposition 3.6. *Let $\mathcal{A}(w_0,\chi)$ be the operator defined above. Assume that χ is dominant.*

(1) *The operator $\mathcal{A}(w_0,\chi)$ has no poles.*
(2) *The image of $\mathcal{A}(w_0,\chi)$ is $L(\chi)$.*
(3) *The Hermitian dual of $L(\chi)$ is $L(-\chi)$. Therefore $L(\chi)$ is Hermitian if and only if $w_0\chi = -\chi$. In this case, if $(\ ,\ )_h$ denotes the Hermitian pairing, then the Hermitian form on $L(\chi)$ is*

$$(t_x\otimes 1\!\!1_\chi, t_y\otimes 1\!\!1_\chi) := (t_x\otimes 1\!\!1_\chi, \frac{1}{N(\chi)} t_y r_{w_0}(\chi)\otimes 1\!\!1_{-\chi})_h, \quad x,y\in W. \tag{3.21}$$

It is easy to see that $L(0)$ is (irreducible and) unitary. In general, we have the following unitarity criterion.

Corollary 3.7. *Assume χ is dominant and $w_0\chi = -\chi$. The spherical irreducible module $L(\chi)$ is unitary if and only if the operators $\mathcal{A}_\psi(w_0,\chi)$ are positive semidefinite for all W-types ψ.*

Clearly, this shows that the computation of the spherical unitary dual, for any given Hecke algebra $\mathbb{H}$, is a finite problem.

Example.

(a) The operator on the sign W-type is easy to compute:

$$\mathcal{A}_{sign}(w_0,\chi) = \prod_{\alpha\in\Delta^+} \frac{1-\langle\check\alpha,\chi\rangle}{1+\langle\check\alpha,\chi\rangle}. \tag{3.22}$$

If χ is dominant, the module $X(\chi)$ is irreducible if and only if $A_{sign}(\chi)\neq 0$, that is, if and only if

$$\langle\check\alpha,\chi\rangle \neq 1, \text{ for all } \alpha\in\Delta^+. \tag{3.23}$$

This is the Hecke algebra analogue of the statement that the spherical quotient of the unramified principal series of a split (real or adjoint p-adic) group admits Whittaker models if and only if it is the full principal series.

(b) Let us consider the simplest case, *i.e.* the Hecke algebra of type A_1. Then there are two representations of W, the trivial and the sign. There is a single positive root α. From corollary 3.7 and formula (3.22), it follows that $L(\chi)$ is unitary if and only if

$$-1 \leq \langle\check\alpha,\chi\rangle \leq 1. \tag{3.24}$$

3.5. Let $w_0 = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \ldots \cdot s_{\alpha_l}$ be a minimal decomposition of w_0 in W, and let

$$\mathcal{A}_\psi(w_0, \chi) = \mathcal{A}_\psi(\alpha_1, w_1\chi) \cdots \mathcal{A}_\psi(\alpha_\ell, w_\ell\chi) \tag{3.25}$$

be the corresponding decomposition of $\mathcal{A}_\psi(w_0, \chi)$, with $w_j = (s_{\alpha_{j+1}} s_{\alpha_{j+2}} \cdots s_{\alpha_\ell})$. Then

$$\mathcal{A}_\psi(\alpha_j, \nu) = \begin{cases} 1 & \text{on the } (+1)\text{-eigenspace of } s_{\alpha_j} \text{ on } V_\psi^* \\ \frac{1-\langle\check{\alpha}_j,\nu\rangle}{1+\langle\check{\alpha}_j,\nu\rangle} & \text{on the } (-1)\text{-eigenspace of } s_{\alpha_j} \text{ on } V_\psi^*. \end{cases} \tag{3.26}$$

If α is a simple root, we have the formula

$$t_{s_\alpha} r_w = r_w t_{s_{w^{-1}\alpha}}. \tag{3.27}$$

From this, since $s_{w^{-1}\alpha} = w^{-1} s_\alpha w$, it follows that

$$t_w r_w = r_w t_w, \text{ for all } w \in W. \tag{3.28}$$

Remark 3.8. In particular, for $w = w_0$, we conclude that every operator $\mathcal{A}_\psi(w_0, \chi)$ preserves the $(+1)$ and the (-1) eigenspaces of w_0 on ψ^*.

3.6. Relevant W-types. In light of corollary 3.7, a spherical module $L(\chi)$ is unitary if and only if $\mathcal{A}_\psi(w_0, \chi)$ is positive semidefinite, for *all* W-types $\psi \in \widehat{W}$.

In fact, one determines in [**Ba1**],[**BC2**],[**Ci**] a strict subset of $\widehat{W}$ which is sufficient to detect unitarity. We call this set *the relevant W-types*. Of course, relevant W-types are interesting if and only if they form a small set. (This is the case, for example, for $W(E_8)$: while there are 112 irreducible representations of $W(E_8)$, our set of relevant $W(E_8)$-types contains only nine representations.)

In general, it is still unclear what the best way to define uniformly this set would be. For example, one can notice, after the calculations are done, that a possible set of relevant W-types consists of representations which in the Springer's correspondence ([**Sp**]) are attached to nilpotent orbits of level 4 or less. (One says that a nilpotent element $\check{e}$ has level k if $ad(\check{e})^{k+1} = 0$, but $ad(\check{e})^k \neq 0$.)

We now provide lists of relevant W-types. They will play a role in section 5.7.

Definition 3.9. The following sets of W-types are called *relevant*:

A_{n-1} : $\{(n-m, m) :\ 0 \le m \le [n/2]\}$;
B_n, C_n : $\{(n-m, m) \times (0) :\ 0 \le m \le [n/2]\} \cup \{(n-m) \times (m) : 0 \le m \le n\}$;
D_n : $\{(n-m, m) \times (0) :\ 0 \le m \le [n/2]\} \cup \{(n-m) \times (m) : 0 \le m \le [n/2]\}$;
G_2 : $\{1_1, 2_1, 2_2\}$;
F_4 : $\{1_1, 4_2, 2_3, 8_1, 9_1\}$;
E_6 : $\{1_p, 6_p, 20_p, 30_p, 15_q\}$;
E_7 : $\{1_a, 7'_a, 27_a, 56'_a, 21'_b, 35_b, 105_b\}$;
E_8 : $\{1_x, 8_z, 35_x, 50_x, 84_x, 112_z, 400_z, 300_x, 210_x\}$.

The notation for Weyl group representations is as in [**Car**].

3.7. Extended Hecke algebras. We present some elements of the construction of extended graded Hecke algebras. This construction will be applied to the setting of a Hecke algebra constructed from the set of good coroots and the (dual of the) R-group, as in section 3.2.

Let $\mathbb{H}$ denote the graded Hecke algebra associated to a root system as in the previous section, and let R be a subgroup of the group of automorphisms of the root system for $\mathbb{H}$.

DEFINITION 3.10. We define $\mathbb{H}'$ to be the semidirect product

$$\mathbb{H}' := \mathbb{C}[R] \ltimes \mathbb{H}, \tag{3.29}$$

where the action of R on $\mathbb{H}$ comes from the action of R by outer automorphisms on the root system of $\mathbb{H}$.

Set

$$W' := R \ltimes W. \tag{3.30}$$

In the same way as for usual graded Hecke algebras, one obtains that:

LEMMA 3.11. *The center of* $\mathbb{H}'$ *is* $\mathbb{A}^{W'}$.

For every $\nu \in \mathfrak{h}^*$, we fix the following notation:

$$\begin{aligned} R(\nu) &= \text{the centralizer of } \nu \text{ in } R, \\ \mathbb{A}'(\nu) &= \mathbb{C}[R(\nu)] \ltimes \mathbb{A}, \\ \mathbb{H}'(\nu) &= \mathbb{C}[R(\nu)] \ltimes \mathbb{H}. \end{aligned} \tag{3.31}$$

Let $triv_W$ be the trivial representation of W. This is stabilized by R, so by Mackey induction, any representation λ of R gives rise to a representation of W', which we denote by $\lambda \ltimes triv_W$; any W'-representation containing $triv_W$ in its restriction to W is obtained in this way.

Call an irreducible W'-representation *fine* if it is of the form $\lambda \ltimes triv_W$ for some $\lambda \in \widehat{R}$. Note that if $\rho = \lambda \ltimes triv_W$ is a fine W'-type, then $\mathrm{Hom}_W[\rho : triv_W] \cong \lambda^*$.

DEFINITION 3.12. We call a module π of $\mathbb{H}'$ *quasi-spherical* if $\mathrm{Hom}_W[\pi : triv_W] \neq 0$.

Clearly, any module π containing a fine W'-type is quasi-spherical. Consider the principal series

$$X'(\nu) = \mathbb{H}' \otimes_{\mathbb{A}'(\nu)} \mathbb{C}_\nu = \mathbb{H}' \otimes_{\mathbb{H}'(\nu)} (\mathbb{H}'(\nu) \otimes_{\mathbb{A}'(\nu)} \mathbb{C}_\nu). \tag{3.32}$$

Every fine W'-type $\rho = \lambda \ltimes triv_W$ appears in $X'(\nu)$ with multiplicity $\dim \lambda$. Our main case of interest is when R is abelian, in this case all the multiplicities are one.

We can extend the definition of intertwining operators to this setting. Assume $uw^0 \in R \ltimes W$. Then, similarly to section 3.4, we define a $\mathbb{H}'$-operator

$$A'(uw^0, \nu)\colon X'(\nu) \to X'(uw^0\nu),\ x \otimes \mathbb{1}_\nu \mapsto xur_{w^0} \otimes \mathbb{1}_{uw^0\nu}. \tag{3.33}$$

We normalize the operator $A'(uw^0, \nu)$ to be the identity on the trivial W'-type, and denote the resulting operator by $\mathcal{A}'(uw^0, \nu)$.

For every W'-type ψ', we obtain an operator

$$\mathcal{A}'_{\psi'}(uw^0, \nu)\colon\ \mathrm{Hom}_{W'}(\psi', X'(\nu)) \longrightarrow \mathrm{Hom}_{W'}(\psi', X'(uw^0\nu)). \tag{3.34}$$

REMARK 3.13. When $w^0\nu = -\nu$, the $\mathbb{H}'$-operator $\mathcal{A}'_{\psi'}(w^0, \nu)$ is block diagonal, each block corresponding to a representation λ of $R(\nu)$, such that it is the restriction to $\mathrm{Hom}_{R(\nu)}(\psi', \lambda)$ of $\mathcal{A}_{\psi'|_W}(w^0, \nu)$.

4. Petite K-types for split real groups

In this section, we discuss the construction of petite K-types and the relation between the real intertwining operators from section 2, and the graded Hecke algebra operators from section 3.

4.1. Operators for real split groups. Assume that G is a real split group, let $P = MAN$ be a minimal parabolic subgroup of G. Choose a (fine) K-type δ and a weakly dominant character ν of A. By the results in section 2.12, a Langlands quotient $L(\delta, \nu)$ is Hermitian if and only if there is a Weyl group element satisfying

$$w \cdot Q = \bar{Q}, \quad w \cdot \delta \simeq \delta \quad \text{and} \quad w \cdot \nu = -\bar{\nu}$$

where Q is the the parabolic defined by ν, as in section 2.9. (If ν is strictly dominant, then $Q = P$ and $w = w_0$, the long Weyl group element.) The intertwining operator

$$\mathcal{A}_\mu(w, \delta, \nu) = \mu_\delta(w) A'_\mu(w, \delta, \nu) \colon \operatorname{Hom}_M(\mu, \delta) \to \operatorname{Hom}_M(\mu, \delta)$$

induces a nondegenerate invariant Hermitian on $L(\delta, \nu)$.

The operator $\mathcal{A}_\mu(w, \delta, \nu)$ has been introduced in sections 2.8 and 2.11. We now describe its properties in greater details.

THEOREM 4.1. *Let $w = s_{\alpha_r} s_{\alpha_{r-1}} \dots s_{\alpha_1}$ be a minimal decomposition of w as a product of simple reflections. Then*

$$\mathcal{A}_\mu(w, \delta, \nu) = \prod_{j=1}^{r} \mathcal{A}_\mu(s_{\alpha_j}, \rho_{j-1}, \nu_{j-1}) \tag{4.1}$$

with

$$\mathcal{A}_\mu(s_{\alpha_j}, \rho_{j-1}, \nu_{j-1}) = \mu_\delta(\sigma_{\alpha_j}) A'_\mu(s_{\alpha_j}, \rho_{j-1}, \nu_{j-1}) \tag{4.2}$$

and ρ_j equal to the unique copy of δ_j inside the fine K-type μ_δ.

PROOF. Recall that $A'_\mu(w, \delta, \nu)$ is a normalization of the standard intertwining operator $A_\mu(w, \delta, \nu)$ introduced in section 2.8. As in (2.31), it has a decomposition of the form:

$$A'_\mu(w, \delta, \nu) = \prod_{j=1}^{r} A'_\mu(s_{\alpha_j}, \delta_{j-1}, \nu_{j-1}) \tag{4.3}$$

with $x_0 = 1$, $\delta_0 = \delta = x_0 \cdot \delta$, $\nu_0 = \nu = x_0 \cdot \nu$, and $\delta_j = \underbrace{s_{\alpha_j} s_{\alpha_{j-1}} \dots s_{\alpha_1}}_{x_j} \cdot \delta = x_j \cdot \delta$, $\nu_j = \underbrace{s_{\alpha_j} s_{\alpha_{j-1}} \dots s_{\alpha_1}}_{x_j} \cdot \nu = x_j \cdot \nu$, for $j \geq 1$. The operator $\mathcal{A}_\mu(w, \delta, \nu) = \mu_\delta(w) A'_\mu(w, \delta, \nu)$

inherits a similar decomposition. We can write:

$$\begin{aligned}
\mathcal{A}_\mu(w,\delta,\nu) &= \mu_\delta(w)\left[\prod_{j=1}^{r} A'_\mu(s_{\alpha_j},\delta_{j-1},\nu_{j-1})\right]\\
&= \mu_\delta(x_r)\left[\prod_{j=1}^{r} A'_\mu(s_{\alpha_j},x_{j-1}\cdot\delta,x_{j-1}\cdot\nu)\right]\mu_\delta(x_0)^{-1}\\
&= \prod_{j=1}^{r}\left[\mu_\delta(x_j)A'_\mu(s_{\alpha_j},x_{j-1}\cdot\delta,x_{j-1}\cdot\nu)\mu_\delta(x_{j-1})^{-1}\right]\\
&= \prod_{j=1}^{r}\left[\mu_\delta(\sigma_{\alpha_j})A'_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})\right]\\
&= \prod_{j=1}^{r}\mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1}).
\end{aligned}$$

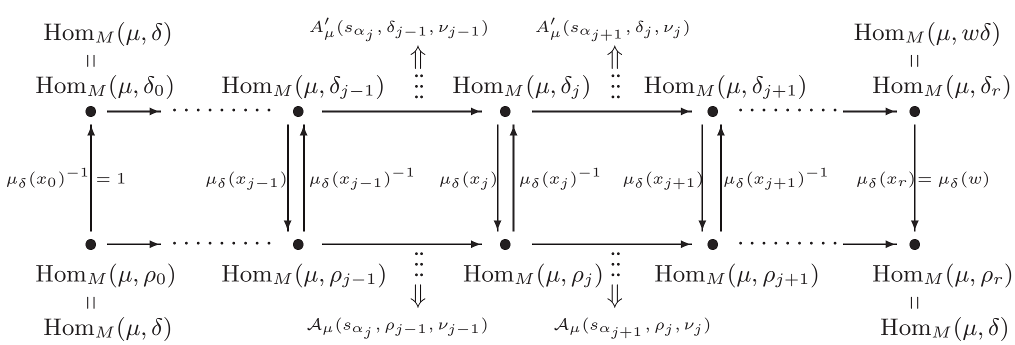
Here ρ_j denotes the unique copy of δ_j inside the (fixed) fine K-type μ_δ. We refer to the factorization

$$\mathcal{A}_\mu(w,\delta,\nu) = \prod_{j=1}^{r}\mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1}) \tag{4.4}$$

as the Gindikin-Karpelevič decomposition of the operator $\mathcal{A}_\mu(w,\delta,\nu)$. Note that every factor

$$\mathcal{A}_\mu(s_{\alpha_i},\rho_{i-1},\nu_{i-1})\colon\ \mathrm{Hom}_M(\mu,\rho_{i-1}) \longrightarrow \mathrm{Hom}_M(\mu,\rho_i)$$

of this decomposition can be regarded as an operator on $\mathrm{Hom}_M(\mu,\mu_\delta)$. We summarize the factorization with the following diagram. The horizontal lines on top represent the decomposition (4.3) of $A'_\mu(w,\delta,\nu)$. The vertical lines are the operators $\mu_\delta(x_j)$'s and their inverses; in particular, the first and last vertical lines are the identity and the operator $\mu_\delta(w)$ respectively. The horizontal lines in the bottom represent the factorization (4.4) of $\mathcal{A}_\mu(w,\delta,\nu)$.



□

Next, we show how to compute the various factors of the decomposition (4.4). Consider the action of $Z^2_{\alpha_j}$ on the space $\mathrm{Hom}_M(\mu,\rho_{j-1})$ defined by

$$T \mapsto T \circ \mathrm{d}\mu(Z_{\alpha_j})^2. \tag{4.5}$$

This action is well defined because $Ad(m)Z_{\alpha_j} = \pm Z_{\alpha_j}$ for all $m \in M$, so $Z^2_{\alpha_j}$ commutes with M. Every eigenvalue of $Z^2_{\alpha_j}$ on $\mathrm{Hom}_M(\mu, \rho_{j-1})$ is of the form $(-l^2)$, with l an integer. Precisely, l is an even integer if α_j is good for ρ_{j-1} and an odd integer otherwise. Write

$$\mathrm{Hom}_M(\mu, \rho_{j-1}) := \bigoplus E^{\alpha_j}(-l^2) \tag{4.6}$$

for the decomposition of $\mathrm{Hom}_M(\mu, \rho_{j-1})$ in generalized eigenspaces for $Z^2_{\alpha_j}$.
Then the operator $\mathcal{A}_\mu(s_{\alpha_j}, \rho_{j-1}, \nu_{j-1})$ acts on $E^{\alpha_j}(-l^2)$ by

$$T \mapsto c_l(\alpha_j, \nu_{j-1})\, \mu_\delta(\sigma_{\alpha_j})\, T\, \mu(\sigma^{-1}_{\alpha_j}), \tag{4.7}$$

where σ_{α_j} is a representative of s_{α_j} in K. The constant $c_l(\alpha_j, \nu_{j-1})$ is equal to 1 for $l = 0$ or 1, and is given by

$$c_{2m+1}(\alpha_j, \nu_{j-1}) = (-1)^m \frac{(2-\lambda)(4-\lambda)\cdots(2m-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2m+\lambda)} \tag{4.8}$$

$$c_{2m}(\alpha_j, \nu_{j-1}) = (-1)^m \frac{(1-\lambda)(3-\lambda)\cdots(2m-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2m-1+\lambda)} \tag{4.9}$$

for $m > 0$. Here $\lambda = \langle \nu_{j-1}, \check{\alpha}_j \rangle$. Notice the similarity with the formulas for the spherical operators of $SL(2, \mathbb{R})$ in section 2.13. Recall from section 2.8.2 that $\mathcal{A}_\mu(s_{\alpha_j}, \rho_{j-1}, \nu_{j-1})$ agrees with a spherical operator for the rank-one group MG^{α_j} on the restriction of μ to MK^{α_j}.

Remark 4.2. For the purpose of computations, it is sometimes convenient to construct simultaneously all the operators $\mathcal{A}_\mu(w, \delta', \nu)$ with δ' an M-type occurring in μ_δ.

Let δ' be any M-type in the W-orbit of δ. The operator $\mathcal{A}_\mu(w, \delta', \nu)$, together with all its factors

$$\mathcal{A}_\mu(s_{\alpha_j}, \rho'_{j-1}, \nu_{j-1})\colon\ \mathrm{Hom}_M(\mu, \rho'_{j-1}) \longrightarrow \mathrm{Hom}_M(\mu, \rho'_j),$$

can be regarded as operators on $\mathrm{Hom}_M(\mu, \mu_\delta)$. Let

$$\mathcal{A}_\mu(w, \nu)\colon\ \mathrm{Hom}_M(E_\mu, E_{\mu_\delta}) \longrightarrow \mathrm{Hom}_M(E_\mu, E_{\mu_\delta}) \tag{4.10}$$

be the direct sum (over all the δ' in the W-orbit of δ) of the operators $\mathcal{A}_\mu(w, \delta', \nu)$.

Via (4.4), the operator $\mathcal{A}_\mu(w, \nu)$ has a decomposition:

$$\mathcal{A}_\mu(w, \nu) = \prod_{j \geq 1} \mathcal{A}_\mu(s_{\alpha_j}, \nu_{j-1}). \tag{4.11}$$

Every factor is an operator on $\mathrm{Hom}_M(E_\mu, E_{\mu_\delta})$, and is easy to compute. Write

$$\mathrm{Hom}_M(E_\mu, E_{\mu_\delta}) = \bigoplus_{l \in \mathbb{N}/2} E^{\alpha_j}(-l^2) \tag{4.12}$$

for the decomposition of $\mathrm{Hom}_M(E_\mu, E_{\mu_\delta})$ into eigenspaces for $Z^2_{\alpha_j}$. Then $\mathcal{A}_\mu(s_{\alpha_j}, \nu_{j-1})$ acts on $E^{\alpha_j}(-l^2)$ by

$$T \mapsto c_l(\alpha_j, \nu_{j-1})\, \mu_\delta(\sigma_{\alpha_j}) T \mu(\sigma^{-1}_{\alpha_j}). \tag{4.13}$$

4.2. Operators on fine K-types. Let μ_δ be our (fixed) fine K-type containing δ (the same one used to define the operator $\mathcal{A}(w,\delta,\nu)$). Our normalization imposes that $\mathcal{A}_{\mu_\delta}(w,\delta,\nu)$ is trivial. We will show that, if μ is *any fine K-type* containing δ, then the operator

$$\mathcal{A}_\mu(w,\delta,\nu) = \prod_{j=1}^{r} \mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})$$

is a (possibly nontrivial) scalar.

Choose μ in $A(\delta)$. To construct the j^{th} factor of the operator $\mathcal{A}_\mu(w,\delta,\nu)$, we look at the action of $Z^2_{\alpha_j}$ on $\mathrm{Hom}_M(\mu,\rho_{j-1})$. Because μ is fine, the only possible eigenvalues of $Z^2_{\alpha_j}$ on this space are 0 and -1, hence

$$\mathrm{Hom}_M(\mu,\rho_{j-1}) = \begin{cases} E^{\alpha_j}_\mu(0) & \text{if } \alpha_j \text{ is good for } \rho_{j-1} \\ E^{\alpha_j}_\mu(-1) & \text{otherwise.} \end{cases} \tag{4.14}$$

In both cases, the operator $\mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})$ acts by $T \mapsto \mu_\delta(\sigma_{\alpha_j})\, T\, \mu(\sigma_{\alpha_j}^{-1})$. This follows immediately from the fact that $c_0(\alpha_j,\nu_{j-1}) = c_1(\alpha_j,\nu_{j-1}) = 1$.

Then, the full intertwining operator

$$\mathcal{A}_\mu(w,\delta,\nu)\colon\ \mathrm{Hom}_M(\mu,\delta) \longrightarrow \mathrm{Hom}_M(\mu,\delta),\ T \mapsto \left[\prod_{j=1}^{r} \mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})\right] T$$

acts by

$$T \mapsto \mu_\delta(\sigma_w)\, T\, \mu(\sigma_w), \tag{4.15}$$

where σ_w is a representative of w in K.

Note that the space $\mathrm{Hom}_M(\mu,\delta)$ is one-dimensional, so $\mathcal{A}_\mu(w,\delta,\nu)$ is a scalar operator. We will now give a different interpretation of this scalar.

For every (not necessarily fine) K-type π containing δ, we define a representation ψ_π of W_δ on $\mathrm{Hom}_M(\pi,\delta)$, as follows. Since δ appears in μ_δ with multiplicity one, we can identify δ with its copy inside μ_δ, and V^δ with $E_{\mu_\delta}(\delta)$ (the isotypic component of δ in μ_δ). Let M'_δ be the preimage of W_δ in the normalizer M' of A in K. The group M'_δ acts on both $E_\pi(\delta)$ and $E_{\mu_\delta}(\delta)$ by restriction of the appropriate action of K, hence it acts on

$$\mathrm{Hom}_M(E_\pi,V^\delta) = \mathrm{Hom}_M(E_\pi,E_{\mu_\delta}(\delta)) = \mathrm{Hom}_M(E_\pi(\delta),E_{\mu_\delta}(\delta))$$

by

$$\sigma\cdot T(v) = \mu_\delta(\sigma)T(\pi(\sigma)^{-1}v) \quad \forall\, v\in E_\pi. \tag{4.16}$$

Because M acts trivially on $\mathrm{Hom}_M(E_\pi,V^\delta)$, we also get a representation ψ_π of W_δ. The restriction of ψ_π to W^0_δ will be denoted by ψ^0_π.

Now assume that π is fine. Let $(M'_\delta)^0$ be the subgroup of M'_δ generated by the elements σ_α for all the good real roots α. Notice that $(M'_\delta)^0$ can be smaller than the preimage of W^0_δ in M', because it may not contain all of M.

The group $(M'_\delta)^0$ acts trivially on the δ-isotypic component of every fine K-type π containing δ. To prove this claim we observe that

$$\pi(\sigma_\alpha^2)v = \pi(m_\alpha)v = \delta(m_\alpha)v = v \quad \forall\, v\in E_\pi(\delta)$$

for all $\sigma_\alpha \in (M'_\delta)^0$ (recall that α is a good real root for δ). So m_α acts trivially on $E_\pi(\delta)$. Because μ is fine and has level at most one, the $(+1)$-eigenspace of

σ_α coincides with the (+1)-eigenspace of m_α (and the (0)-eigenspace to Z_α). In particular, σ_α acts trivially on $E_\pi(\delta)$.

Let us go back to the space

$$\mathrm{Hom}_M(\pi,\delta) = \mathrm{Hom}_M(E_\pi, V^\delta) = \mathrm{Hom}_M(E_\pi(\delta), E_{\mu_\delta}(\delta)).$$

Since π and μ_δ are both fine, the action of $(M'_\delta)^0$ on this space is obviously trivial. The group $W^0_\delta = [M(M'_\delta)^0]/M$ also acts trivially on $\mathrm{Hom}_M(\pi,\delta)$.

Hence, for every fine K-type π containing δ, there is a representation of the R group $R_\delta = W_\delta/W^0_\delta$ on $\mathrm{Hom}_M(E_\pi, V^\delta)$. We denote this representation by ϱ_π. If $[w] \in R_\delta$ and σ is a representative for w in M'_δ, then

$$(\varrho_\pi[w]T)(v) = \mu_\delta(\sigma)T(\pi(\sigma)^{-1}v) \quad \forall\, v \in E_\pi. \tag{4.17}$$

Notice that this is exactly the action of $\mathcal{A}_\pi(w,\delta,\nu)$ on $\mathrm{Hom}_M(E_\pi, V^\delta)$.

Remark 4.3. The representations ψ_μ, ψ^0_μ and ϱ_π depend on the choice of the (fixed) fine K-type $\mu_\delta \in A(\delta)$. This is the same fine K-type used to define the operator $\mathcal{A}(w,\delta,\nu)$.

Proposition 4.4. *Choose a minimal parabolic subgroup $P = MAN$ of G, a representation δ of M and a weakly dominant character ν of A. Suppose that $X(\delta,\nu)$ is Hermitian, and let $w \in W_\delta$ be such that $w\cdot Q = \bar{Q}$ and $w \cdot \nu = -\bar{\nu}$. Denote by σ a representative for w in M'_δ and by $[w]$ the equivalence class of w in R_δ.*

Having fixed a fine K-type $\mu_\delta \in A(\delta)$, we associate a character ϱ_π of R_δ to every fine K-type π containing δ, as above. The intertwining operator

$$\mathcal{A}_\pi(w,\delta,\nu) = \mu_\delta(w)A'_\pi(w,\delta,\nu)$$

acts on $\mathrm{Hom}_M(\pi,\delta)$ *by the scalar* $\varrho_\pi[w]$.

Corollary 4.5. *If $w \in W^0_\delta$, the operator $\mathcal{A}_\pi(w,\delta,\nu)$ is trivial for all $\pi \in A(\delta)$.*

If $w \notin W^0_\delta$ (and $R_\delta \neq \{1\}$), then different fine K-types may get different sign.

4.3. A nonunitarity criterion. A first criterion for nonunitarity is obtained by analyzing the signature of the intertwining operator on the fine K-types.

Proposition 4.6. *Let G be a real split group. Choose a minimal parabolic subgroup $P = MAN$ of G, a representation δ of M and a* strictly *dominant character ν of A. Let $L(\delta,\nu)$ be the corresponding irreducible representation of G. Suppose that $w\cdot P = \bar{P}$ and $w\cdot\nu = -\bar{\nu}$ for some $w \in W_\delta$, so that $L(\delta,\nu)$ is Hermitian, and assume that there is no element $w^0 \in W^0_\delta$ with the same property. Then $L(\delta,\nu)$ is not unitary.*

Proof. Since $A(\delta) = \#R_\delta$ and R_δ is nontrivial, the principal series $X(\delta,\nu)$ contains several fine K-types. Every fine K-type of $X(\delta,\nu)$ is contained in $L(\delta,\nu)$, because $L(\delta,\nu)$ is the unique irreducible subquotient of $X(\delta,\nu)$. So, to prove that $L(\delta,\nu)$ is nonunitary (*i.e.* $\mathcal{A}(w,\delta,\nu)$ is not positive semidefinite), it is sufficient to show that there is a fine K-type π in $A(\delta)$ such that $\mathcal{A}_\pi(w,\delta,\nu) = \varrho_\pi[w] < 0$.

Because R_δ is an abelian two-group, the dual $\widehat{R_\delta}$ is a group isomorphic to R_δ. In particular, the number of distinct characters of R_δ equals the number of fine K-types containing δ. Also notice that, with the notation of the previous section, $\varrho_{\pi_1} = \varrho_{\pi_2}$ if and only if $\pi_1 = \pi_2$. So every character of R_δ is of the form ϱ_π for some $\pi \in A(\delta)$. We conclude that there must be at least one fine K-type π in $A(\delta)$ such that $\varrho_\pi[w] = -1$, and we are done. ∎

Another explanation (suggested by D. Vogan) for why this result should hold is as follows. Choose $G_2 \supset G$ as disconnected as possible (if G is the real points of an algebraic group, then G_2 is the preimage of the real points of the adjoint group of G), and let $M_2A_2N_2$ be a minimal parabolic subgroup of G_2 containing MAN. The group M_2 is still abelian and contains M; we can choose an extension δ_2 of δ to M_2 such that $W_{\delta_2} = W_\delta^0$. Then the principal series $X(\delta_2, \nu)$ of G_2 is not Hermitian, and its restriction to G cannot be unitary. (Note that every fine K-type containing δ appears in the restriction of the unique fine K_2-type containing δ_2.)

We now give some applications of the proposition.

(1) If $\nu = a > 0$, the nonspherical Langlands quotient $L(sign, a)$ of $SL(2,\mathbb{R})$ is Hermitian. The element $w = s_\alpha \in W_{sign}$ carries P into $\bar{P}$ and ν into $-\nu$. Because $W^0_{sign} = \{1\}$, there is no element in W^0_{sign} with the same property. It is not hard to check that the operator $\mathcal{A}(s_\alpha, sign, a)$ takes opposite sign on the two fine $SO(2)$-types ($+1$ and -1) containing $sign$. Both fine $SO(2)$-types are contained in $L(sign, a)$, hence $L(sign, a)$ is not unitary. Notice that the corresponding representation of $G_2 = SL^{\pm}(2,\mathbb{R})$ is not even Hermitian.

(2) If ν is real and strictly dominant, the nonspherical Langlands quotient $L(\delta_p, \nu)$ of $Sp(2n,\mathbb{R})$ is Hermitian (*cf.* section 2.6). The element $w = -I \in W_{\delta_p} = C_{n-p} \times C_p$ carries P into $\bar{P}$ and ν into $-\nu$. Because $W^0_{\delta_p} = C_{n-p} \times D_p$, when p is odd there is no element of $W^0_{\delta_p}$ that can change sign to ν (only an even number of sign changes can occur in the last p entries of ν). It is not hard to check that the operator $\mathcal{A}(-1, \delta_p, \nu)$ takes opposite sign on the two fine $U(n)$-types ($\Lambda^p(\mathbb{C}_n)$ and $\Lambda^p(\mathbb{C}_n)^*$) containing δ_p. Both fine $U(n)$-types are contained in $L(\delta_p, \nu)$, hence $L(\delta_p, \nu)$ is nonunitary.

In these examples $R_\delta = \mathbb{Z}/2\mathbb{Z}$ and $A(\delta)$ contains two fine K-types (which correspond to the trivial and the sign representation of $\mathbb{Z}/2\mathbb{Z}$ respectively). Because $[w]$ is a generator for the R_δ group, the operator $\mathcal{A}(w, \delta, \nu)$ takes opposite sign on the two fine K-types. Hence the Langlands quotient $L(\delta, \nu)$ (which includes both K-types) is nonunitary.

Remark 4.7. In proposition 4.6, ν is required to be dominant, so $X(\delta, \nu)$ has a unique irreducible subquotient (which must of course contain all the fine K-types for δ). If ν is only weakly dominant, the lowest K-types are distributed among the various subquotients. If one of these subquotients contains at least two fine K-types, then a similar nonunitarity test may apply.

Proposition 4.8. *Suppose that*

(1) $R_\delta \neq \{1\}$, *and*
(2) $R_\delta(\nu) \neq R_\delta$

so that there is a subquotient of $X(\delta, \nu)$ *containing (at least) two fine* K*-types* π_1 *and* π_2. *Also assume that* $w \cdot \nu = -\bar{\nu}$ *for some* $w \in W_\delta$, *and that there is no element* $w_0 \in W^0_\delta$ *with the same property. If* $\varrho_{\pi_1}[w] \neq \varrho_{\pi_2}[w]$, *then the Langlands subquotient* $L(\delta, \nu)(\pi_1) = L(\delta, \nu)(\pi_2)$ *is not unitary.*

Petite K-types are meant to provide additional nonunitarity certificates for Langlands quotients of principal series.

4.4. Definition of petite K-type. The definition of petite K-types is rather technical. It is intended to provide a natural relation between intertwining operators for real groups and intertwining operators for graded Hecke algebras. This relation will be detailed in the next few sections.

DEFINITION 4.9. Let μ be a K-type containing δ. We say that μ is "petite for δ" if the following conditions hold.

(1) If α is a good root for δ, and α is simple in both W_δ^0 and W, then the only possible eigenvalues of Z_α^2 on $\mathrm{Hom}_M(\mu,\delta)$ are 0 and -4. This gives:

$$\mathrm{Hom}_M(\mu,\delta) \equiv E^\alpha(0) \oplus E^\alpha(-4).$$

(2) If α is a good root for δ, and α is simple in W_δ^0 but not in W, choose a *minimal* decomposition of s_α in W of the form

$$s_\alpha = (s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_l}) s_\xi (s_{\gamma_l} \cdots s_{\gamma_2} s_{\gamma_1})$$

with
- ξ good for $\tau = (s_{\gamma_l} \cdots s_{\gamma_2} s_{\gamma_1})\delta$, and
- γ_j bad for $\tau_j^- = (s_{\gamma_{j-1}} \cdots s_{\gamma_2} s_{\gamma_1})\delta$ and $\tau_j^+ = (s_{\gamma_{j+1}} \cdots s_{\gamma_l} s_\xi s_{\gamma_l} \cdots s_{\gamma_1})\delta$, for all $j = 1 \dots l$.

Then

(2a) the only possible eigenvalues of Z_ξ^2 on $\mathrm{Hom}_M(\mu,\tau)$ are 0 and -4.

(2b) the only possible eigenvalue of $Z_{\gamma_j}^2$ on $\mathrm{Hom}_M(\mu,\tau_j^\pm)$ is -1.

Note that, in these cases,

$$\mathrm{Hom}_M(\mu,\tau) \equiv E^\xi(0) \oplus E^\xi(-4)$$

and

$$\mathrm{Hom}_M(\mu,\tau^\pm) \equiv E^{\gamma_j}(-1) \quad \forall j = 1 \dots l.$$

(3) If α is a bad root for δ but s_α is in the stabilizer of δ, then the only possible eigenvalue of Z_α^2 on $\mathrm{Hom}_M(\mu,\delta)$ is -1. This gives:

$$\mathrm{Hom}_M(\mu,\delta) \equiv E^\alpha(-1).$$

PROPOSITION 4.10. *K-types of level at most 2 are petite.*

DEFINITION 4.11. Let μ be a K-type containing δ. We say that μ is "quasi-petite for δ" if conditions (1) and (2) hold.

PROPOSITION 4.12. *K-types of level at most 3 are quasi-petite.*

4.5. Spherical Petite K-types. Assume that δ is trivial and ν is weakly dominant. Choose $w \in W$ such that $w \cdot Q = \bar{Q}$, $w \cdot \delta = \delta$ and $w \cdot \nu = -\bar{\nu}$. Note that $W_\delta^0 = W_\delta = W$ in this case.

THEOREM 4.13. *Let δ be the trivial character of M. For every spherical K-type μ, let ψ_μ^0 be the representation of W_δ^0 on $\mathrm{Hom}_M(\mu,\delta)$ introduced in section 4.2. If μ is petite, then*

$$\mathcal{A}_\mu(w,\delta,\nu) = \mathcal{A}_{(\psi_\mu^0)^*}(w,\nu)$$

for all weakly dominant ν.

The operator on the left is an operator for the real group G; the operator on the right is an operator for the affine graded Hecke algebra corresponding to W (*cf.* section 3.3).

Proof. Consider first the Hecke algebra operator

$$\mathcal{A}_{(\psi_\mu^0)^*}(w,\nu)\colon \left(V_{(\psi_\mu^0)^*}\right)^* = \mathrm{Hom}_M(\mu,\delta) \to \left(V_{(\psi_\mu^0)^*}\right)^* = \mathrm{Hom}_M(\mu,\delta) \tag{4.18}$$

(Note that domain and codomain of $\mathcal{A}_{(\psi_\mu^0)^*}$ coincide with the ones of $\mathcal{A}_\mu(w,\delta,\nu)$). If

$$w = s_{\beta_r}\cdots s_{\beta_2}s_{\beta_1} \tag{4.19}$$

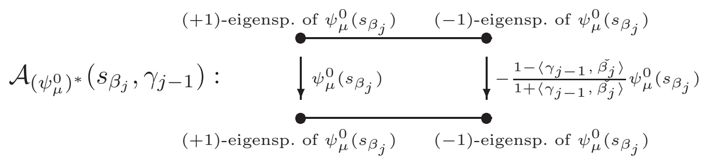
is a minimal decomposition of w in W, then $\mathcal{A}_{(\psi_\mu^0)^*}$ has a Gindikin-Karpelevič factorization of the form

$$\mathcal{A}_{(\psi_\mu^0)^*}(w,\nu) = \prod_{j=1}^{r}\mathcal{A}_{(\psi_\mu^0)^*}(s_{\beta_j},\gamma_{j-1}) \tag{4.20}$$

with $\gamma_{j-1} = s_{\beta_{j-1}}\cdots s_{\beta_2}s_{\beta_1}\nu$ for all $j \geq 1$ and $\gamma_0 = \nu$. The j^{th}-factor of the operator acts by

$$\mathcal{A}_{(\psi_\mu^0)^*}(s_{\beta_j},\gamma_{j-1}) := \begin{cases} 1 & \text{on the } (+1)\text{-eigenspace of } \psi_\mu^0(s_{\beta_j}) \\ \frac{1-\langle\gamma_{j-1},\check{\beta}_j\rangle}{1+\langle\gamma_{j-1},\check{\beta}_j\rangle} & \text{on the } (-1)\text{-eigenspace of } \psi_\mu^0(s_{\beta_j}). \end{cases} \tag{4.21}$$

Here is a picture of the action of $\mathcal{A}_{(\psi_\mu^0)^*}(s_{\beta_j},\gamma_{j-1})$:



Notice that

$$\mathcal{A}_{(\psi_\mu^0)^*}(s_{\beta_j},\gamma_{j-1}) \equiv \frac{1+\langle\gamma_{j-1},\,\check{\beta}_j\rangle\psi_\mu^0(s_{\beta_j})}{1+\langle\gamma_{j-1},\,\check{\beta}_j\rangle}. \tag{4.22}$$

Now look at the real operator. Corresponding to the same minimal decompositions of w in W, there is the factorization (4.4) for $\mathcal{A}_\mu(w,\delta,\nu)$:

$$\mathcal{A}_\mu(w,\delta,\nu) = \prod_{j=1}^{r}\mathcal{A}_\mu(s_{\beta_j},\rho_{j-1},\nu_{j-1}). \tag{4.23}$$

Observe that $\rho_{j-1} = \delta$ and $\nu_{j-1} = \gamma_{j-1}$, for all $j = 1\ldots r$. Most importantly, both $\mathcal{A}_{(\psi_\mu^0)^*}(s_{\beta_j},\gamma_{j-1})$ and $\mathcal{A}_\mu(s_{\beta_j},\delta,\gamma_{j-1})$ are operators on $\mathrm{Hom}_M(\mu,\delta)$. We will prove that the two factors match.

Recall from section 4.1 that the action of $\mathcal{A}_\mu(s_{\beta_j},\delta,\gamma_{j-1})$ on $\mathrm{Hom}_M(\mu,\delta)$ depends on the decomposition of $\mathrm{Hom}_M(\mu,\delta)$ into $Z_{\beta_j}^2$ eigenspaces. In particular, $\mathcal{A}_\mu(s_{\beta_j},\delta,\gamma_{j-1})$ acts on $E^{\beta_j}(-4m^2)$ by

$$T \mapsto c_{2m}(\beta_j,\gamma_{j-1})\,\mu_\delta(\sigma_{\beta_j})\,T\,\mu(\sigma_{\beta_j}^{-1}). \tag{4.24}$$

Because β_j is a good root for δ, we can re-write this action as

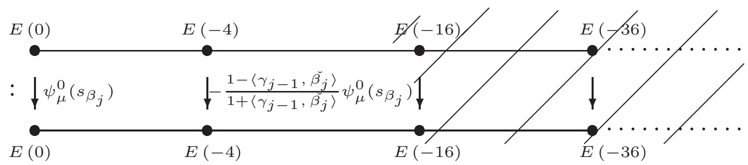
$$T \mapsto c_{2m}(\beta_j,\gamma_{j-1})\,\psi_\mu^0(s_{\beta_j})T. \tag{4.25}$$

Now, because μ is petite, and β_j is a good root for δ (simple in both W and W_δ^0), the only eigenvalues of $Z_{\beta_j}^2$ on $\mathrm{Hom}_M(\mu,\delta)$ are 0 and −4:

$$\mathrm{Hom}_M(\mu,\delta) := E^{\beta_j}(0)\oplus E^{\beta_j}(-4).$$

Hence only the constants $c_0(\beta_j, \gamma_{j-1}) = 1$ and $c_2(\beta_j, \gamma_{j-1}) = -\frac{1-\langle \gamma_{j-1}, \check{\beta}_j \rangle}{1+\langle \gamma_{j-1}, \check{\beta}_j \rangle}$ appear:

$\mathcal{A}_\mu(s_{\beta_j}, \delta, \gamma_{j-1}) :$



Notice that

$$E^{\beta_j}(0) = (+1)\text{-eigenspace of } \psi^0_\mu(s_{\beta_j}) \tag{4.26}$$

$$E^{\beta_j}(-4) = (-1)\text{-eigenspace of } \psi^0_\mu(s_{\beta_j}) \tag{4.27}$$

so

$$\mathcal{A}_\mu(s_{\beta_j}, \delta, \gamma_{j-1}) = \mathcal{A}_{(\psi^0_\mu)^*}(s_{\beta_j}, \gamma_{j-1}) \quad \forall j = 1 \dots r. \tag{4.28}$$

We conclude that

$$\mathcal{A}_\mu(w, \delta, \nu) = \mathcal{A}_{(\psi^0_\mu)^*}(w, \nu). \tag{4.29}$$

□

4.6. Nonspherical petite *K*-types. The matching of operators in the *non-spherical* case is much harder, especially if the K-type is not pseudospherical. In this case, W^0_δ is a proper subgroup of W and the two intertwining operators cannot be compared term by term. We point out some of the difficulties:

- The element w is in the stabilizer of δ in W, but does not necessarily belong to W^0_δ, so the operator $\mathcal{A}_{(\psi^0_\mu)^*}(w, \nu)$ might be meaningless.
- If α is a bad root for δ, and is simple in W, then the α-factor of $\mathcal{A}_\mu(w, \delta, \nu)$ does not have an immediate correspondent in $\mathcal{A}_{(\psi^0_\mu)^*}(w, \nu)$.
- If α is a good root for δ, and is simple in W^0_δ but not in W, then the α-factor of $\mathcal{A}_{(\psi^0_\mu)^*}(w, \nu)$ does not have an immediate correspondent in $\mathcal{A}_\mu(w, \delta, \nu)$.

It is convenient to proceed by increasing level of difficulty. At this stage we will assume that $w \in W^0_\delta$, so that we can choose a minimal decomposition of w in W^0_δ which is "compatible" with the one of w in W.

LEMMA 4.14. *Let $w = s_{\beta_r} \cdots s_{\beta_2} s_{\beta_1}$ be a minimal decomposition of w in W^0_δ. Notice that the roots occurring in this factorization are obviously simple in W^0_δ, but need* not *to be simple in W. If β_i is not simple, we can choose a* minimal *decomposition of s_{β_i} in W such that:*

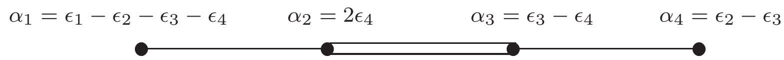
$$s_{\beta_i} = (s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_l}) s_\xi (s_{\gamma_l} \cdots s_{\gamma_2} s_{\gamma_1}) \tag{4.30}$$

with

- *ξ good for $(s_{\gamma_l} \cdots s_{\gamma_2} s_{\gamma_1})\delta$, and*
- *γ_j bad for both $(s_{\gamma_{j-1}} \cdots s_{\gamma_2} s_{\gamma_1})\delta$ and $(s_{\gamma_{j+1}} \cdots s_{\gamma_l} s_\xi s_{\gamma_l} \cdots s_{\gamma_1})\delta$.*

We can choose these minimal decompositions in a way that, after replacing every non simple reflection s_{β_i} by its expression in 4.30, we obtain a minimal decomposition of w in W.

The following example will clarify the lemma. Let G be the split linear group of type F_4, with simple roots:



There is a one-dimensional nongenuine character δ of M (with a three-dimensional orbit $\{\delta, \delta', \delta''\}$ under W) that admits

as a basis for the good roots. Then W_δ^0 is a Weyl group of type C_4. The long Weyl group element $w = -1$ has length 16 in W_δ^0, and length 24 in W. We choose

(4.31) $$w = s_{\alpha_2}(s_{\alpha_3}s_{\alpha_2}s_{\alpha_3})(s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4})(s_\beta s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_\beta)$$

to be a minimal decomposition of w in W_δ^0. Note that β is the only good root which is simple in W_δ^0 but not in W; we decompose it as:

(4.32) $$s_\beta = (s_{\alpha_1}s_{\alpha_2})s_{\alpha_3}(s_{\alpha_2}s_{\alpha_1}).$$

Then

- α_1 is bad for δ. The reflection s_{α_1} carries δ into $\delta' = s_{\alpha_1}\delta$ and δ' into δ.
- α_2 is bad for δ'. The reflection s_{α_2} carries δ' into $\delta'' = s_{\alpha_2}\delta'$ and δ'' into δ'.
- α_3 is good for δ''. . The reflection s_{α_3} obviously stabilizes δ''.
- The composition $s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}$ stabilizes δ.

Replacing every occurrence of s_β in w by the product $s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}$, we obtain a minimal decomposition of w in W:

$$w = s_{\alpha_2}(s_{\alpha_3}s_{\alpha_2}s_{\alpha_3})(s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4})\cdot$$
$$(\underbrace{s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}}_{=s_\beta}s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}\underbrace{s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}}_{=s_\beta}).$$

THEOREM 4.15. *Let δ be a nontrivial representation of M and let ν be a weakly dominant character of A. Assume that there exists $w \in W_\delta^0$ such that $w \cdot Q = \bar{Q}$ and $w \cdot \nu = -\bar{\nu}$. For every K-type μ containing δ, let ψ_μ^0 be the representation of W_δ^0 on $\mathrm{Hom}_M(\mu, \delta)$ introduced in section 4.2. If μ is petite, then*

$$\mathcal{A}_\mu(w, \delta, \nu) = \mathcal{A}_{(\psi_\mu^0)^*}(w, \nu).$$

The operator on the left is an operator for the real group G; the operator on the right is an operator for the affine graded Hecke algebra corresponding to W_δ^0 (*cf.* section 3.3).

PROOF. Suppose that

(4.33) $$w = s_{\beta_r} \cdots s_{\beta_2}s_{\beta_1}$$

and

(4.34)
$$w = s_{\beta_r} \cdots (\underbrace{s_{\gamma_1}s_{\gamma_2} \cdots s_{\gamma_l}s_\xi s_{\gamma_l} \cdots s_{\gamma_2}s_{\gamma_1}}_{s_{\beta_i}})s_{\beta_{i-1}} \cdots (\underbrace{s_{\tau_1}s_{\tau_2} \cdots s_{\tau_l}s_\zeta s_{\tau_l} \cdots s_{\tau_2}s_{\tau_1}}_{s_{\beta_j}}) \cdots s_{\beta_1}$$

are minimal decompositions of w in W_δ^0 and W respectively, that are compatible in the sense of the previous lemma.

Recall that the Hecke algebra operator

$$\mathcal{A}_{(\psi^0_\mu)^*}(w,\nu)\colon \left(V_{(\psi^0_\mu)^*}\right)^* = \mathrm{Hom}_M(\mu,\delta) \to \left(V_{(\psi^0_\mu)^*}\right)^* = \mathrm{Hom}_M(\mu,\delta) \tag{4.35}$$

has the factorization (4.20):

$$\mathcal{A}_{(\psi^0_\mu)^*}(w,\nu) = \prod_{j=1}^{r} \mathcal{A}_{(\psi^0_\mu)^*}(s_{\beta_j},\gamma_{j-1}). \tag{4.36}$$

The j^{th}-factor of $\mathcal{A}_{(\psi^0_\mu)^*}(w,\nu)$ is again an operator on $\mathrm{Hom}_M(\mu,\delta)$, and acts by

$$\mathcal{A}_{(\psi^0_\mu)^*}(s_{\beta_j},\gamma_{j-1}) := \begin{cases} 1 & \text{on the } (+1)\text{-eigenspace of } \psi^0_\mu(s_{\beta_j}) \\ \frac{1-\langle\gamma_{j-1},\check{\beta}_j\rangle}{1+\langle\gamma_{j-1},\check{\beta}_j\rangle} & \text{on the } (-1)\text{-eigenspace of } \psi^0_\mu(s_{\beta_j}). \end{cases} \tag{4.37}$$

The real operator $\mathcal{A}_\mu(w,\delta,\nu)$, on the other hand, has the factorization (4.20). This factorization involves more factors; every factor corresponds to a simple root in W (which might, of course, be either good or bad for δ). We will prove that:

(a) If β is a good root for δ, and β is simple in both W^0_δ and W, then the β-factor of $\mathcal{A}_{(\psi^0_\mu)^*}(w,\nu)$ matches the corresponding β-factor of $\mathcal{A}_\mu(w,\delta,\nu)$.

(b) If β is a good root for δ, and β is simple in W^0_δ but not in W, write

$$s_\beta = (s_{\gamma_1}s_{\gamma_2}\cdots s_{\gamma_l})s_\xi(s_{\gamma_l}\cdots s_{\gamma_2}s_{\gamma_1})$$

for a minimal decomposition of s_β in W as in lemma 4.14. Then the β-factor of $\mathcal{A}_{(\psi^0_\mu)^*}(w,\nu)$ matches the product of the all the factors of $\mathcal{A}_\mu(w,\delta,\nu)$ coming from $(s_{\gamma_1}s_{\gamma_2}\cdots s_{\gamma_l})s_\xi(s_{\gamma_l}\cdots s_{\gamma_2}s_{\gamma_1})$.

In our F_4 example, the matchings are as follows:

(a') $\mathcal{A}_{(\psi^0_\mu)^*}(\alpha_i,\gamma) = \mathcal{A}_\mu(\alpha_i,\delta,\gamma)$ for all $i = 2,3,4$, and all γ.

(b') $\mathcal{A}_{(\psi^0_\mu)^*}(\beta,\gamma) = \mathcal{A}_\mu(\alpha_1,\delta',s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma) \circ \mathcal{A}_\mu(\alpha_2,\delta'',s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\circ$

$$\circ\mathcal{A}_\mu(\alpha_3,\delta'',s_{\alpha_2}s_{\alpha_1}\gamma) \circ \mathcal{A}_\mu(\alpha_2,\delta',s_{\alpha_1}\gamma) \circ \mathcal{A}_\mu(\alpha_1,\delta,\gamma)$$

for all γ.

Instead of proving claims (a) and (b), we will prove (a') and (b') instead. The general idea will emerge from this simpler case.

Condition (a') is easy to prove. Let $\beta = \alpha_i$, for $i = 2,3,4$. Then β is a good root for δ, and is simple in both W and W^0_δ. Because μ is petite, $\mathrm{Hom}_M(\mu,\delta) = E^\beta(0) \oplus E^\beta(-4)$. The same argument used in the spherical case shows that the β-factor of the Hecke algebra operator matches the β-factor of the real operator:

$$\mathcal{A}_{(\psi^0_\mu)^*}(\beta,\gamma) = \mathcal{A}_\mu(\beta,\delta,\gamma) \tag{4.38}$$

for all γ. Notice that both factors act on $\mathrm{Hom}_M(\mu,\delta)$.

Condition (b') is more delicate. Choose $\beta = \alpha_1$ and $s_\beta = (s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1})$. The Hecke algebra operator $\mathcal{A}_{(\psi^0_\mu)^*}(\beta,\gamma)$ acts on $\mathrm{Hom}_M(\mu,\delta)$ in the usual way. The composition of the factors of the real operator corresponding to $(s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1})$ also acts on $\mathrm{Hom}_M(\mu,\delta)$; the single factors, however, do not. Set $\delta' = s_{\alpha_1}\delta$ and

$\delta'' = s_{\alpha_2} s_{\alpha_1}\delta$. Then

$$\mathcal{A}_\mu(s_{\alpha_1},\delta,\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta) \longrightarrow \mathrm{Hom}_M(\mu,\delta') \tag{4.39}$$

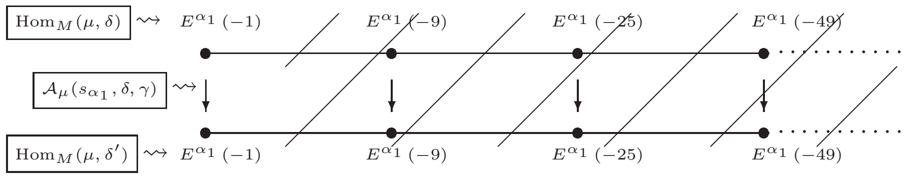
$$\mathcal{A}_\mu(s_{\alpha_2},\delta',s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta') \longrightarrow \mathrm{Hom}_M(\mu,\delta'') \tag{4.40}$$

$$\mathcal{A}_\mu(s_{\alpha_3},\delta'',s_{\alpha_2}s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta'') \longrightarrow \mathrm{Hom}_M(\mu,\delta'') \tag{4.41}$$

$$\mathcal{A}_\mu(s_{\alpha_2},\delta'',s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta'') \longrightarrow \mathrm{Hom}_M(\mu,\delta') \tag{4.42}$$

$$\mathcal{A}_\mu(s_{\alpha_1},\delta',s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta') \longrightarrow \mathrm{Hom}_M(\mu,\delta). \tag{4.43}$$

Because μ is petite and α_1 is bad for both δ and δ', serious restrictions are imposed on the eigenvalues of $\mathrm{d}\mu(Z_{\alpha_1})^2$ on both $\mathrm{Hom}_M(\mu,\delta)$ and $\mathrm{Hom}_M(\mu,\delta')$:



Then, for all γ, we have:

$$\mathcal{A}_\mu(s_{\alpha_1},\delta,\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta) \longrightarrow \mathrm{Hom}_M(\mu,\delta'),\ T \mapsto \mu_\delta(\sigma_{\alpha_1})T\mu(\sigma_{\alpha_1})^{-1} \tag{4.44}$$

and

$$\mathcal{A}_\mu(s_{\alpha_1},\delta',s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta') \longrightarrow \mathrm{Hom}_M(\mu,\delta),\ T \mapsto \mu_\delta(\sigma_{\alpha_1})T\mu(\sigma_{\alpha_1})^{-1}.$$

Note that, because T is M-invariant and $\sigma_{\alpha_1}^2 \in M$, we can also write

$$\mathcal{A}_\mu(s_{\alpha_1},\delta',s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)T = \mu_\delta(\sigma_{\alpha_1})^{-1}T\mu(\sigma_{\alpha_1}). \tag{4.45}$$

There are similar restrictions on the eigenvalues of $\mathrm{d}\mu(Z_{\alpha_2})^2$ on $\mathrm{Hom}_M(\mu,\delta')$ and $\mathrm{Hom}_M(\mu,\delta'')$. Hence

(4.46)

$$\mathcal{A}_\mu(s_{\alpha_2},\delta',s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta') \longrightarrow \mathrm{Hom}_M(\mu,\delta''),\ T \mapsto \mu_\delta(\sigma_{\alpha_2})T\mu(\sigma_{\alpha_2})^{-1}$$

and

(4.47)

$$\mathcal{A}_\mu(s_{\alpha_2},\delta'',s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta'') \longrightarrow \mathrm{Hom}_M(\mu,\delta'),\ T \mapsto \mu_\delta(\sigma_{\alpha_2})^{-1}T\mu(\sigma_{\alpha_2}).$$

We are only missing the central factor, $\mathcal{A}_\mu(s_{\alpha_3},\delta'',s_{\alpha_2}s_{\alpha_1}\gamma)$. Because μ is petite, and α_3 is good for δ'', $Z_{\alpha_3}^2$ acts on $\mathrm{Hom}_M(\mu,\delta'')$ with eigenvalues 0 and (-4). Then

$$\mathcal{A}_\mu(s_{\alpha_3},\delta'',s_{\alpha_2}s_{\alpha_1}\gamma)\colon\ \mathrm{Hom}_M(\mu,\delta'') \longrightarrow \mathrm{Hom}_M(\mu,\delta'') \tag{4.48}$$

acts by

$$T \mapsto \frac{T + \langle s_{\alpha_2}s_{\alpha_1}\gamma, \check{\alpha_3}\rangle \mu_\delta(\sigma_{\alpha_3})T\mu(\sigma_{\alpha_3})^{-1}}{1+\langle s_{\alpha_2}s_{\alpha_1}\gamma,\check{\alpha_3}\rangle}. \tag{4.49}$$

Finally, we look at the composition of the 5 factors:

$$\mathcal{A}_\mu(\alpha_1,\delta',s_{\alpha_2}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\circ\mathcal{A}_\mu(\alpha_2,\delta'',s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\gamma)\circ$$

$$\circ\mathcal{A}_\mu(\alpha_3,\delta'',s_{\alpha_2}s_{\alpha_1}\gamma)\circ\mathcal{A}_\mu(\alpha_2,\delta',s_{\alpha_1}\gamma)\circ\mathcal{A}_\mu(\alpha_1,\delta,\gamma),$$

which acts by

$$T\mapsto\frac{T+\langle s_{\alpha_2}s_{\alpha_1}\gamma,\check{\alpha_3}\rangle\,\mu_\delta(\sigma_\beta)T\mu(\sigma_\beta)^{-1}}{1+\langle s_{\alpha_2}s_{\alpha_1}\gamma,\check{\alpha_3}\rangle}. \tag{4.50}$$

Because β is a good root for δ and

$$\langle s_{\alpha_2}s_{\alpha_1}\gamma,\check{\alpha_3}\rangle=\langle\gamma,s_{\alpha_1}s_{\alpha_2}\check{\alpha_3}\rangle=\langle\gamma,\check{\beta}\rangle,$$

we can re-write this action as

$$T\mapsto\frac{T+\langle\gamma,\check{\beta}\rangle\,\psi^0_\mu(s_\beta)T}{1+\langle\gamma,\check{\beta}\rangle}.$$

Hence the product of operators behaves exactly like $\mathcal{A}_{(\psi^0_\mu)^*}(\beta,\gamma)$. This concludes the proof of the theorem. □

We now go one step forward, and discuss the case $w\in W^\delta$ (not necessarily in W^0_δ). Recall that W is the semidirect product of W^0_δ with

$$R^c_\delta=\{w\in W\colon w({}^\vee\Delta^+_\delta)={}^\vee\Delta^+_\delta\}.$$

R^c_δ is an abelian two group, and is generated by reflections through strongly orthogonal bad roots perpendicular to $\rho(\Delta_\delta)$.

Theorem 4.16. *Let δ be a (nontrivial) representation of M and let ν be a weakly dominant character of A. Assume that there exists $w\in W_\delta$ such that $w\cdot Q=\bar{Q}$ and $w\cdot\nu=-\bar{\nu}$. Write $w=uw^0$, with $u\in R^c_\delta$ and $w^0\in W^0_\delta$. For every K-type μ containing δ, let ψ_μ be the representation of W_δ on $\mathrm{Hom}_M(\mu,\delta)$ introduced in section 4.2, and let ψ^0_μ be its restriction to W^0_δ. If μ is petite, then*

$$\mathcal{A}_\mu(w,\delta,\nu)=\psi_\mu(u)\mathcal{A}_{(\psi^0_\mu)^*}(w^0,\nu).$$

The operator on the left is an operator for the real group G; the operator on the right is an operator for the extended Hecke algebra corresponding to W_δ (*cf.* section 3.7).

Proof. Choose a minimal decomposition of $w=uw^0$ in W of the form

$$w=\underbrace{s_{\zeta_t}\cdots s_{\zeta_1}}_{u}\underbrace{s_{\alpha_s}\cdots s_{\alpha_1}}_{w^0} \tag{4.51}$$

with $s_{\zeta_i}\in R^c_\delta$ and $(s_{\alpha_s}\cdots s_{\alpha_1})$ a minimal decomposition of w^0 in W. Notice that $\zeta_1\ldots\zeta_t$ are bad roots for δ, but the corresponding reflections stabilize δ.

The intertwining operator $\mathcal{A}_\mu(w,\delta,\nu)$ factors:

(4.52)

$$\mathcal{A}_\mu(w,\delta,\nu)=\mathcal{A}_\mu(s_{\zeta_t},\delta,\nu_{s+t-1})\cdots\mathcal{A}_\mu(s_{\zeta_1},\delta,\nu_s)\left[\prod_{j=1}^{s}\mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})\right].$$

If μ is petite, then

$$\left[\prod_{j=1}^{s}\mathcal{A}_\mu(s_{\alpha_j},\rho_{j-1},\nu_{j-1})\right]=\mathcal{A}_{(\psi^0_\mu)^*}(w^0,\nu) \tag{4.53}$$

by theorem 4.15.

Let us look at the remaining factors of the operator $\mathcal{A}_\mu(w,\delta,\nu)$. Since s_{ζ_i} stabilizes δ, each $\mathcal{A}_\mu(s_{\zeta_i},\delta,\nu_{r+i-1})$ is an operator on $\mathrm{Hom}_M(\mu,\delta)$. Because μ is petite, $Z^2_{\gamma_i}$ acts on $\mathrm{Hom}_M(\mu,\delta)$ with eigenvalue (-1). Then

$$\mathcal{A}_\mu(s_{\zeta_i},\delta,\nu_{s+i-1})\colon \mathrm{Hom}_M(\mu,\delta)\longrightarrow \mathrm{Hom}_M(\mu,\delta),\; T\mapsto \mu_\delta(\sigma_{\zeta_i})T\mu(\sigma_{\zeta_i})^{-1} \tag{4.54}$$

for all $i=1\dots t$. We can re-write this action in terms of the W_δ-representation ψ_μ introduced in section 4.2:

$$\mathcal{A}_\mu(s_{\zeta_i},\delta,\nu_{s+i-1})\colon \mathrm{Hom}_M(\mu,\delta)\longrightarrow \mathrm{Hom}_M(\mu,\delta),\; T\mapsto \psi_\mu(s_{\zeta_i})T. \tag{4.55}$$

Composing the various factors of the operator we find:

$$\mathcal{A}_\mu(w,\delta,\nu)=\underbrace{\psi_\mu(s_{\zeta_{s+t-1}})\cdots\psi_\mu(s_{\zeta_1})}_{\psi_\mu(u)}\mathcal{A}_{(\psi_\mu^0)*}(w^0,\nu)=\psi_\mu(u)\mathcal{A}_{(\psi_\mu^0)*}(w^0,\nu). \tag{4.56}$$

This conclues the proof of the theorem. □

Remark 4.17. If μ is fine, then

$$\mathcal{A}_\mu(w,\delta,\nu)=\psi_\mu(u)\mathcal{A}_{(\psi_\mu^0)*}(w,\nu)=\psi_\mu(u)\psi_\mu^0(w^0)=\psi_\mu(w)=\varrho_\mu[w]$$

(in accordance with proposition 4.4).

4.7. Matching of unitary duals. Let G be a split real group. Suppose, for simplicity, that the principal series $X^G(\delta,\nu)$ has a unique irreducible Langlands subquotient $L^G(\delta,\nu)$. Let $w\in W$ be such that

$$w\cdot\delta\simeq\delta\quad w\cdot\nu=-\bar\nu\quad\text{and}\quad w\cdot Q=\bar Q.$$

The unitarity of the Langlands quotient $L^G(\delta,\nu)$ depends on the signature of the Hermitian intertwining operator $\mathcal{A}(w,\delta,\nu)$. More precisely, $L^G(\delta,\nu)$ is unitary if and only if $\mathcal{A}_\mu(w,\delta,\nu)$ is positive-semidefinite for all $\mu\in\widehat{K}$ containing δ.

Because the Weyl group element w stabilizes δ, we can write $w=uw^0$, with $u\in R^c_\delta$ and $w^0\in W^0_\delta$. If μ is a *petite* K-type containing δ, then

$$\mathcal{A}_\mu(w,\delta,\nu)=\psi_\mu(u)\mathcal{A}_{(\psi_\mu^0)*}(w^0,\nu). \tag{4.57}$$

Here ψ_μ is the W_δ-representation on $\mathrm{Hom}_M(\mu,\delta)$ and ψ^0_μ is its restriction to W^0_δ. The operator $\mathcal{A}_{(\psi_\mu^0)*}(w^0,\nu)$ is a spherical intertwining operator for a graded Hecke algebra $\mathbb{H}^0$ defined as in section 3.3, with $W{=}W^0_\delta$ and Π the simple roots of Δ_δ.

Recall from section 3.6 that the unitarity of the irreducible spherical $\mathbb{H}^0$-module $L^{\mathbb{H}^0}(\nu)$ is detected by a finite number of *relevant* W^0_δ-types.

If $A(\delta)$ has cardinality one, then $W^0_\delta=W_\delta$, R_δ is trivial and $u=1$. Hence

$$\mathcal{A}_\mu(w,\delta,\nu)=\mathcal{A}_{(\psi_\mu^0)*}(w,\nu) \tag{4.58}$$

for all μ petite.

Suppose that every relevant W^0_δ type comes from a petite K-type via the correspondence $\mu\to\psi^0_\mu$. In this case, the unitarity of the G-module $L^G(\delta,\nu)$ implies the unitarity of Hecke algebra-module $L^{\mathbb{H}^0}(\nu)$.

$L^G(\delta,\nu)$ is unitary	=====================>	$L^{\mathbb{H}^0}(\nu)$ is unitary
$\Updownarrow$		$\Updownarrow$

$\mathcal{A}_\mu(w,\delta,\nu)\geq 0$ $\forall\,\mu\in\widehat{K}$	$\Rightarrow$	$\mathcal{A}_\mu(w,\delta,\nu)\geq 0$ $\forall\,\mu$ petite	$\Rightarrow$	$\mathcal{A}_\tau(w,\nu)\geq 0$ $\forall\,\tau$ relevant	$\Leftrightarrow$	$\mathcal{A}_\tau(w,\nu)\geq 0$ $\forall\,\tau\in\widehat{W^0_\delta}$

It follows that the portion of the unitary dual of G induced by δ is embedded in the spherical unitary dual of $\mathbb{H}^0$.

THEOREM 4.18 ([**Ba1**],[**Ba2**]). *Let G be any real split group and let δ be the trivial representation of M. Then $R_\delta=\{1\}$, $W^0_\delta=W_\delta=W$ and every relevant W-type comes from a petite K-type (via $\mu\to\psi_\mu$).*

As a consequence, the spherical unitary dual of G is embedded in the spherical unitary dual of the corresponding Hecke algebra $\mathbb{H}$.

Suppose that $A(\delta)$ has cardinality bigger than one, so that R_δ is nontrivial. Because we are assuming the existence of a unique irreducible subquotient, every fine K-type in $A(\delta)$ is contained in $L^G(\delta,\nu)$. We distinguish two cases:

(a) $w\in W^0_\delta$
(b) $w\in W_\delta\diagdown W^0_\delta$, and there is no $w^0\in W^0_\delta$ satisfying

$$w^0\cdot\nu=-\bar{\nu}\quad\text{and}\quad w^0\cdot Q=\bar{Q}.$$

If $w\in W^0_\delta$, the operator $\mathcal{A}(w,\delta,\nu)$ acts trivially on every fine K-type contained in $L^G(\delta,\nu)$, and acts by

$$\mathcal{A}_{(\psi^0_\mu)^*}(w,\nu)$$

on every petite K-type containing δ. Assume that every relevant W^0_δ type can be matched with a petite K-type. Then the same analysis performed above shows that $L^G(\delta,\nu)$ is unitary only if $L^{\mathbb{H}^0}(\delta,\nu)$ is unitary. Hence we obtain an embedding of unitary duals.

If $w\in W_\delta\diagdown W^0_\delta$, and there is no $w^0\in W^0_\delta$ satisfying

$$w^0\cdot\nu=-\bar{\nu}\quad\text{and}\quad w^0\cdot Q=\bar{Q},$$

then there are at least two fine K-types π_1 and π_2 such that

$$\psi_{\pi_1}(w)=\varrho_{\pi_1}[w]=-\varrho_{\pi_2}[w]=-\psi_2(w). \tag{4.59}$$

Because the operators $\mathcal{A}_{\pi_1}(w,\delta,\nu)$ and $\mathcal{A}_{\pi_1}(w,\delta,\nu)$ have opposite sign, the Langlands quotient $L^G(\delta,\nu)$ is not unitary.

4.7.1. *Example:* $Sp(2n,\mathbb{R})$. Let G be the real split group $Sp(2n,\mathbb{R})$. Then $K=U(n)$ and $M=(\mathbb{Z}_2)^n$. There are $(n+1)$ W-conjugacy classes of M-types: the spherical M-type

$$\delta_0=\textit{trivial}$$

and the nonspherical M-types

$$\delta_p=(\underbrace{+,+,\dots,+}_{n-p},\underbrace{-,-,\dots,-}_{p})\qquad p=1\dots n.$$

Recall from section 2.6 that $W^0_{\delta_0}=W_{\delta_0}=W$, while $W^0_{\delta_p}=C_{n-p}\times D_p$ and $W_{\delta_p}=C_{n-p}\times C_p$ for $p\geq 1$. The R_{δ_p}-group is $\frac{\mathbb{Z}}{2\mathbb{Z}}$, and indeed δ_p is contained into two fine K-types:

$$\mu_p(+)=(\underbrace{1,\dots,1}_{p},\underbrace{0,\dots,0}_{n-p})=\Lambda^p(\mathbb{C}^n)$$

and

$$\mu_p(-) = (\underbrace{0, \dots, 0}_{n-p}, \underbrace{-1, \dots, -1}_{p}) = [\Lambda^p(\mathbb{C}^n)]^*.$$

If $p = 1 \dots n$, we will assume that the last entry of

$$\nu = (a_1, a_2, \dots, a_n)$$

is nonzero and that p is even. Under these assumptions, the principal series $X(\delta_p, \nu)$ has a unique irreducible subquotient and there exists $w \in W^0_{\delta_p}$ mapping P into $\bar{P}$ and ν into $-\bar{\nu}$.

The case of $Sp(2n, \mathbb{R})$ is very special, because every relevant W_δ-type can be matched with a petite K-type (via $\psi_\mu \leftrightarrow \mu$). The matching is as follows:

relevant $W(C_n)$-type ψ	spherical petite K-type such that $\psi_\mu = \psi$
$(n-k) \times (k)$	$(\underbrace{2, \dots, 2}_{k}, 0, \dots, 0)$
$(n-k, k) \times (0)$	$(\underbrace{1, \dots, 1}_{k}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{k})$

relevant $W(C_{n-p} \times C_p)$-type ψ	nonspherical petite K-type such that $\psi_\mu = \psi$
$(triv) \otimes [(a, p-a) \times (0)]$	$(\underbrace{1, \dots, 1}_{a}, \underbrace{1, \dots, 1}_{n-p}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{a})$
$(triv) \otimes [(p-c) \times (c)]$	$(\underbrace{2, \dots, 2}_{c}, \underbrace{1, \dots, 1}_{n-p}, 0, \dots, 0)$
$[(n-p-c) \times (c)] \otimes (triv)$	$(\underbrace{1, \dots, 1}_{n-p-c}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{c})$
$[(c, n-p-c) \times (0)] \otimes (triv)$	$(\underbrace{2, \dots, 2}_{c}, \underbrace{1, \dots, 1}_{n-p-2c}, \underbrace{0, \dots, 0}_{p+c})$

To prove that the K-types in these tables are actually petite, we observe that they all appear in the tensor products

$$\mu_+(p) \otimes \mu_-(p) = \sum_{2a+b=2p} (\underbrace{1,\dots,1}_{a}, \underbrace{0,\dots,0}_{b}, \underbrace{-1,\dots,-1}_{n-a-b}),$$
$$\mu_+(p) \otimes \mu_+(k) = \sum_{2a+b=2p} (\underbrace{2,\dots,2}_{a}, \underbrace{1,\dots,1}_{b}, \underbrace{0,\dots,0}_{n-a-b}).$$

Because $\mu_\pm(p)$ has level 1, every summand of these decompositions has level at most two and is automatically petite. We get another set of petite K-types by changing all the signs to minuses, *i.e.* by passing to the dual.

The existence of this matching between relevant W_δ-types and petite K-types containing δ allows us to draw the following conclusions.

- If $\delta = \delta_0$, let $\mathbb{H}$ be the affine graded Hecke algebra corresponding to W (defined as in section 3.3). Then the spherical Langlands quotient $L^G(\delta_0, \nu)$ is unitary only if the spherical Hecke algebra-module $L^{\mathbb{H}}(\nu)$ is unitary.
- If $\delta = \delta_p$, assume that $a_n \neq 0$ and p is even. Let $\mathbb{H}^0$ be the affine graded Hecke algebra corresponding to W_δ^0 (defined as in section 3.3). Then the nonspherical Langlands quotient $L^G(\delta_p, \nu)$ is unitary only if the spherical Hecke algebra-module $L^{\mathbb{H}^0}(\nu)$ is unitary.

4.7.2. *Remarks.* So far we assumed that the principal series $X(\delta, \nu)$ had a unique irreducible subquotient. We conclude this section with some brief remarks on the general case.

The dual R-group $\widehat{R_\delta}$ acts simply transitively on $A(\delta)$, so the number of fine K-types containing δ is equal to the cardinality of R_δ (recall that R_δ is an abelian group, isomorphic to its dual). Two fine K-types occur in the same irreducible subquotient if and only if they lie in the same orbit of $R_\delta^\perp(\nu)$ (the annihilator of $R_\delta(\nu)$ inside $\widehat{R_\delta}$). Every subquotient contains the same number of fine K-types, equal to the cardinality of $R_\delta^\perp(\nu)$. We conclude that there are exactly $\frac{\#\widehat{R_\delta}}{\#R_\delta^\perp(\nu)}$ irreducible subquotients. It is easy to see that $R_\delta^\perp(\nu)$ is the Kernel of the restriction map

$$\widehat{R_\delta} \to \widehat{R_\delta(\nu)},\ \chi \mapsto \chi|_{R_\delta(\nu)}.$$

(Since the groups are abelian, all irreducible representations are one-dimensional, so this is map is well defined and surjective.) Hence

$$\#\text{ irreducible subquotients } = \# \frac{\widehat{R_\delta}}{R_\delta^\perp(\nu)} = \#\widehat{R_\delta(\nu)} = \#R_\delta(\nu).$$

Now suppose that $R_\delta(\nu)$ is not trivial. Then the principal series $X(\delta, \nu)$ contains several irreducible subquotients, and the intertwining operator $\mathcal{A}_\mu(w, \delta, \nu)$ has a block diagonal structure (with one block per subquotient). The blocks are in one-to-one correspondence the set of orbits of $R_\delta^\perp(\nu)$ in $A(\delta)$.

If μ is petite, the operator $\mathcal{A}_\mu(w, \delta, \nu)$ matches the operator $\mathcal{A}'_{\psi_\mu^*}(w, \nu)$ for the quasi-spherical module $X'(\nu)$ for the extended graded Hecke algebra $\mathbb{H}'$ corresponding to W_δ (*cf.* section 3.7). The operator $\mathcal{A}'_{\psi_\mu^*}(w, \nu)$ also has a block diagonal structure, with one block per character of $R_\delta(\nu)$.

We note that there is a one-to-one correspondence between blocks of $\mathcal{A}_\mu(w,\delta,\nu)$ and blocks of $\mathcal{A}'_{\psi_\mu^*}(w,\nu)$. Equivalently, there is a bijection between orbits of $R_\delta^\perp(\nu)$ in $A(\delta)$ and characters of $R_\delta(\nu)$: fix a base point μ_δ in $A(\delta)$ (*i.e.* a fine K-type containing δ) and identify $A(\delta)$ with $\widehat{R_\delta}$ via $\mu \mapsto \psi_\mu$. The set of orbits is then identified with $\frac{\widehat{R_\delta}}{R_\delta^\perp(\nu)}$, which we know is isomorphic to $\widehat{R_\delta(\nu)}$.

Then, to check the signature of the Hermitian form on a petite K-type μ occurring in the subquotient $L(\delta,\nu)(\mu_\delta)$, one can look at the the signature of the appropriate block of the Hecke algebra operator $\mathcal{A}'_{\psi_\mu^*}(w,\nu)$. Similarly for the other subquotients (but an issue with the normalization may arise).

5. Spherical unitary dual

In this section we describe the spherical unitary modules, with real infinitesimal (central) character. To distinguish between the real and the p-adic case, we will denote by $L^{\mathbb{R}}(\chi)$ and $L^{\mathbb{H}}(\chi)$ the corresponding spherical modules for $G(\mathbb{R})$, respectively Hecke algebra $\mathbb{H}$, and similarly for all related notation.

5.1. Parameters. To every $\chi \in \check{\mathfrak{h}}_{\mathbb{R}}$ we attach uniquely a nilpotent $\check{G}$-orbit $\check{\mathcal{O}}(\chi)$ in $\check{\mathfrak{g}}$ as follows. Consider

$$\check{\mathfrak{g}}_{1,\chi} = \{x \in \check{\mathfrak{g}} : ad(\chi)(x) = x\}, \quad \check{G}_{0,\chi} = \{g \in \check{G} : Ad(g)(\chi) = \chi\}. \tag{5.1}$$

It is known that $\check{G}_{0,\chi}$ acts with finitely many orbits on $\check{\mathfrak{g}}_{1,\chi}$, and as a consequence, there is a unique open orbit in $\mathfrak{g}_{1,\chi}$. Let $\check{\mathcal{O}}(\chi)$ denote the $\check{G}$-saturation of this open orbit. For the classification and relevant facts about nilpotent orbits in complex Lie algebras, we refer the reader to [**Car**] and [**CM**].

Let $\check{\mathcal{O}}$ be a nilpotent $\check{G}$-orbit in $\check{\mathfrak{g}}$.

Definition 5.1 (1). The $\check{\mathcal{O}}$*-complementary series* are the sets

$$CS^{\mathbb{H}}(\check{\mathcal{O}}) = \{\chi : L^{\mathbb{H}}(\chi) \text{ is unitary and } \check{\mathcal{O}}(\chi) = \check{\mathcal{O}}\}, \text{ in the } p\text{-adic case,} \tag{5.2}$$

$$CS^{\mathbb{R}}(\check{\mathcal{O}}) = \{\chi : L^{\mathbb{R}}(\chi) \text{ is unitary and } \check{\mathcal{O}}(\chi) = \check{\mathcal{O}}\}, \text{ in the real case.} \tag{5.3}$$

When we want to refer to both sets simultaneously, we will just use the notation $CS(\check{\mathcal{O}})$.

The spherical unitary parameters are the disjoint union of the corresponding complementary series $\sqcup_{\check{\mathcal{O}}} CS(\check{\mathcal{O}})$.

Fix a Lie triple $\{\check{e},\check{h},\check{f}\}$ in $\check{\mathcal{O}}$. Let $\mathfrak{z}(\check{e},\check{h},\check{f})$ denote the centralizer in $\check{\mathfrak{g}}$ of $\check{e},\check{h},\check{f}$. Then the orbit $\check{\mathcal{O}}(\chi)$ can be described differently. By [**BM1**], the orbit $\check{\mathcal{O}}(\chi)$ is the unique one satisfying the conditions:

(5.4)
(1) $w\chi = \check{h}/2 + \nu$, for some $\nu \in \mathfrak{z}(\check{e},\check{h},\check{f})$, and
(2) $\check{\mathcal{O}}(\chi)$ is maximal with respect to condition (1).

Clearly $\check{\mathcal{O}}(\check{h}/2) = \check{\mathcal{O}}$. In fact these parameters are special, they are instances of *Arthur parameters*.

Definition 5.2 (2). The modules $L^{\mathbb{H}}(\check{h}/2)$ and $L^{\mathbb{R}}(\check{h}/2)$ are called *special unipotent*.

The conjectures in [**Ar**] suggest that the special unipotent parameters should be unitary. In the Hecke algebra case, there exists the Iwahori-Matsumoto involution IM which preserves unitarity and takes the special unipotent modules to tempered modules. The unitarity of the corresponding group representations when G is p-adic is then implied by [**BM1**].

In the real case, the unitarity of special unipotent modules is proved in [**Ba1**] for split classical groups. This is beyond the scope of this exposition, and we refer the reader to section 9 in [**Ba1**] for details.

Returning to the $\check{\mathcal{O}}$-complementary series, we see that $CS^{\mathbb{H}}(\check{\mathcal{O}})$ contains at least one element, $\check{h}/2$, corresponding to the special unipotent. In fact, when $\check{\mathcal{O}}$ is a distinguished orbit, the conditions (5.4) imply that $\check{h}/2$ is the unique element of $CS^{\mathbb{H}}(\check{\mathcal{O}})$.

5.2. 0-complementary series. The basic case one needs to compute is when $\check{\mathcal{O}}$ is the trivial nilpotent orbit. The parameters χ, which we will assume dominant, such that $\check{\mathcal{O}}(\chi)=0$ are precisely those such that

$$\langle\check{\alpha},\chi\rangle\neq 1,\ \text{for any } \alpha\in\Delta^+. \tag{5.5}$$

In the (adjoint) p-adic case, these parameters correspond to the modules which are both spherical and admit Whittaker models, in other words, to the irreducible principal series $X(\chi)$.

It is clear that the operators $\mathcal{A}_\mu(w_0,\chi)$ in section 3.4 are isomorphisms in any open region of the complement of the hyperplane arrangement given by (5.5) in the dominant Weyl chamber in $\check{\mathfrak{h}}_{\mathbb{R}}$. Due to the Hermitian condition, we may only consider χ lying in the (-1)-eigenspace of w_0:

$$E_0=\{\chi\in\check{\mathfrak{h}}_{\mathbb{R}}: w_0\chi=-\chi\}. \tag{5.6}$$

Consequently, these operators have constant signature inside any such open regions, and we see that $CS(0)$ is a union of open regions intersected with E_0.

Let $\mathcal{C}_0$ denote the *fundamental alcove*:

$$\mathcal{C}_0=\{\chi\in\check{\mathfrak{h}}_{\mathbb{R}}: 0\leq\langle\check{\alpha},\chi\rangle<1,\ \text{for all } \alpha\in\Pi\}. \tag{5.7}$$

Any open region conjugate to $\mathcal{C}_0$ by the affine Weyl group is called an alcove.

The following result is well-known. We will regard $CS(0)$ in this section as a subset of the fundamental Weyl chamber, *i.e.* we will only consider dominant χ.

LEMMA 5.3. (1) $\mathcal{C}_0\cap E_0\subset CS(0)$.
(2) *Every open region contributing to $CS(0)$ is bounded.*

Part (1) is implied by the fact that $X(0)$ is irreducible and unitary. Part (2) is a signature calculation. In the Hecke algebra setting, one can easily show that $\mathcal{A}_\mu(w_0,\nu)$, with μ the reflection W-representation, can only be positive definite in the bounded regions.

The first result is the determination of the 0-complementary series. The proof for classical groups is in [**BM3**] and [**Ba1**], with a different proof in [**BC3**], while for exceptional groups it is in [**Ci**] for F_4 and [**BC1**] for E_6,E_7,E_8. Earlier, for type A, this result was obtained in [**Vo1**] in the real case and [**Ta**] in the p-adic case, while for G_2, it is part of [**Vo2**] for the real case, and [**Mu**] for the p-adic.

THEOREM 5.4. *The 0-complementary series $CS(0)$ is formed of:*

A. $\mathcal{C}_0 \cap E_0$;
B. $\mathcal{C}_0$;
C,D. $2^{[n/2]}$ *alcoves intersected with* E_0, *where* n *is the rank of the group;*
G2. 2 *alcoves;*
F4. 2 *alcoves;*
E. 2 *alcoves intersected with* E_0 *for* E_6, 2^3 *alcoves for* E_7, *and* 2^4 *alcoves for* E_8.

In explicit coordinates, in type C_n *or* D_n, *if* $\chi = (\nu_1, \dots, \nu_n)$, *with* $w_0\chi = -\chi$, *the set* $CS(0)$ *is formed of the parameters satisfying the following condition: there exists an index* i *such that*

$$0 \le \nu_1 \le \cdots \le \nu_i < 1 - \nu_{i-1} < \nu_{i+1} < \cdots < \nu_n < 1, \text{ and between any } \nu_j < \nu_{j+1}, \tag{5.8}$$
$$i \le j < n, \text{ there is an odd number of } (1 - \nu_l),\ 1 \le l < i.$$

The explicit description for the exceptional types is in section 5.3.

In the p-adic case, the proof has two components:

(1) in any region which has a wall of the form $\langle \check\alpha, \chi \rangle = 0$, for some simple coroot $\check\alpha$, one knows by unitary induction the signature of the operators $\mathcal{A}^{\mathbb{H}}_{\psi}(w_0, \nu)$ on every W-type ψ.
(2) any open region for which all the walls are of the form $\langle \check\alpha, \chi \rangle = 1$, for $\alpha \in \Delta^+$, cannot be unitary. One can use a signature argument here, for example in the classical cases one can show that the signature of the operator $\mathcal{A}^{\mathbb{H}}_{\psi}(w_0, \chi)$, where $\psi = (\text{refl}) + Sym^2(\text{refl})$ is indefinite (see [**BC3**]). For E_7 and E_8, a different argument is used in [**BC1**], namely it is shown that any such region must have a codimension two edge given by two coroots which form an A_2, and by a simple signature argument, the parameter cannot be unitary in such a case.

    Explicit calculations showed that in fact in all cases, classical and exceptional, a parameter χ is in $CS(0)$ if and only if the operator $\mathcal{A}^{\mathbb{H}}_{\psi}(w_0, \chi)$ is positive definite for $\psi = (\text{refl}) + Sym^2(\text{refl})$, but we do not have a conceptual proof of this fact for E_7 and E_8. However the outline above gives conceptual proofs in all cases of the fact that χ is in $CS(0)$ if and only if

$$\mathcal{A}^{\mathbb{H}}_{\psi}(w_0, \chi) \text{ is positive definite for all relevant } W\text{-types } \psi. \tag{5.9}$$

In the real case, the same result holds by the following extra argument:

(3) by (5.9), the unitary spherical and generic parameters for $G(\mathbb{R})$ are a subset of $CS(0)$;
(4) any $\chi \in CS(0)$ has the property that $\langle \check\alpha, \chi \rangle \ne m$, for all roots $\alpha \in \Delta^+$, and all $m \in \mathbb{Z}_{>0}$. Therefore the spherical minimal principal series $X(\chi)$ in the real case is irreducible as well for all $\chi \in CS(0)$;
(5) also in the real case, any $\chi \in CS(0)$ can be proven unitary by irreducible deformations and unitary induction.

5.3. We record next the precise description of the 0-complementary series (that is, the generic spherical unitary parameters) for simple exceptional split groups. We use the Bourbaki simple roots in type E. To simplify the notation, we will write $\check\alpha < 1$ instead of $\langle \check\alpha, \chi \rangle < 1$ in the description of the unitary regions.

5.3.1. *G_2*. The parameter is $\chi = (\nu_1, \nu_1 + \nu_2, -2\nu_1 - \nu_2)$, with $\nu_1 \geq 0$, $\nu_2 \geq 0$. The 0-complementary series is:

(1) $\check{\alpha}_6 < 1$, and $\check{\alpha}_1, \check{\alpha}_2 \geq 0$.
(2) $\check{\alpha}_4 < 1$, $\check{\alpha}_5 > 1$, and $\check{\alpha}_1 \geq 0$.

We use the coroots $\check{\alpha}_1 = (2/3, -1/3, -1/3)$, $\check{\alpha}_2 = (-1, 1, 0)$, $\check{\alpha}_6 = (0, 1, -1)$, $\check{\alpha}_5 = (1, 0, -1)$, and $\check{\alpha}_4 = (1/3, 1/3, -2/3)$.

5.3.2. *F_4*. The parameter is $\chi = (\nu_1, \nu_2, \nu_3, \nu_4)$ assumed dominant. The 0-complementary series is:

(1) $\check{\alpha}_{24} < 1$, and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4 \geq 0$.
(2) $\check{\alpha}_{22} < 1$, $\check{\alpha}_{23} > 1$, and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_4 \geq 0$.

We use the coroots $\check{\alpha}_1 = (1, -1, -1, -1)$, $\check{\alpha}_2 = (0, 0, 0, 2)$, $\check{\alpha}_3 = (0, 0, 1, -1)$, $\check{\alpha}_4 = (0, 1, -1, 0)$, and $\check{\alpha}_{24} = (2, 0, 0, 0)$, $\check{\alpha}_{23} = (1, 1, 1, 1)$, $\check{\alpha}_{22} = (1, 1, 1, -1)$.

5.3.3. *E_6*. In $W(E_6)$, the longest Weyl group element w_0 does not act by -1. Therefore, we only consider dominant parameters χ such that $w_0\chi = -\chi$. In coordinates,

$$\begin{aligned}\chi = \Big(&\frac{\nu_1 - \nu_2}{2} - \nu_3, \frac{\nu_1 - \nu_2}{2} - \nu_4, \frac{\nu_1 - \nu_2}{2} + \nu_4, \frac{\nu_1 - \nu_2}{2} + \nu_3, \frac{\nu_1 + \nu_2}{2}, \\ &-\frac{\nu_1 + \nu_2}{2}, -\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1 + \nu_2}{2}\Big). \end{aligned} \tag{5.10}$$

The 0-complementary series is:

(1) $\check{\alpha}_{36} < 1$, and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6 \geq 0$.
(2) $\check{\alpha}_{34} < 1$, $\check{\alpha}_{35} > 1$, and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_5, \check{\alpha}_6 \geq 0$.

5.3.4. *E_7*. The parameters are $\chi = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, -\nu_7, \nu_7)$, assumed dominant.

The 0-complementary series is:

(1) $\check{\alpha}_{63} < 1$, and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7 \geq 0$.
(2) $\check{\alpha}_{61} < 1$, $\check{\alpha}_{62} > 1$ and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7 \geq 0$.
(3) $\check{\alpha}_{58} < 1$, $\check{\alpha}_{59} < 1$, $\check{\alpha}_{60} > 1$ and $\check{\alpha}_1, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_6, \check{\alpha}_7 \geq 0$.
(4) $\check{\alpha}_{53} < 1$, $\check{\alpha}_{54} < 1$, $\check{\alpha}_{55} < 1$, $\check{\alpha}_{56} > 1$, $\check{\alpha}_{57} > 1$ and $\check{\alpha}_1, \check{\alpha}_3, \check{\alpha}_5 \geq 0$.
(5) $\check{\alpha}_{46} < 1$, $\check{\alpha}_{47} < 1$, $\check{\alpha}_{48} < 1$, $\check{\alpha}_{49} < 1$, $\check{\alpha}_{50} > 1$, $\check{\alpha}_{51} > 1$, $\check{\alpha}_{52} > 1$ and $\check{\alpha}_2 \geq 0$.
(6) $\check{\alpha}_{53} < 1$, $\check{\alpha}_{59} < 1$, $\check{\alpha}_{56} > 1$ and $\check{\alpha}_1, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6 \geq 0$.
(7) $\check{\alpha}_{49} < 1$, $\check{\alpha}_{53} < 1$, $\check{\alpha}_{54} < 1$, $\check{\alpha}_{52} > 1$, $\check{\alpha}_{56} > 1$ and $\check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5 \geq 0$.
(8) $\check{\alpha}_{47} < 1$, $\check{\alpha}_{48} < 1$, $\check{\alpha}_{49} < 1$, $\check{\alpha}_{53} < 1$, $\check{\alpha}_{51} > 1$, $\check{\alpha}_{52} > 1$ and $\check{\alpha}_2, \check{\alpha}_4 \geq 0$.

5.3.5. *E_8*. The parameters are $\chi = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8)$, assumed dominant.

The 0-complementary series is:

(1) $\check{\alpha}_{120} < 1$ and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.
(2) $\check{\alpha}_{113} < 1, \check{\alpha}_{114} < 1$; $\check{\alpha}_{115} > 1$ and $\check{\alpha}_1, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.
(3) $\check{\alpha}_{109} < 1, \check{\alpha}_{110} < 1$; $\check{\alpha}_{111} > 1, \check{\alpha}_{112} > 1$ and $\check{\alpha}_3, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.
(4) $\check{\alpha}_{91} < 1, \check{\alpha}_{92} < 1, \check{\alpha}_{97} < 1, \check{\alpha}_{98} < 1$; $\check{\alpha}_{95} > 1, \check{\alpha}_{96} > 1, \check{\alpha}_{101} > 1$ and $\check{\alpha}_3, \check{\alpha}_4 \geq 0$.
(5) $\check{\alpha}_{90} < 1, \check{\alpha}_{91} < 1, \check{\alpha}_{92} < 1, \check{\alpha}_{97} < 1$; $\check{\alpha}_{94} > 1, \check{\alpha}_{95} > 1, \check{\alpha}_{96} > 1$ and $\check{\alpha}_1, \check{\alpha}_3 \geq 0$.
(6) $\check{\alpha}_{89} < 1, \check{\alpha}_{90} < 1, \check{\alpha}_{91} < 1, \check{\alpha}_{92} < 1$; $\check{\alpha}_{93} > 1, \check{\alpha}_{94} > 1, \check{\alpha}_{95} > 1$, $\check{\alpha}_{96} > 1$ and $\check{\alpha}_1 \geq 0$.

(7) $\check{\alpha}_{104} < 1, \check{\alpha}_{110} < 1$; $\check{\alpha}_{107} > 1, \check{\alpha}_{112} > 1$ and $\check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.
(8) $\check{\alpha}_{104} < 1, \check{\alpha}_{105} < 1, \check{\alpha}_{106} < 1$; $\check{\alpha}_{107} > 1, \check{\alpha}_{108} > 1$ and $\check{\alpha}_2, \check{\alpha}_4, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.
(9) $\check{\alpha}_{118} < 1$; $\check{\alpha}_{119} > 1$ and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_8 \geq 0$.
(10) $\check{\alpha}_{97} < 1, \check{\alpha}_{110} < 1$; $\check{\alpha}_{101} > 1, \check{\alpha}_{112} > 1$ and $\check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7 \geq 0$.
(11) $\check{\alpha}_{97} < 1, \check{\alpha}_{105} < 1, \check{\alpha}_{106} < 1$; $\check{\alpha}_{101} > 1, \check{\alpha}_{108} > 1$ and $\check{\alpha}_2, \check{\alpha}_4, \check{\alpha}_6, \check{\alpha}_7 \geq 0$.
(12) $\check{\alpha}_{116} < 1$; $\check{\alpha}_{117} > 1$ and $\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_6, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.
(13) $\check{\alpha}_{97} < 1, \check{\alpha}_{98} < 1, \check{\alpha}_{106} < 1$; $\check{\alpha}_{101} > 1, \check{\alpha}_{102} > 1$ and $\check{\alpha}_2, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6 \geq 0$.
(14) $\check{\alpha}_{97} < 1, \check{\alpha}_{98} < 1, \check{\alpha}_{99} < 1$; $\check{\alpha}_{96} > 1, \check{\alpha}_{101} > 1, \check{\alpha}_{102} > 1$ and $\check{\alpha}_2, \check{\alpha}_4, \check{\alpha}_5 \geq 0$.
(15) $\check{\alpha}_{97} < 1, \check{\alpha}_{98} < 1, \check{\alpha}_{99} < 1, \check{\alpha}_{100} < 1$; $\check{\alpha}_{101} > 1, \check{\alpha}_{102} > 1, \check{\alpha}_{103} > 1$ and $\check{\alpha}_2, \check{\alpha}_5 \geq 0$.
(16) $\check{\alpha}_{114} < 1$; $\check{\alpha}_{112} > 1$ and $\check{\alpha}_1, \check{\alpha}_3, \check{\alpha}_4, \check{\alpha}_5, \check{\alpha}_6, \check{\alpha}_7, \check{\alpha}_8 \geq 0$.

5.3.6. *Roots for type E.* The notation for the positive coroots which appear in the 0-complementary series is as follows.

E_6.

$$\check{\alpha}_{34} = \tfrac{1}{2}(-1,1,-1,1,1,-1,-1,1) \quad \check{\alpha}_{35} = \tfrac{1}{2}(-1,-1,1,1,1,-1,-1,1)$$
$$\check{\alpha}_{36} = \tfrac{1}{2}(1,1,1,1,1,-1,-1,1)$$

E_7.

$$\check{\alpha}_{46} = \tfrac{1}{2}(-1,1,-1,1,1,-1,-1,1) \quad \check{\alpha}_{47} = \tfrac{1}{2}(-1,1,1,-1,-1,1,-1,1)$$
$$\check{\alpha}_{48} = \tfrac{1}{2}(1,-1,-1,1,-1,1,-1,1) \quad \check{\alpha}_{49} = \epsilon_5 + \epsilon_6$$
$$\check{\alpha}_{50} = \tfrac{1}{2}(-1,-1,1,1,1,-1,-1,1) \quad \check{\alpha}_{51} = \tfrac{1}{2}(-1,1,-1,1,-1,1,-1,1)$$
$$\check{\alpha}_{52} = \tfrac{1}{2}(1,-1,-1,-1,1,1,-1,1) \quad \check{\alpha}_{53} = \tfrac{1}{2}(1,1,1,1,1,-1,-1,1)$$
$$\check{\alpha}_{54} = \tfrac{1}{2}(-1,-1,1,1,-1,1,-1,1) \quad \check{\alpha}_{55} = \tfrac{1}{2}(-1,1,-1,-1,1,1,-1,1)$$
$$\check{\alpha}_{56} = \tfrac{1}{2}(1,1,1,1,-1,1,-1,1) \quad \check{\alpha}_{57} = \tfrac{1}{2}(-1,-1,1,-1,1,1,-1,1)$$
$$\check{\alpha}_{58} = \tfrac{1}{2}(1,1,1,-1,1,1,-1,1) \quad \check{\alpha}_{59} = \tfrac{1}{2}(-1,-1,-1,1,1,1,-1,1)$$
$$\check{\alpha}_{60} = \tfrac{1}{2}(1,1,-1,1,1,1,-1,1) \quad \check{\alpha}_{61} = \tfrac{1}{2}(1,-1,1,1,1,1,-1,1)$$
$$\check{\alpha}_{62} = \tfrac{1}{2}(-1,1,1,1,1,1,-1,1) \quad \check{\alpha}_{63} = -\epsilon_7 + \epsilon_8$$

E_8.

$$\check{\alpha}_{89} = \tfrac{1}{2}(1,-1,1,1,1,1,-1,1) \quad \check{\alpha}_{90} = \tfrac{1}{2}(1,1,-1,1,1,-1,1,1)$$
$$\check{\alpha}_{91} = \tfrac{1}{2}(1,1,1,-1,-1,1,1,1) \quad \check{\alpha}_{92} = \tfrac{1}{2}(-1,-1,-1,1,-1,1,1,1)$$
$$\check{\alpha}_{93} = \tfrac{1}{2}(-1,1,1,1,1,1,-1,1) \quad \check{\alpha}_{94} = \tfrac{1}{2}(1,-1,1,1,1,-1,1,1)$$
$$\check{\alpha}_{95} = \tfrac{1}{2}(1,1,-1,1,-1,1,1,1) \quad \check{\alpha}_{96} = \tfrac{1}{2}(-1,-1,-1,-1,1,1,1,1)$$
$$\check{\alpha}_{97} = -\epsilon_7 + \epsilon_8 \quad \check{\alpha}_{98} = \tfrac{1}{2}(-1,1,1,1,1,-1,1,1)$$
$$\check{\alpha}_{99} = \tfrac{1}{2}(1,-1,1,1,-1,1,1,1) \quad \check{\alpha}_{100} = \tfrac{1}{2}(1,1,-1,-1,1,1,1,1)$$
$$\check{\alpha}_{101} = -\epsilon_6 + \epsilon_8 \quad \check{\alpha}_{102} = \tfrac{1}{2}(-1,1,1,1,-1,1,1,1)$$
$$\check{\alpha}_{103} = \tfrac{1}{2}(1,-1,1,-1,1,1,1,1) \quad \check{\alpha}_{104} = -\epsilon_5 + \epsilon_8$$
$$\check{\alpha}_{105} = \tfrac{1}{2}(-1,1,1,-1,1,1,1,1) \quad \check{\alpha}_{106} = \tfrac{1}{2}(1,-1,-1,1,1,1,1,1)$$
$$\check{\alpha}_{107} = -\epsilon_4 + \epsilon_8 \quad \check{\alpha}_{108} = \tfrac{1}{2}(-1,1,-1,1,1,1,1,1)$$
$$\check{\alpha}_{109} = -\epsilon_3 + \epsilon_8 \quad \check{\alpha}_{110} = \tfrac{1}{2}(-1,-1,1,1,1,1,1,1)$$
$$\check{\alpha}_{111} = -\epsilon_2 + \epsilon_8 \quad \check{\alpha}_{112} = \tfrac{1}{2}(1,1,1,1,1,1,1,1)$$
$$\check{\alpha}_{113} = \epsilon_1 + \epsilon_8 \quad \check{\alpha}_{114} = -\epsilon_1 + \epsilon_8$$
$$\check{\alpha}_{115} = \epsilon_2 + \epsilon_8 \quad \check{\alpha}_{116} = \epsilon_3 + \epsilon_8$$
$$\check{\alpha}_{117} = \epsilon_4 + \epsilon_8 \quad \check{\alpha}_{118} = \epsilon_5 + \epsilon_8$$
$$\check{\alpha}_{119} = \epsilon_6 + \epsilon_8 \quad \check{\alpha}_{120} = \epsilon_7 + \epsilon_8$$

5.4. $\mathcal{O}$-complementary series. In this section we state the main theorems as they follow from [**Ba1**], [**BC1**].

5.4.1.

Theorem 5.5 ([**Ba1**],[**BC1**],[**Ci**]). *Fix $\check{\mathcal{O}}$ a nilpotent $\check{G}$-orbit in $\check{\mathfrak{g}}$ and a Lie triple $\{\check{e}, \check{h}, \check{f}\}$. Let χ be a (hyperbolic) semisimple element such that $\check{\mathcal{O}}(\chi) = \check{\mathcal{O}}$, and which we write as $\chi = \check{h}/2 + \nu$, with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$. Then*

$$\chi \text{ is in } \check{C}S^{\mathbb{H}}(\check{\mathcal{O}}) \text{ if and only if } \nu \text{ is in } CS^{\mathbb{H}(\mathfrak{z}(\check{e},\check{h},\check{f}))}(0), \tag{5.11}$$

unless $\check{\mathcal{O}}$ is one of the following orbits:

- *$A_1 + \widetilde{A}_1$ in F_4,*
- *$A_2 + 3A_1$ in E_7,*
- *$A_4 + A_2 + A_1$, $A_4 + A_2$, $D_4(a_1) + A_2$, $A_3 + 2A_1$, $A_2 + 2A_1$, and $4A_1$ in E_8.*

The explicit description of the complementary series is in section 6.

In the case of the exceptions, unless the orbit is $4A_1$ in E_8, the complementary series is smaller than the one for the centralizer, and for $4A_1$, it is larger.

5.4.2.

Theorem 5.6 ([**Ba1**],[**BC2**]). *A spherical module $L^{\mathbb{H}}(\chi)$ is unitary if and only if the operators $\mathcal{A}^{\mathbb{H}}_{\psi}(w_0, \chi)$ are positive semidefinite for all relevant W-types ψ.*

5.4.3. We record the results for the spherical principal series of a split real group.

Theorem 5.7 ([**Ba1**],[**Ba2**]). *Every relevant W-type ψ appears as a W subrepresentation of the $(V_\mu^M)^*$ space of a petite K-type (μ, V_μ).*

The construction of petite K-types was explained in section 4.

Corollary 5.8. *For every nilpotent orbit $\check{\mathcal{O}}$, one has*

$$CS^{\mathbb{R}}(\check{\mathcal{O}}) \subseteq CS^{\mathbb{H}}(\check{\mathcal{O}}). \tag{5.12}$$

5.4.4. We have already seen in section 5.2 that in fact $CS^{\mathbb{R}}(0) = CS^{\mathbb{H}}(0)$.

Theorem 5.9 ([**Ba1**]). *If $G(\mathbb{R})$ is split classical, then*

$$CS^{\mathbb{R}}(\check{\mathcal{O}}) = CS^{\mathbb{H}}(\check{\mathcal{O}}), \tag{5.13}$$

for every nilpotent orbit $\check{\mathcal{O}}$.

In addition to the unitarity of the special unipotent representations already mentioned in section 5.1, in order to establish this theorem, [**Ba1**] needs to analyze the irreducibility of parabolically induced representations (see section 10 in [**Ba1**]). One of the ideas which makes this tractable is a combinatorial parameterization of the spherical representations, which we recall next.

5.5. Strings. In this section, we give a description of the parameterization of the spherical dual as in [**Ba1**], section 2, by means of a generalization of the *multisegments* or *strings* introduced for $GL(n)$ in [**Ze**].

To every parameter χ, one attaches a multisegment, so that the orbit $\check{\mathcal{O}}(\chi)$ and the decomposition $\chi = \check{h}/2 + \nu$ used in the previous section can be read off easily. These multisegments arise naturally in the setting of the geometry for the Hecke algebra. More precisely, they parameterize the unique open $\check{G}_{0,\chi}$-orbit in $\check{\mathfrak{g}}_{1,\chi}$ (see section 5.1.)

Let $[a] = (a_1, \dots, a_k)$ be a set of numbers.

DEFINITION 5.10. We call $[a]$ an *increasing (respectively decreasing) string* if $-a_{i-1} + a_i = 1$ (respectively $a_{i-1} - a_i = 1$) for all i.

We will explain next how one builds from χ: the multisegment, the orbit $\check{\mathcal{O}}(\chi)$, the middle element $\check{h}$ and the parameter ν in the centralizer $\mathfrak{z}(\check{e}, \check{h}, \check{f})$. Recall that for simple classical types the complex nilpotent orbits $\check{\mathcal{O}}$ are parameterized by partitions as follows:

- partitions of n, when $\check{\mathfrak{g}} = sl(n, \mathbb{C})$;
- partitions of $2n$, with odd parts occurring with even multiplicity, when $\check{\mathfrak{g}} = sp(2n, \mathbb{C})$;
- partitions of $2n+1$, with even parts occurring with even multiplicity, when $\check{\mathfrak{g}} = so(2n+1, \mathbb{C})$;
- partitions of $2n$, with even parts occurring with even multiplicity, when $\check{\mathfrak{g}} = so(2n, \mathbb{C})$. In this case, when the partition is very even, *i.e.* all parts are even, there are two distinct orbits corresponding to it.

5.5.1. *G of type B_n.* We partition the entries of the character $\chi = (\nu_1, \dots, \nu_n)$ into subsets A_τ where $0 \le \tau \le 1/2$ and

$$A_\tau = \{\nu_i : \nu_i \text{ or } -\nu_i \equiv 1/2 + \tau (mod\ \mathbb{Z})\}. \tag{5.14}$$

There are two cases $0 \le \tau < 1/2$ and $\tau = 1/2$.

When $0 \le \tau < 1/2$, form $A'_\tau = A_\tau \sqcup (-A_\tau)$. We partition A'_τ into a disjoint union of increasing strings $M^+_{\tau,1}, \dots, M^+_{\tau,\ell}$ and decreasing strings $M^-_{\tau,1}, \dots, M^-_{\tau,\ell}$, where $M^+_{\tau,i} = -M^-_{\tau,i}$ as follows. Remove the smallest entry, say $-a$ in A'_τ and place it in $M^+_{1,\tau}$, and the largest entry a and place it in $M^-_{1,\tau}$. If $a-1$ appears in A'_τ, remove it from A'_τ and place it $M^-_{1,\tau}$ and similarly, remove $1-a$ and place it in $M^-_{\tau,1}$. Continue with $a-2, a-3, \dots$ until this is not possible. This completes the construction of $M^\pm_{\tau,1}$. Then repeat the process with the remaining entries in A'_τ to construct $M^\pm_{\tau,2}$, $M^\pm_{\tau,3}$, etc. Once this is finish, every pair $(M^+_{\tau,i}, M^-_{\tau,i})$ adds:

1. a pair (l_i, l_i), where $l_i = length(M^+_{\tau,i}) = length(M^-_{\tau,i}))$ to the partition of $\check{\mathcal{O}}(\chi)$;
2. the entries $[l_i] = (-(l_i - 1), -(l_i - 1) + 2, \dots, (l_i - 1))$ to $\check{h}$ (this is the middle element of the principal orbit in $gl(l_i)$);
3. the entry $|\nu_i|$, where $\nu_i(1, \dots, 1) = M^+_{\tau,i} - 1/2[l_i]$ to ν.

We give two examples of this process. For example, if $\tau = 0$ and $A_0 = (0, 0, 1, 1, 1, 1, 2, 3, 3, 4, 5)$, then $M^+_{0,1} = (-5, -4, -3, -2, -1, 0, 1)$, $M^+_{0,2} = (-4, -3)$, and $M^+_{0,3} = (-1, 0, 1)$. Of course, always, $M^-_{\tau,i} = -M^+_{\tau,i}$. This means we add to the partition of $\check{\mathcal{O}}(\chi)$ the entries $(2, 2, 3, 3, 7, 7)$. To $\check{h}$ we add $(-1, 1)$, $(-2, 0, 2)$ and $(-6, -4, -2, 0, 2, 4, 6)$, and to ν we add $7/2$, 0, and 2.

If $\tau = 1/4$, and $A_{1/4} = (1/4, 1/4, 3/4, 5/4, 5/4)$, then $M^+_{\tau,1} = (-5/4, -1/4, 3/4)$ and $M^+_{\tau,2} = (-5/4, -1/4)$. This adds the entries $(2, 2, 3, 3)$ to the partition of $\check{\mathcal{O}}(\chi)$. To $\check{h}$ we add $(-1, 1)$ and $(-2, 0, 2)$, and to ν we add $3/4$ and $1/4$.

Now assume $\tau = 1/2$. Form again $A'_{1/2} = A_{1/2} \sqcup (-A_{1/2})$. We only construct increasing strings $M^+_{1/2,i}$, $i = 1, \ell$ in this case. Remove the smallest entry b in $A'_{1/2}$

and place it in $M^+_{1/2,1}$. Continue with $b+1, b+2, \dots$ until this is not possible. This concludes the construction of $M^+_{1/2,1}$. We repeat the process with the remaining entries in $A'_{1/2}$, until we remove all of them. Any two strings $M^+_{1/2,i}$ and $M^+_{1/2,j}$ such that $M^+_{1/2,i} = -M^+_{1/2,j}$ contribute:

(1) a pair (k_i, k_i), where $k_i = \mathit{length}(M^+_{1/2,i}) = \mathit{length}(M^+_{1/2,j})$ to $\check{\mathcal{O}}(\chi)$;
(2) the entries $[k_i] = (-(k_i-1) - (k_i-1)+2, \dots, (k_i-1))$ to $\check{h}$;
(3) the entry $|\nu_i|$, where $\nu_i(1,\dots,1) = M^+_{1/2,i} - 1/2[k_i]$.

Remove them from the list of strings and repeat. If the number of strings was odd, there is one remaining string at the end, say $M^+_{1/2,k}$. We call this string *distinguished*. The motivation is that the positive part of this string is 1/2 the middle element of a distinguished nilpotent orbit in a symplectic complex algebra as in [**CM**]. Add the partition corresponding to that orbit to $\check{\mathcal{O}}(\chi)$. The contribution to $\check{h}$ is twice the positive part of $M^+_{1/2,k}$, while there is no contribution to ν.

For example, if $A_{1/2} = (1/2, 1/2, 1/2, 3/2, 3/2, 3/2, 5/2, 5/2, 5/2, 7/2)$, then we extract five strings: $M^+_{1/2,1} = (-7/2, -5/2, -3/2, -1/2, 1/2, 3/2, 5/2, 7/2)$, $M^+_{1/2,2} = M^+_{1/2,3} = (-5/2, -3/2, -1/2, 1/2, 3/2, 5/2)$, $M^+_{1/2,4} = (5/2)$, and $M^+_{1/2,5} = (-5/2)$. Then the distinguished string is $M^+_{1/2,1}$. Its positive part is $(1/2, 3/2, 5/2, 7/2)$, which is 1/2 the middle element of the principal nilpotent orbit in $sp(8)$. It adds (8) to $\check{\mathcal{O}}(\chi)$, $(1,3,5,7)$ to $\check{h}$, and nothing to ν. The other four strings add $(6,6)$ and $(1,1)$ respectively to $\check{\mathcal{O}}(\chi)$. Their contribution to $\check{h}$ are $(-5,-3,-1,1,3,5)$, and (0), and to ν, they contribute 0, respectively 5/2.

In conclusion, if our χ where the disjoint union

$$\chi = A_0 \sqcup A_{1/4} \sqcup A_{1/2}, \tag{5.15}$$

with A_τ as above, then we rearrange the entries of the nilpotent orbit increasingly, *e.g.*

$$\check{\mathcal{O}}(\chi) = (1,1;2,2,2,2;3,3,3,3;6,6;7,7;;8), \tag{5.16}$$

and $\check{h}$ and ν are permuted accordingly. We separate the distinguished part by ; ; and the groups of identical entries by ;. After this arrangement

$$\nu = (5/2; 3/4, 7/2; 0, 1/4; 0; 2; ;\ ). \tag{5.17}$$

5.5.2. *G of type C_n.* The algorithm of forming strings is the same as the one for B_n except that $A'_\tau = A_\tau \sqcup (-A_\tau) \sqcup \{0\})$, and the special case is $\tau = 0$. For $\tau = 0$ we apply the algorithm as in the case $\tau = 1/2$ for B_n, and the distinguished string corresponds to a distinguished nilpotent orbit in $so(2k+1,\mathbb{C})$. In the case $0 < \tau \le 1/2$ the algorithm is identical to $0 \le \tau < 1/2$ in B_n.

5.5.3. *G of type D_n.* The algorithm of forming strings is the same as the one for B_n except that the special case is $\tau = 0$. For $\tau = 0$ we apply the algorithm as in the case $\tau = 1/2$ for B_n, and the distinguished string corresponds to a distinguished nilpotent orbit in $so(2k,\mathbb{C})$. In the case $0 < \tau \le 1/2$ the algorithm is identical to $0 \le \tau < 1/2$ in B_n. There is a minor complication when the parameter belongs to a very even nilpotent orbit which we ignore here. We refer the reader to section 2.7 in [**Ba1**] for the details of this case.

5.6. Testing unitarity. Once the decomposition of a character χ into strings is completed as in section 5.5, testing unitarity by theorem 5.5 is easy.

Let assume that from a character (spherical parameter) χ we obtained in section 5.5 the strings and the parameter ν for the centralizer $\mathfrak{z}(\check{e}, \check{h}, \check{f})$. Let us assume

$$\check{\mathcal{O}}(\chi) = (\underbrace{1, \dots, 1}_{2\ell_1}; \underbrace{2, \dots, 2}_{2\ell_2}; \dots; \underbrace{k \dots k}_{2\ell_k}; ; [\lambda]), \tag{5.18}$$
$$\nu = (\nu_{1,1}, \dots, \nu_{1,\ell_1}; \nu_{2,1}, \dots, \nu_{2,\ell_2}; \dots; \nu_{k,1}, \dots, \nu_{k,\ell_k}; ; \ ),$$

where $\ell_i \geq 0$, $i = 1, k$, and the conventions for notation are as in (5.16) and (5.17). Moreover, we may permute the entries in ν between any two consecutive ;'s to be increasing.

The type of the centralizer $\mathfrak{z}(\check{e}, \check{h}, \check{f})$ is well-known (see [**CM**] or [**Car**]). It is a product of types B_{ℓ_i}, D_{ℓ_i} or C_{ℓ_i}, depending on the type of G. Then one checks if the corresponding entries $(\nu_{i,1}, \dots, \nu_{i,\ell_i})$ in ν satisfy the conditions in (5.8) for the corresponding (dual) type.

Example. If $(\check{\mathcal{O}}(\chi), \nu)$ are as in (5.16) and (5.17), then $\check{\mathfrak{g}} = sp(32, \mathbb{C})$, and $\mathfrak{z}(\check{e}, \check{h}, \check{f})$ has type $C_1 \times D_2 \times C_2 \times D_1 \times C_1$, in the same order as $\check{\mathcal{O}}(\chi)$ is written, and where by D_1 we mean a one-dimensional torus. We test the corresponding ν_i's against (5.8) (for a torus D_1, the only unitary parameter is 0):

ν	centralizer	unitary?
$(5/2)$	C_1	no
$(3/4, 7/2)$	D_2	no
$(0, 1/4)$	C_2	yes
(0)	D_1	yes
(2)	C_1	no

In conclusion, χ is not unitary.

5.7. Maximal parabolic cases. In the next sections we give a sketch of some of the ideas involved in the proofs of the results stated in section 5.4. We will be concerned with theorems 5.5 and 5.6, which are proved in [**Ba1**] for classical split groups,[**Ci**] for G_2 and F_4, and [**BC2**] for types E. The method for proving theorem 5.5, but not the statement about relevant W-types, was used for the first time in [**BM3**] for classical split groups.

Recall that we are in the setting of the affine graded Hecke algebra $\mathbb{H} = \mathbb{H}_G$. The method consists in a double induction:

(a) an upward induction, by the rank of the group G, and
(b) a downward induction, in the closure ordering for nilpotent orbits $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$.

In this scheme, one determines the 0-complementary series last. We have seen in section 5.2 that there is an alternate method for finding the 0-complementary series directly.

As remarked in section 5.1, there is nothing to do for distinguished orbits $\check{\mathcal{O}}$. Therefore, the first basic cases to do are when the nilpotent $\check{\mathcal{O}}$ is parameterized in the Bala-Carter classification by the Levi component of a maximal parabolic subalgebra. We call them *maximal parabolic cases.*

Let us assume therefore that the Lie triple $\{\check{e}, \check{h}, \check{f}\}$ of $\check{\mathcal{O}}$ is contained in the Levi subalgebra $\check{\mathfrak{m}}$ of a maximal parabolic subalgebra $\check{\mathfrak{p}} = \check{\mathfrak{m}} + \check{\mathfrak{n}} \subset \check{\mathfrak{g}}$. The graded Hecke algebra corresponding to $\check{\mathfrak{m}}$, which we denote $\mathbb{H}_M$, is naturally a subalgebra of $\mathbb{H}$. Let $\chi = \check{h}/2 + \nu$ be the character as before. Note that $\check{h}/2$ defines a special unipotent representation $L_M(\check{h}/2)$ for M.

LEMMA 5.11. *With the notation as above, assume that ν is dominant with respect to the roots in $\mathfrak{n}$. Then $L(\chi)$ is the unique irreducible quotient of*

$$X(M, \check{h}/2, \nu) := \mathrm{Ind}_{\mathbb{H}_M}^{\mathbb{H}_G}(L_M(\check{h}/2) \otimes \mathbb{C}_\nu). \tag{5.19}$$

We refer the reader to [**Ba1**] (also [**BM3**]) for the proof of this result. It is proved using the Iwahori-Matsumoto involution and the geometric classification of $\mathbb{H}$-modules as it follows from [**KL**] and [**Lu1**].

One knows that the module $L(\chi)$ is Hermitian if and only if there exits $w \in W$ such that

$$wM = M, \quad w\check{h} = \check{h}, \quad \text{and } w\nu = -\nu. \tag{5.20}$$

If there exists such a w, we choose w_m to be a minimal element in the double coset $W(M)wW(M)$. Then one can define a generalization of the intertwining operators from section 3.4:

$$\begin{aligned} &A(w_m, \nu) : X(M, \check{h}/2, \nu) \longrightarrow X(M, \check{h}/2, -\nu), \quad \text{and} \\ &A_\psi(w_m, \nu) : \mathrm{Hom}_{W(M)}[V_\psi : L_M(\check{h}/2)] \longrightarrow \mathrm{Hom}_{W(M)}[V_\psi : L_M(\check{h}/2)], \end{aligned} \tag{5.21}$$

for every W-type (ψ, V_ψ).

In section 6 of [**Ba1**], these operators are computed explicitly for all relevant W-types in the classical groups as in definition 3.9. One can reduce the calculation, so that the only cases that one considers there are for $\check{\mathfrak{h}}$ the middle element of the principal nilpotent orbit on $\check{\mathfrak{m}}$. In those cases, the dimension of the Hom spaces in equation (5.21) is always 1. That makes $A_\psi(w_m, \nu)$ a scalar, which is normalized so that it is $+1$ when $\psi = triv$.

Example 4.6.1. Let us assume that G is of type B_{n+k}. Then $\check{\mathfrak{g}} = sp(2n+2k, \mathbb{C})$, and we consider $\check{\mathcal{O}} = (k, k, 2n)$, sothat $\chi = (\frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2}, -\frac{k-1}{2} + \nu, \dots, \frac{k-1}{2} + \nu)$, and $\check{\mathfrak{m}} = sp(2n, \mathbb{C}) \times gl(k, \mathbb{C})$. The calculation in [**Ba1**] gives:

relevant W-type	$A_\psi(w_m, \nu)$
$(n+k-m) \times (m)$	$\displaystyle\prod_{0 \le j \le m-1} \frac{n + \frac{k}{2} - j - \nu}{n + \frac{k}{2} - j + \nu}$
$(m, n+k-m) \times (0)$	$\displaystyle\prod_{0 \le j \le m-1} \frac{n + \frac{k}{2} - j - \nu}{n + \frac{k}{2} - j + \nu} \cdot \frac{\frac{k}{2} - j - \nu}{\frac{k}{2} - j + \nu}$

,

where $0 \le m \le k$.

For example, when $n = 1$, $k = 1$, we are in the case $sp(2) \times gl(1) \subset sp(4)$. The only W-types in the $\mathrm{Ind}_{\mathbb{H}_{C_1}}^{\mathbb{H}_{C_2}}(triv \otimes \mathbb{C}_\nu)$ are 2×0, 1×1 and 11×0. Then $\chi = (\nu, \frac{1}{2})$, $\nu > 0$. The operators are $+1$, $\frac{3/2-\nu}{3/2+\nu}$, and $\frac{(1/2-\nu)(3/2-\nu)}{(1/2+\nu)(3/2+\nu)}$ respectively. Since $X(M, \check{h}/2, \nu)$ is irreducible at $\nu = 0$ in this case, we deduce that, in $sp(4, \mathbb{C})$, $CS^{\mathbb{H}}((1,1,2)) = \{\chi = (\nu, 1/2) : 0 \le \nu < 1/2\}$, which is identical with the 0-complementary series of the centralizer of type A_1 of the nilpotent $(1,1,2)$.

On the other hand, if one considers $n = 0$, $k = 2$, we are in the case $gl(2) \subset sp(4)$. The only W-types in the $\mathrm{Ind}_{\mathbb{H}_{A_1}}^{\mathbb{H}_{C_2}}(triv \otimes \mathbb{C}_\nu)$ are 2×0, 1×1 and 0×2. Then $\chi = (-\frac{1}{2} + \nu, \frac{1}{2} + \nu)$, $\nu > 0$. The operators are $+1$, $\frac{1-\nu}{1+\nu}$, and $-\frac{(1-\nu)}{(1+\nu)}$ respectively. This shows $X(M, \check{h}/2, \nu)$, $\nu > 0$ if irreducible, it must be nonunitary. In this case, we know that $X(M, \check{h}/2, \nu)$ is reducible at $\nu = 0$. We deduce that, in $sp(4, \mathbb{C})$, $CS^{\mathbb{H}}((2,2)) = \{\chi = (-1/2+\nu, 1/2+\nu) : \nu = 0\}$, which is the same as the unitary set for the centralizer, which is of type D_1, *i.e.* a one-dimensional torus.

For exceptional groups, it is not true anymore that the relevant W-types appear with multiplicity 1, even if we induce from the trivial representation of $\mathbb{H}_M$. Explicit calculations of the operators $A_\psi(w_m, \nu)$ are done in [**BC2**].

Example 4.6.2. Let us consider G of type E_8, $\check{\mathcal{O}} = E_6 + A_1$, so that $\check{\mathfrak{m}} = E_6 + A_1$. The character is $\chi = (0, 1, 2, 3, 4, -\frac{9}{2}, -\frac{7}{2}, 4) + \nu\omega_7$, $\nu > 0$, where ω_7 is the fundamental weight corresponding to the coroot $\check{\alpha}_7$. (Recall that we use Bourbaki's notation.) In this example, computing the determinants of $A_\psi(w_m, \nu)$ turns out to be sufficient. we give below the table with the relevant W-types which appear in this case.

W-type	dim Hom-space	Determinant of $A_\psi(w_m, \nu)$
8_z	1	$\frac{\frac{19}{2}-\nu}{\frac{19}{2}+\nu}$
35_x	3	$\frac{(\frac{19}{2}-\nu)^2(\frac{17}{2}-\nu)(\frac{11}{2}-\nu)(\frac{9}{2}-\nu)(\frac{5}{2}-\nu)}{(\frac{19}{2}+\nu)^2(\frac{17}{2}+\nu)(\frac{11}{2}+\nu)(\frac{9}{2}+\nu)(\frac{5}{2}+\nu)}$
112_z	2	$\frac{(\frac{19}{2}-\nu)^2(\frac{17}{2}-\nu)(\frac{11}{2}-\nu)^2(\frac{9}{2}-\nu)(\frac{5}{2}-\nu)(\frac{3}{2}-\nu)}{(\frac{19}{2}+\nu)^2(\frac{17}{2}+\nu)(\frac{11}{2}+\nu)^2(\frac{9}{2}+\nu)(\frac{5}{2}+\nu)(\frac{3}{2}+\nu)}$
84_x	2	$\frac{(\frac{19}{2}-\nu)^2(\frac{17}{2}-\nu)(\frac{11}{2}-\nu)^2(\frac{9}{2}-\nu)(\frac{5}{2}-\nu)(\frac{3}{2}-\nu)(\frac{1}{2}-\nu)}{(\frac{19}{2}+\nu)^2(\frac{17}{2}+\nu)(\frac{11}{2}+\nu)^2(\frac{9}{2}+\nu)(\frac{5}{2}+\nu)(\frac{3}{2}+\nu)(\frac{1}{2}+\nu)}$

We plot the signatures of these determinants in a table:

ν		$\frac{1}{2}$		$\frac{3}{2}$		$\frac{5}{2}$		$\frac{9}{2}$		$\frac{11}{2}$		$\frac{17}{2}$		$\frac{19}{2}$	
8_z	+	+	+	+	+	+	+	+	+	+	+	+	+	0	−
35_x	+	+	+	+	+	0	−	0	+	0	−	0	+	0	+
84_x	+	0	−	0	+	0	−	0	+	0	+	0	−	0	−
112_z	+	+	+	0	−	0	+	0	−	0	−	0	+	0	+

Since $X(M, \check{h}/2, \nu)$ is irreducible at $\nu = 0$, we conclude that, in E_8, $CS(E_6 + A_1) = \{\chi : 0 \le \nu < 1/2\}$, which is identical with the 0-complementary series of the centralizer which is type A_1.

An example of a maximal parabolic nilpotent orbit with torus centralizer in E_8, and therefore with the unitary set formed only of $\nu = 0$, is $\check{\mathcal{O}} = D_5 + A_2$. Similar tables as above are available in that case, but we skip the details here.

Remarks. There are three important remarks to be made which were assumed implicitly in the examples above:

(1) The reducibility points of the induced module $X(M, \check{h}/2, \nu)$ for $\nu > 0$ are known a priori, by different methods (see [**BC1**] and the references therein). Therefore, we can compare with the explicit calculations of operators $A_\psi(w_m, \nu)$, and see that the relevant W-types do not miss any reducibility points.

(2) The reducibility of $X(M, \check{h}/2, \nu)$, at $\nu = 0$ is known by geometric considerents ([**KL**]).

(3) Whenever $X(M, \check{h}/2, \nu)$ is reducible at $\nu > 0$, the spherical quotient is parameterized by a larger nilpotent $\check{\mathcal{O}}' \supsetneq \check{\mathcal{O}}$. Therefore, whenever one is concerned with the orbit $\check{\mathcal{O}}$, these reducibility points do not need to be considered. The unitarity of the corresponding modules was *already* checked in our inductive procedure.

We summarize now the main consequence of these types of calculations.

PROPOSITION 5.12 ([**Ba1**],[**BC2**]). *Assume $\check{\mathcal{O}}$ is a maximal parabolic nilpotent orbit. Then a spherical module parameterized by χ such that $\check{\mathcal{O}}(\chi)=\check{\mathcal{O}}$ is unitary if and only if the intertwining operators are positive semidefinite on the relevant W-types.*

We remark that one can obtain the unitarity results for maximal parabolic cases more easily if the constraint of working with relevant W-types only is removed. This is the approach of [**BM3**] and [**BC1**]. In there, one obtains the complementary series by checking the signature of two W-types which are Springer representations for $\check{\mathcal{O}}$ and for an orbit $\check{\mathcal{O}}'$ consecutive to $\check{\mathcal{O}}$ in the closure ordering.

5.8. Induction. In this section, we exemplify the inductive step in the proof. We are still in the setting of the affine graded Hecke algebra. The main idea is as follows: assume we start with a parameter $\chi=\check{h}/2+\nu$ associated to $\check{\mathcal{O}}$, where $\check{\mathcal{O}}$ is not a maximal parabolic nilpotent. Let M and $X(M,\check{h}/2,\nu)$ be as in lemma 5.11. One knows explicitly for which values of ν the module $X(M,\check{h}/2,\nu)$ is reducible. If $\nu=\nu_1$ is such a value (and ν_1 is dominant with respect to the roots in $\check{\mathfrak{n}}$), then the spherical module $L(\chi_1)$, where $\chi_1=\check{h}/2+\nu_1$, is attached to a nilpotent orbit $\mathcal{O}_1$, with the property that

$$\check{\mathcal{O}}_1\supset\check{\mathcal{O}} \text{ and } \check{\mathcal{O}}_1\neq\check{\mathcal{O}}. \tag{5.22}$$

By induction, we already know if $L(\chi_1)$ is unitary or not.

LEMMA 5.13 (1). *Let $\chi(t)=\check{h}/2+\nu(t):[0,1]\to\check{\mathfrak{h}}$ be a continuous function, such that $\chi(t)=\chi_0$ and $\chi(1)=\chi_1$. Assume that $X(M,\check{h}/2,\nu(t))$ is irreducible for $0\le t<1$ and $X(M,\check{h}/2,\nu_1)$ is reducible.*

If $L(\chi_1)$ is not unitary, then $L(\chi_0)=X(M,\check{h}/2,\nu_0)$ is not unitary as well.

This well-known criterion for nonunitarity can be applied therefore, and it is the main tool for ruling out nonunitary parameters. Of course, in general, there are many delicate combinatorial issues that arise; in the classical cases, one needs to choose carefully how to deform parameters $\chi=\check{h}/2+\nu$. In the exceptional groups, another complication arises: there are cases (*e.g.* in $\check{\mathcal{O}}=2A_2\subset E_8$) of parameters $\chi=\chi_0$, which turn out to be nonunitary, but in all possible deformations as in lemma 5.13(1), the modules $L(\chi_1)$ are unitary. In those cases we apply some ad-hoc signature arguments.

The second (again well-known) criterion which we employ is the following *complementary series* method.

LEMMA 5.14 (2). *Let $\chi(t)=\check{h}/2+\nu(t):[0,1]\to\check{\mathfrak{h}}$ be a continuous function, such that $\chi(t)=\chi_0$ and $\chi(1)=\chi_1$. Assume that*

(a) *$X(M,\check{h}/2,\nu(t))$ is Hermitian and irreducible for $0\le t\le 1$, and*
(1) *$X(M,\check{h}/2,\nu_1)$ is unitarily induced, i.e. $X(M,\check{h}/2,\nu_1)=\mathrm{Ind}_{\mathbb{H}_{M'}}^{\mathbb{H}_G}(V_{M'})$, where M' is a Levi component, $V_{M'}$ is a Hermitian spherical module for $\mathbb{H}_{M'}$.*

Then $L(\chi_0)=X(M,\check{h}/2,\nu_0)$ is unitary in $\mathbb{H}_G$ if and only if $V_{M'}$ is unitary in $\mathbb{H}_{M'}$.

Since, by induction the unitarity of $\mathbb{H}_{M'}$-modules is known, we can apply this criterion. This is our main criterion for proving the unitarity of modules $L(\chi)$.

Example. To illustrate this discussion, we conclude the section with a simple example. Consider $\check{\mathcal{O}} = (1^4, 2)$ in $sp(6,\mathbb{C})$, so $\check{\mathfrak{m}} = gl(1)^2 \times sp(2)$. We write $\chi = (\nu_1, \nu_2, 1/2)$, where by conjugation with the Weyl group, we can assume that $0 \le \nu_2 \le \nu_1$. The centralizer has type C_2 in this case. The lines where $X(M, \check{h}/2, (\nu_1, \nu_2))$ becomes reducible are drawn in figure 5.8. In this picture, the solid lines denote unitary spherical parameters, while the dotted lines denote nonunitary parameters. The reducibility lines (in the Hecke algebra case) are:

- $\nu_1 = 1/2,\ \nu_2 = 1/2$: $\mathcal{O}(\chi) = (2211)$;
- $\nu_1 = 3/2,\ \nu_2 = 3/2$: $\mathcal{O}(\chi) = (411)$;
- $\nu_1 \pm \nu_2 = 1$: $\mathcal{O}(\chi) = (222)$.

On the lines $\nu_2 = 0$, and $\nu_1 = \nu_2$, the module is unitarily induced from a spherical module parameterized by (211) in $sp(4,\mathbb{C})$, respectively (11) in $gl(2,\mathbb{C})$.

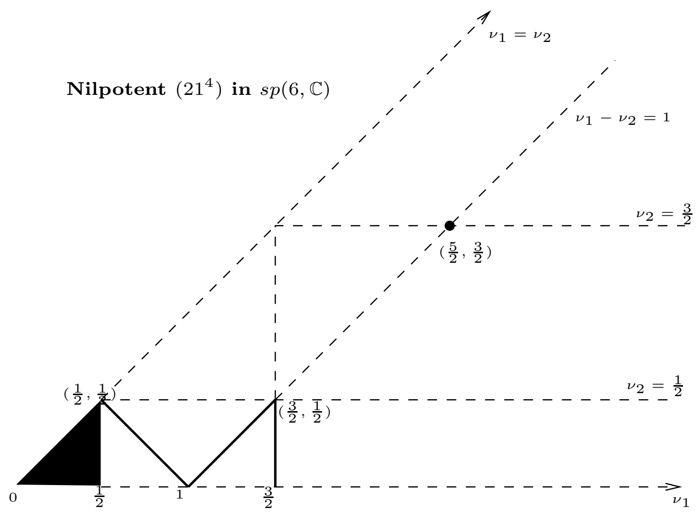


FIGURE 2. Example: $\check{\mathcal{O}} = (21^4)$ for the Hecke algebra of type C_3

6. Lists of unitary spherical parameters

In this section, we give tables with the explicit description of the spherical unitary parameters for split p-adic groups as in theorem 5.5. By theorems 5.7 and 5.9, these lists also describe the spherical unitary dual for split real classical groups, and, by [**Vo2**], also for G_2. For real split F_4, E_6, E_7, E_8, the tables represent the only spherical parameters which could be unitary, and it is natural to expect that, in fact, this is the spherical unitary dual in these cases as well.

6.1. Examples of classical groups.

6.1.1. A_4.

Table 5: Table of spherical unitary parameters for A_4

$\check{\mathcal{O}}$	χ	unitary ν
(5)	$(-2,-1,0,1,2)$	
(41)	$(-3/2,-1/2,0,1/2,3/2)$	
(32)	$(-1,-1/2,0,1/2,1)$	
(311)	$(-1,0,1,-\nu,\nu)$	$0 \leq \nu < 1/2$
(221)	$(-1/2-\nu,1/2-\nu,0,-1/2+\nu,1/2+\nu)$	$0 \leq \nu < 1/2$
(21^3)	$(-1/2,0,1/2,-\nu,\nu)$	$0 \leq \nu < 1/2$
(1^5)	$(-\nu_2,-\nu_1,0,\nu_1,\nu_2)$	$0 \leq \nu_2 \leq \nu_2 < 1/2$

6.1.2. B_4.

Table 6: Table of spherical unitary parameters for B_4

$\check{\mathcal{O}}$	χ	unitary ν
(8)	$(1/2,3/2,5/2,7/2)$	
(62)	$(1/2,1/2,3/2,5/2)$	
(611)	$(\nu,1/2,3/2,5/2)$	$0 \leq \nu < 1/2$
(44)	$(-3/2+\nu,-1/2+\nu,1/2+\nu,3/2+\nu)$	$\nu = 0$
(422)	$(-1/2+\nu,1/2+\nu,1/2,3/2)$	$0 \leq \nu < 1$
(4211)	$(\nu,1/2,1/2,3/2)$	$0 \leq \nu < 1/2$
(41^4)	$(\nu_1,\nu_2,1/2,3/2)$	$0 \leq \nu_1 \leq \nu_2 < 1/2$
(332)	$(-1+\nu,\nu,1+\nu,1/2)$	$0 \leq \nu < 1/2$
(3311)	$(-1+\nu_1,\nu_1,1+\nu_1,\nu_2)$	$0 \leq \nu_1 < 1/2, 0 \leq \nu_2 < 1/2$
(2^4)	$(-1/2+\nu_1,1/2+\nu_1,-1/2+\nu_2,1/2+\nu_2)$	$0 \leq \nu_1 \leq \nu_2 < 1/2$
(2^311)	$(-1/2+\nu_1,1/2+\nu_1,\nu_2,1/2)$	$0 \leq \nu_1 < 1/2, 0 \leq \nu_2 < 1/2$
(221^4)	$(\nu_1,\nu_2,-1/2+\nu_3,1/2+\nu_3)$	$\nu_3 = 0,\ 0 \leq \nu_1 \leq \nu_2 < 1/2$
(21^6)	$(\nu_1,\nu_2,\nu_3,1/2)$	$0 \leq \nu_1 \leq \nu_2 \leq \nu_3 < 1/2$
(1^8)	$(\nu_1,\nu_2,\nu_3,\nu_4)$	$0 \leq \nu_1 \leq \nu_2 \leq \nu_3 \leq \nu_4 < 1/2$

6.1.3. C_4.

Table 7: Table of spherical unitary parameters for C_4

$\check{\mathcal{O}}$	χ	unitary ν
(9)	$(1,2,3,4)$	
(711)	$(\nu,1,2,3)$	$\nu = 0$

Table 7 – continued from previous page

$\check{\mathcal{O}}$	χ	unitary ν
(531)	$(0, 1, 1, 2)$	
(522)	$(-1/2+\nu, 1/2+\nu, 1, 2)$	$0 \le \nu < 1/2$
(441)	$(-3/2+\nu, -1/2+\nu, 1/2+\nu, 3/2+\nu)$	$0 \le \nu < 1/2$
(51^4)	$(\nu_1, \nu_2, 1, 2)$	$0 \le \nu_1 \le \nu_2 < 1-\nu_1$
(333)	$(-1+\nu, \nu, 1+\nu, 1)$	$0 \le \nu < 1$
(331^3)	$(-1+\nu_1, \nu_1, 1+\nu_1, \nu_2)$	$\nu_1 = 0,\ 0 \le \nu_2 < 1$
(32211)	$(-1/2+\nu_1, 1/2+\nu_1, \nu_2, 1)$	$\nu_2 = 0,\ 0 \le \nu_1 < 1/2$
(31^6)	$(\nu_1, \nu_2, \nu_3, 1)$	$\nu_3 = 0,\ 0 \le \nu_1 \le \nu_2 < 1-\nu_1$
(2^41)	$(-1/2+\nu_1, 1/2+\nu_1, -1/2+\nu_2, 1/2+\nu_2)$	$0 \le \nu_1 \le \nu_2 < 1/2$
(221^5)	$(-1/2+\nu_1, 1/2+\nu_1, \nu_2, \nu_3)$	$0 \le \nu_1 < 1/2,\ 0 \le \nu_2 < \nu_3 < 1-\nu_2$
(1^9)	$(\nu_1, \nu_2, \nu_3, \nu_4)$	$0 \le \nu_1 \le \nu_2 \le \nu_3 \le \nu_4 \le 1-\nu_3$ $0 \le \nu_1 \le \nu_2 \le \nu_3 < 1-\nu_2 < \nu_4 < 1-\nu_1$

6.1.4. D_4.

Table 8: Table of spherical unitary parameters for D_4

$\check{\mathcal{O}}$	χ	unitary ν
(71)	$(0, 1, 2, 3)$	
(53)	$(0, 1, 1, 2)$	
(51^3)	$(\nu, 0, 1, 2)$	$0 \le \nu < 1$
$(44)_+$	$(-3/2+\nu, -1/2+\nu, 1/2+\nu, 3/2+\nu)$	$0 \le \nu < 1/2$
$(44)_-$	$(3/2-\nu, -1/2+\nu, 1/2+\nu, 3/2+\nu)$	$0 \le \nu < 1/2$
(3311)	$(\nu_1, -1+\nu_2, \nu_2, 1+\nu_2)$	$\nu_1 = 0,\ \nu_2 = 0$
(31^5)	$(\nu_1, \nu_2, 0, 1)$	$0 \le \nu_1 \le \nu_2 < 1-\nu_1$
$(2^4)_+$	$(-1/2+\nu_1, 1/2+\nu_1, -1/2+\nu_2, 1/2+\nu_2)$	$0 \le \nu_1, \nu_2 < 1/2$
$(2^4)_-$	$(1/2-\nu_1, 1/2+\nu_1, -1/2+\nu_2, 1/2+\nu_2)$	$0 \le \nu_1, \nu_2 < 1/2$
(221^4)	$(-1/2+\nu_1, 1/2+\nu_2, \nu_3, \nu_4)$	$0 \le \nu_1 < 1/2,\ 0 \le \lvert\nu_2\rvert \le \nu_3 < 1-\lvert\nu_2\rvert$
(1^8)	$(\nu_1, \nu_2, \nu_3, \nu_4)$	$0 \le \lvert\nu_1\rvert \le \nu_2 \le \nu_3 \le \nu_4 < 1-\nu_3$ $0 \le \lvert\nu_1\rvert \le \nu_2 \le \nu_3 < 1-\nu_2 < \nu_4 < 1-\lvert\nu_1\rvert$

6.2. Exceptional groups. The tables contain the nilpotent orbits $\check{\mathcal{O}} \neq 0$ in the notation of [**Car**] (the case $\check{\mathcal{O}} = 0$ was recorded in section 5.3), the Hermitian infinitesimal character, and the coordinates and type of the centralizer.

The nilpotent orbits which are exceptions are marked with $*$ in the tables. The complementary series for them are listed after the tables. For the rest of the nilpotent orbits, an infinitesimal character χ is in the complementary series if and only if the corresponding parameter ν is in the 0-complementary series for $\mathfrak{z}(\check{\mathcal{O}})$. The parameter ν is given by a string $(\nu_1, \dots, \nu_k)$, and the order agrees with the way the centralizer $\mathfrak{z}(\check{\mathcal{O}})$ is written in the tables. The parts of ν corresponding to a torus T_1 or T_2 in $\mathfrak{z}(\check{\mathcal{O}})$ must be 0, in order for χ to be unitary. In addition, if ν corresponds to A_1, the complementary series is $0 \le \nu < \frac{1}{2}$, while the notation A_1^ℓ means that it is $0 \le \nu < 1$. If a string $(\nu_1, \dots, \nu_k)$ of ν corresponds to type A_k, the last $k - [\frac{k}{2}]$ entries must be 0 in order for χ to be unitary. For example, in the table for E_8, for the nilpotent $A_4 + A_1$, the ν-string is (ν_1, ν_2, ν_3) and the centralizer is

A_2+T_1. This means that the unitary parameters are those for which $\nu_3=0$ (this is the T_1-piece), $\nu_2=0$ and $0\le\nu_1<\frac{1}{2}$ (this is the 0-complementary series of A_2).

There is one difference in E_6 due to the fact that we only consider Hermitian spherical infinitesimal characters χ. In this table, the ν-string already refers to the semisimple and Hermitian spherical parameter of the centralizer. For example, the nilpotent A_2+A_1 in E_6 has centralizer A_2+T_1, and the corresponding χ has a single ν. This ν corresponds to the Hermitian parameter in the A_2 part of $\mathfrak{z}(\check{\mathcal{O}})$, so it must satisfy $0\le\nu<\frac{1}{2}$.

6.2.1. G_2.

Table 9: Table of parameters $(\check{\mathcal{O}},\nu)$ for G_2

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
G_2	$(1,2,-3)$	1
$G_2(a_1)$	$(0,1,-1)$	1
$\widetilde{A}_1$	$(1,-\frac{1}{2}+\nu,-\frac{1}{2}-\nu)$	A_1
A_1	$(-\frac{1}{2}+\nu,\frac{1}{2}+\nu,-2\nu)$	A_1

6.2.2. F_4.

Table 10: Table of parameters $(\check{\mathcal{O}},\nu)$ for F_4

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
F_4	$(\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$	1
$F_4(a_1)$	$(\frac{7}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	1
$F_4(a_2)$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	1
C_3	$(\nu,\frac{5}{2},\frac{3}{2},\frac{1}{2})$	A_1
B_3	$(\frac{3}{2}+\nu,-\frac{3}{2}+\nu,\frac{3}{2},\frac{1}{2})$	A_1^ℓ
$F_4(a_3)$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	1
$C_3(a_1)$	$(\nu,\frac{3}{2},\frac{1}{2},\frac{1}{2})$	A_1
$A_1+\widetilde{A}_2$	$(\frac{1}{4}+\frac{3\nu}{2},\frac{3}{4}+\frac{\nu}{2},-\frac{1}{4}+\frac{\nu}{2},-\frac{5}{4}+\frac{\nu}{2})$	A_1
B_2	$(\nu_1,\nu_2,\frac{3}{2},\frac{1}{2})$	$2A_1$
$\widetilde{A}_1+A_2$	$(\frac{1}{2}+2\nu,\nu,-1+\nu,\frac{1}{2})$	A_1
$\widetilde{A}_2$	$(\nu_2+\frac{3\nu_1}{2},1+\frac{\nu_1}{2},\frac{\nu_1}{2},-1+\frac{\nu_1}{2})$	G_2
A_2	$(\frac{1}{2}+\nu_1+\nu_2,-\frac{1}{2}+\nu_1,-\frac{1}{2}+\nu_2,\frac{1}{2})$	A_1+T_1
$*A_1+\widetilde{A}_1$	$(\nu_1,\frac{1}{2}+\nu_2,-\frac{1}{2}+\nu_2,\frac{1}{2})$	$A_1+A_1^\ell$
$\widetilde{A}_1$	$(\nu_1+\nu_2,\nu_1-\nu_2,\frac{1}{2}+\nu_3,-\frac{1}{2}+\nu_3)$	B_2+T_1
A_1	$(\nu_1,\nu_2,\nu_3,\frac{1}{2})$	C_3

$\mathbf{F_4}$ exception:

$\mathbf{A_1+\widetilde{A}_1}$: $\{\nu_1+2\nu_2<\frac{3}{2},\ \nu_1<\frac{1}{2}\}\cup\{2\nu_2-\nu_1>\frac{3}{2},\nu_2<1\}$.

6.2.3. E_6.

Table 11: Table of Hermitian parameters $(\check{\mathcal{O}},\nu)$ for E_6

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
E_6	$(0,1,2,3,4,-4,-4,4)$	1
$E_6(a_1)$	$(0,1,1,2,3,-3,-3,3)$	1

Table 11 – continued from previous page

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
D_5	$(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, \frac{5}{2})$	T_1
$E_6(a_3)$	$(0, 0, 1, 1, 2, -2, -2, 2)$	1
$D_5(a_1)$	$(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, -\frac{7}{4}, -\frac{7}{4}, \frac{7}{4})$	T_1
A_5	$(-\frac{11}{4}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{5}{4}, -\frac{5}{4}, \frac{5}{4}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	A_1
$A_4 + A_1$	$(0, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$	T_1
D_4	$(0, 1, 2, 3, \nu, -\nu, -\nu, \nu)$	A_2
A_4	$(-2, -1, 0, 1, 2, 0, 0, 0) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	A_1T_1
$D_4(a_1)$	$(0, 0, 1, 1, 1, -1, -1, 1)$	T_2
$A_3 + A_1$	$(-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{5}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	A_1T_1
$2A_2 + A_1$	$(0, 1, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu(0, 0, 1, 1, 1, -1, -1, 1)$	A_1
A_3	$(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 0, 0, 0, 0) + (\frac{\nu_1}{2}, \frac{\nu_1}{2}, \frac{\nu_1}{2}, \frac{\nu_1}{2}, \frac{\nu_2}{2}, -\frac{\nu_2}{2}, -\frac{\nu_2}{2}, \frac{\nu_2}{2})$	B_2T_1
$A_2 + 2A_1$	$(\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}) + \nu(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$	A_1T_1
$2A_2$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) +$ $(\frac{\nu_2}{2}, \frac{\nu_2}{2}, \frac{2\nu_1+\nu_2}{2}, \frac{2\nu_1+\nu_2}{2}, \frac{2\nu_1+\nu_2}{2}, -\frac{2\nu_1+\nu_2}{2}, -\frac{2\nu_1+\nu_2}{2}, \frac{2\nu_1+\nu_2}{2})$	G_2
$A_2 + A_1$	$(-\frac{1}{2}, \frac{1}{2}, -1, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	A_2T_1
A_2	$(0, -1, 0, 1, 0, 0, 0, 0) + (\frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2},$ $\frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2})$	$2A_2$
$3A_1$	$(0, 1, -\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0) + (0, 0, \nu_1, \nu_2, \nu_1, -\nu_1, -\nu_1, \nu_1)$	A_2A_1
$2A_1$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0) +$ $(\frac{-\nu_1+\nu_2}{2}, \frac{-\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, -\frac{\nu_1}{2}, -\frac{\nu_1}{2}, \frac{\nu_1}{2})$	B_3T_1
A_1	$(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0) + (\frac{-\nu_1+\nu_2}{2}, \frac{\nu_1-\nu_2}{2},$ $\frac{-\nu_1+\nu_2}{2} + \nu_3, \frac{\nu_1-\nu_2}{2} + \nu_3, \frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, -\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2}{2})$	A_5

6.2.4. E_7.

Table 12: Table of parameters $(\check{\mathcal{O}}, \nu)$ for E_7

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
E_7	$(0, 1, 2, 3, 4, 5, -\frac{17}{2}, \frac{17}{2})$	1
$E_7(a_1)$	$(0, 1, 1, 2, 3, 4, -\frac{13}{2}, \frac{13}{2})$	1
$E_7(a_2)$	$(0, 1, 1, 2, 2, 3, -\frac{11}{2}, \frac{11}{2})$	1
$E_7(a_3)$	$(0, 0, 1, 1, 2, 3, -\frac{9}{2}, \frac{9}{2})$	1
E_6	$(0, 1, 2, 3, 4, -4, -4, 4) + \nu(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	A_1^ℓ
D_6	$(0, 1, 2, 3, 4, 5, 0, 0) + \nu(0, 0, 0, 0, 0, 0, -1, 1)$	A_1
$E_6(a_1)$	$(0, 1, 1, 2, 3, -3, -3, 3) + \nu(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	T_1
$E_7(a_4)$	$(0, 0, 1, 1, 1, 2, -\frac{7}{2}, \frac{7}{2})$	1
$D_6(a_1)$	$(0, 1, 1, 2, 3, 4, 0, 0) + \nu(0, 0, 0, 0, 0, 0, -1, 1)$	A_1
A_6	$(-\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2}) + \nu(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, 1)$	A_1^ℓ
$D_5 + A_1$	$(0, 1, 2, 3, -\frac{5}{2}, -\frac{3}{2}, -2, 2) + \nu(0, 0, 0, 0, 1, 1, -1, 1)$	A_1
$E_7(a_5)$	$(0, 0, 1, 1, 1, 2, -\frac{5}{2}, \frac{5}{2})$	1
$D_6(a_2)$	$(0, 1, 1, 2, 2, 3, 0, 0) + \nu(0, 0, 0, 0, 0, 0, -1, 1)$	A_1
$A_5 + A_1$	$(\frac{11}{4}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, -\frac{1}{4}, \frac{1}{4}) + \nu(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{3}{2})$	A_1
D_5	$(0, 1, 2, 3, -2, -2, -2, 2) + \nu_1(0, 0, 0, 0, 1, 1, -1, 1)$ $+\nu_2(0, 0, 0, 0, -1, 1, 0, 0)$	$2A_1$
$E_6(a_3)$	$(0, 0, 1, 1, 2, -2, -2, 2) + \nu(0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$	A_1^ℓ
$D_5(a_1)A_1$	$(0, 1, 1, 2, -2, -1, -\frac{3}{2}, \frac{3}{2}) + \nu(0, 0, 0, 0, 1, 1, -1, 1)$	A_1^ℓ
$(A_5)'$	$(-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 0, 0) + \nu_1(0, 0, 0, 0, 0, 0, -1, 1)$ $+\nu_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$	$A_1A_1^\ell$

Table 12 – continued from previous page

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
A_4+A_2	$(0,1,2,-2,-1,0,-1,1)+\nu(0,0,0,1,1,1,-\frac{3}{2},\frac{3}{2})$	A_1^ℓ
$(A_5)''$	$(\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},0,0)+\nu_2(0,0,0,0,0,0,-1,1)$ $+\nu_1(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$	G_2
$D_5(a_1)$	$(0,1,1,2,3,0,0,0)+\nu_1(0,0,0,0,0,0,-1,1)$ $+\nu_2(0,0,0,0,0,1,-\frac{1}{2},\frac{1}{2})$	A_1T_1
A_4+A_1	$(\frac{9}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},\frac{7}{4},-\frac{1}{4},-\frac{1}{2},\frac{1}{4})+\nu_1(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-1,1)$ $+\nu_2(0,0,0,0,0,1,-\frac{1}{2},\frac{1}{2})$	T_2
D_4+A_1	$(0,1,2,3,-\frac{1}{2},\frac{1}{2},0,0)+\nu_1(0,0,0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$ $+\nu_2(0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	B_2
$A_3A_2A_1$	$(0,1,-2,-1,0,1,-\frac{1}{2},\frac{1}{2})+\nu(0,0,1,1,1,1,-2,2)$	A_1
A_4	$(0,-2,-1,0,1,2,0,0)+\nu_1(0,0,0,0,0,0,-1,1)$ $+\nu_2(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})+\nu_3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-1,1)$	A_2T_1
A_3+A_2	$(0,1,2,-1,0,1,0,0)+\nu_1(0,0,0,0,0,0,-1,1)$ $+\nu_2(0,0,0,1,1,1,0,0)$	A_1T_1
D_4	$(0,1,2,3,\nu_2-\nu_1,\nu_2+\nu_1,-\nu_3,\nu_3)$	C_3
$D_4(a_1)A_1$	$(0,1,1,2,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,0,\nu_2,\nu_2,-\nu_1,\nu_1)$	$2A_1$
A_3+2A_1	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0)+(0,0,\nu_2,\nu_2,\nu_2,\nu_2,-\nu_1,\nu_1)$	$2A_1$
$D_4(a_1)$	$(0,1,1,2,\nu_2-\nu_3,\nu_2+\nu_3,-\nu_1,\nu_1)$	$3A_1$
$(A_3+A_1)'$	$(0,1,2,0,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,2\nu_2,\nu_3,\nu_3,-\nu_1,\nu_1)$	$3A_1$
$2A_2+A_1$	$(\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},-\frac{1}{4},\frac{1}{4})+\nu_1(1,-1,-1,1,1,1,0,0)$ $+\nu_2(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$	$2A_1$
$(A_3+A_1)''$	$(\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},-\frac{1}{2},\frac{1}{2},0,0)+$ $(-\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_1}{2},\frac{\nu_3-\nu_2}{2},\frac{\nu_3-\nu_2}{2},-\frac{\nu_3+\nu_2}{2},\frac{\nu_3+\nu_2}{2})$	B_3
A_2+3A_1	$(0,1,-1,0,-1,0,-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,1,-2,2)$ $+\nu_2(0,0,0,0,1,1,-1,1)$	G_2
$2A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,-1,-1,1)$ $+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_3(0,0,0,0,0,1,-\frac{1}{2},\frac{1}{2})$	G_2A_1
A_3	$(0,1,2,\nu_1,\nu_2,\nu_3,\nu_4)$	B_3A_1
$*A_2+2A_1$	$(0,1,-1,0,1,0,0,0)+(0,0,\nu_2,\nu_2,\nu_2,\nu_3,-\nu_1,\nu_1)$	$A_12A_1^\ell$
A_2+A_1	$(1,0,1,0,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,0,\nu_2,\nu_2,-\nu_1,\nu_1)$ $+\nu_3(0,0,0,1,1,1,-\frac{3}{2},\frac{3}{2})+\nu_4(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$	A_3T_1
$4A_1$	$(0,1,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(0,0,\nu_3,\nu_3,\nu_2,\nu_2,-\nu_1,\nu_1)$	C_3
A_2	$(1,0,1,0,0,0,0,0)+(0,0,0,0,\nu_2-\nu_3,\nu_2+\nu_3,-\nu_1,\nu_1)$ $+\nu_4(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})+\nu_5(0,0,0,1,1,1,-\frac{3}{2},\frac{3}{2})$	A_5
$(3A_1)'$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(\nu_1,\nu_1,\nu_2,\nu_2,\nu_3,\nu_3,-\nu_4,\nu_4)$	C_3A_1
$(3A_1)''$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(-\nu_4,\nu_4,\nu_3,\nu_3,\nu_2,\nu_2,-\nu_1,\nu_1)$	F_4
$2A_1$	$(0,1,\nu_1,\nu_2,\nu_3,\nu_4,-\nu_5,\nu_5)$	B_4A_1
A_1	$(\frac{\nu_1+\nu_2+\nu_3-\nu_4}{2},\frac{\nu_1+\nu_2-\nu_3+\nu_4}{2},\frac{\nu_1-\nu_2+\nu_3+\nu_4}{2},\frac{-\nu_1+\nu_2+\nu_3+\nu_4}{2},$ $-\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},-\frac{\nu_5+\nu_6}{2},\frac{\nu_5+\nu_6}{2})$	D_6

$\mathbf{E_7}$ exception:

$\mathbf{A_2+2A_1}$. Three regions: $\{0\le\nu_1<\frac{1}{2},0\le\nu_2<1,0\le\nu_3<1,\nu_1+\frac{3\nu_2}{2}+\frac{\nu_3}{2}<\frac{3}{2}\}$, $\{0\le\nu_1<\frac{1}{2},0\le\nu_2<1,0\le\nu_3<1,-\nu_1+\frac{3\nu_2}{2}+\frac{\nu_3}{2}<\frac{3}{2},\nu_1+\frac{3\nu_2}{2}-\frac{\nu_3}{2}>\frac{3}{2}\}$, and $\{0\le\nu_1<\frac{1}{2},0\le\nu_2<1,0\le\nu_3<1,\frac{3\nu_2}{2}+\frac{\nu_3}{2}>\frac{3}{2},\nu_1+\frac{3\nu_2}{2}-\frac{\nu_3}{2}<\frac{3}{2}\}$.

6.2.5. E_8.

Table 13: Table of parameters $(\check{\mathcal{O}}, \nu)$ for E_8

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
E_8	$(0,1,2,3,4,5,6,23)$	1
$E_8(a_1)$	$(0,1,1,2,3,4,5,18)$	1
$E_8(a_2)$	$(0,1,1,2,2,3,4,15)$	1
$E_8(a_3)$	$(0,0,1,1,2,3,4,13)$	1
$E_8(a_4)$	$(0,0,1,1,2,2,3,11)$	1
E_7	$(0,1,2,3,4,5,-\frac{17}{2},\frac{17}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_8(b_4)$	$(0,0,1,1,1,2,3,10)$	1
$E_8(a_5)$	$(0,0,1,1,1,2,2,9)$	1
$E_7(a_1)$	$(0,1,1,2,3,4,-\frac{13}{2},\frac{13}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_8(b_5)$	$(0,0,1,1,1,2,3,8)$	1
D_7	$(0,1,2,3,4,5,6,0)+\nu(0,0,0,0,0,0,0,2)$	A_1
$E_8(a_6)$	$(0,0,1,1,1,2,2,7)$	1
$E_7(a_2)$	$(0,1,1,2,2,3,-\frac{11}{2},\frac{11}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
E_6+A_1	$(0,1,2,3,4,-\frac{9}{2},-\frac{7}{2},4)+\nu(0,0,0,0,0,1,1,2)$	A_1
$D_7(a_1)$	$(0,1,1,2,3,4,5,0)+\nu(0,0,0,0,0,0,0,2)$	T_1
$E_8(b_6)$	$(0,0,1,1,1,1,2,6)$	1
$E_7(a_3)$	$(0,0,1,1,2,3,-\frac{9}{2},\frac{9}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_6(a_1)A_1$	$(0,1,1,2,3,-\frac{7}{2},-\frac{5}{2},3)+\nu(0,0,0,0,0,1,1,2)$	T_1
A_7	$(-\frac{17}{4},-\frac{13}{4},-\frac{9}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{7}{4})+\nu(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{2})$	T_1
E_6	$(0,1,2,3,4,-4,-4,4)+\nu_1(0,0,0,0,0,1,1,2)$ $+\nu_2(0,0,0,0,0,0,1,1)$	G_2
D_6	$(0,1,2,3,4,5,\nu_1,\nu_2)$	B_2
D_5+A_2	$(0,1,2,3,-3,-2,-1,2)+\nu(0,0,0,0,1,1,1,3)$	T_1
$E_6(a_1)$	$(0,1,1,2,3,-3,-3,3)+\nu_2(0,0,0,0,0,1,1,2)$ $+\nu_1(0,0,0,0,0,0,1,1)$	A_2
$E_7(a_4)$	$(0,0,1,1,1,2,-\frac{7}{2},\frac{7}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
A_6+A_1	$(\frac{13}{4},-\frac{9}{4},-\frac{5}{4},-\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{11}{4},\frac{1}{4})+\nu(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{2})$	A_1
$D_6(a_1)$	$(0,1,1,2,3,4,0,0)+\nu_1(0,0,0,0,0,0,-1,1)$ $+(0,0,0,0,0,0,1,1)$	$2A_1$
A_6	$(-3,-2,-1,0,1,2,3,0)+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $+\nu_1(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{7}{2})$	$2A_1$
$E_8(a_7)$	$(0,0,0,1,1,1,1,4)$	1
D_5+A_1	$(0,1,2,3,4,-\frac{1}{2},\frac{1}{2},0)+\nu_1(0,0,0,0,0,0,0,2)$ $+\nu_2(0,0,0,0,0,1,1,0)$	$2A_1$
$E_7(a_5)$	$(0,0,1,1,1,2,-\frac{5}{2},\frac{5}{2})+\nu(0,0,0,0,0,0,1,1)$	A_1
$E_6(a_3)A_1$	$(0,0,1,1,2,-\frac{5}{2},-\frac{3}{2},2)+\nu(0,0,0,0,0,1,1,2)$	A_1
$D_6(a_2)$	$(0,1,1,2,2,3,-\nu_1+\nu_2,\nu_1+\nu_2)$	$2A_1$
$D_5(a_1)A_2$	$(0,1,1,2,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{3}{2})+\nu(0,0,0,0,1,1,1,3)$	A_1
A_5+A_1	$(\frac{1}{4},-\frac{11}{4},-\frac{7}{4},-\frac{3}{4},\frac{1}{4},\frac{5}{4},\frac{9}{4},\frac{1}{4})+\nu_2(-1,0,0,0,0,0,0,1)$ $+\nu_1(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$	$2A_1$
A_4+A_3	$(0,1,2,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},1)+\nu(0,0,0,1,1,1,1,4)$	A_1
D_5	$(0,1,2,3,4,\nu_1,\nu_2,\nu_3)$	B_3
$E_6(a_3)$	$(0,0,1,1,2,-2,-2,2)+\nu_1(0,0,0,0,0,1,1,2)$ $+\nu_2(0,0,0,0,0,0,1,1)$	G_2
D_4+A_2	$(0,1,2,3,-1,0,1,0)+\nu_2(0,0,0,0,1,1,1,3)$ $+\nu_1(0,0,0,0,0,0,0,2)$	A_2
$*A_4A_2A_1$	$(0,1,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2})+\nu(0,0,1,1,1,1,1,5)$	A_1
$*D_5(a_1)A_1$	$(0,1,1,2,3,-\frac{1}{2}+\nu_2,\frac{1}{2}+\nu_2,2\nu_1)$	$A_1^{\ell}A_1$

Table 13 – continued from previous page

$\check{\mathcal{O}}$	χ	$\mathfrak{z}(\check{\mathcal{O}})$
A_5	$(\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},0,0)+\nu_1(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{3}{2})$ $+\nu_2(0,0,0,0,0,0,-1,1)+\nu_3(0,0,0,0,0,0,1,1)$	G_2A_1
$*A_4+A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2})+\nu_2(1,1,0,0,0,0,0,0)$ $+\nu_1(0,0,1,1,1,1,1,5)$	$2A_1$
A_4+2A_1	$(0,1,-2,-1,0,1,2,0)+\nu_1(0,0,0,0,0,0,0,2)$ $\nu_2(0,0,1,1,1,1,1,0)$	A_1T_1
$D_5(a_1)$	$(0,1,1,2,3,\nu_3,\nu_2,\nu_1)$	A_3
$2A_3$	$(0,1,2,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0)+\nu_2(0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1)$	B_2
A_4+A_1	$(0,1,2,-\frac{3}{2},-\frac{1}{2},-1,-1,1)+\nu_2(0,0,0,0,0,1,1,2)$ $+\nu_1(0,0,0,0,0,0,1,1)+\nu_3(0,0,0,1,1,1,1,4)$	A_2T_1
$D_4(a_1)A_2$	$(0,1,1,2,-1,0,1,0)+\nu_1(0,0,0,0,1,1,1,3)$ $+\nu_2(0,0,0,0,0,0,0,2)$	A_2
D_4+A_1	$(0,1,2,3,-\frac{1}{2},\frac{1}{2},0,0)+$ $(0,0,0,0,\nu_1,\nu_1,-\nu_2+\nu_3,\nu_2+\nu_3)$	C_3
$A_3A_2A_1$	$(0,1,-2,-1,0,1,-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,1,-2,2)$ $+\nu_2(0,0,0,0,0,0,1,1)$	$2A_1$
A_4	$(0,-2,-1,0,1,2,0,0)+$ $(\nu_4,-\nu_1+\nu_2,\nu_3,\nu_3,\nu_3,\nu_3,\nu_3,\nu_1+\nu_2)$	A_4
A_3+A_2	$(0,1,2,-1,0,1,0,0)+(0,0,0,\nu_3,\nu_3,\nu_3,\nu_1,\nu_2)$	B_2T_1
$D_4(a_1)A_1$	$(0,1,1,2,-\frac{1}{2},\frac{1}{2},0,0)+(0,0,0,0,\nu_1,\nu_1,-\nu_2+\nu_3,\nu_2+\nu_3)$	$3A_1$
A_3+2A_1	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},0,0)+(0,0,\nu_1,\nu_1,\nu_1,\nu_1,\nu_2,\nu_3)$	A_1B_2
$2A_2+2A_1$	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-1,0,\frac{1}{2})+\nu_1(0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},1,1,\frac{1}{2})$ $+\nu_2(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,\frac{3}{2})$	B_2
D_4	$(0,1,2,3,\nu_3-\nu_4,\nu_3+\nu_4,\nu_1-\nu_2,\nu_1+\nu_2)$	F_4
$D_4(a_1)$	$(0,1,1,2,\nu_4,\nu_3,\nu_2,\nu_1)$	D_4
A_3+A_1	$(0,1,2,-\frac{1}{2},\frac{1}{2},0,0,0)+(0,0,0,\nu_1,\nu_1,\nu_2,\nu_3,\nu_4)$	A_1B_3
$2A_2+A_1$	$(0,1,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,-1,-1,1)$ $+\nu_2(0,0,0,0,0,1,1,2)+\nu_3(0,0,0,0,0,0,1,1)$	A_1G_2
$2A_2$	$(-\frac{1}{2},\frac{1}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,-1,-1,1)$ $+\nu_2(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_3(0,0,0,0,0,1,1,2)$ $+\nu_4(0,0,0,0,0,0,1,1)$	$2G_2$
$*A_2+3A_1$	$(0,1,-1,0,-1,0,-\frac{1}{2},\frac{1}{2})+\nu_1(0,0,1,1,1,1,-2,2)$ $+\nu_2(0,0,0,0,1,1,-1,1)+\nu_3(0,0,0,0,0,0,1,1)$	G_2A_1
A_3	$(0,1,2,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$	B_5
$*A_2+2A_1$	$(0,1,-1,0,1,0,0,0)+(0,0,\nu_1,\nu_1,\nu_1,\nu_2,\nu_3,\nu_4)$	A_1B_3
A_2+A_1	$(1,0,1,0,-\frac{1}{2},\frac{1}{2},0,0)+$ $(-\nu_5,\nu_5,\nu_5,\nu_4,\nu_3,\nu_3,-\nu_2+\nu_1,\nu_2+\nu_1)$	A_5
$*4A_1$	$(0,1,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+$ $(0,0,\nu_1,\nu_1,\nu_2,\nu_2,-\nu_3+\nu_4,\nu_3+\nu_4)$	C_4
A_2	$(\frac{\nu_1-\nu_2-\nu_3+\nu_4}{2},\frac{-\nu_1+\nu_2-\nu_3+\nu_4}{2},\frac{-\nu_1-\nu_2+\nu_3+\nu_4}{2},$ $\frac{\nu_1+\nu_2+\nu_3+\nu_4}{2},-1+\frac{\nu_5-\nu_6}{2},\frac{\nu_5-\nu_6}{2},1+\frac{\nu_5-\nu_6}{2},\frac{\nu_5+3\nu_6}{2})$	E_6
$3A_1$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},0,0)+(-\nu_4,\nu_4,\nu_3,\nu_3,\nu_2,\nu_2,-\nu_1,\nu_1)$ $+\nu_5(0,0,0,0,0,0,1,1)$	F_4A_1
$2A_1$	$(0,1,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)$	B_6
A_1	$(\frac{\nu_1+\nu_2+\nu_3-\nu_4}{2},\frac{\nu_1+\nu_2-\nu_3+\nu_4}{2},\frac{\nu_1-\nu_2+\nu_3+\nu_4}{2},\frac{-\nu_1+\nu_2+\nu_3+\nu_4}{2},$ $\frac{-\nu_5-\nu_6+2\nu_7}{2},-\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},\frac{1}{2}+\frac{-\nu_5+\nu_6}{2},\frac{\nu_5+\nu_6+2\nu_7}{2})$	E_7

E_8 exceptions:
$A_4 + A_2 + A_1$. $\{0 \le \nu < \frac{3}{10}\}$.
$D_5(a_1) + A_1$. Two regions: $\{0 \le \nu_2 < \frac{1}{2}, 2\nu_1 + \nu_2 < \frac{3}{2}\}$, and $\{0 \le \nu_1 < 1, 2\nu_1 - \nu_2 > \frac{3}{2}\}$.
$A_4 + A_2$. Two regions: $\{0 \le \nu_2 < \frac{1}{2}, 5\nu_1 + \nu_2 < 2\}$, and $\{0 \le \nu_1 < \frac{1}{2}, 5\nu_1 - \nu_2 > 2\}$.
$A_2 + 3A_1$. Four regions: $\{3\nu_1 + 2\nu_2 < 1, 0 \le \nu_3 < \frac{1}{2}\}$, $\{2\nu_1 + \nu_2 < 1 < 3\nu_1 + \nu_2, 0 \le \nu_3 < \frac{1}{2}, 3\nu_1 + 2\nu_2 + \nu_3 < \frac{3}{2}\}$, $\{2\nu_1 + \nu_2 < 1 < 3\nu_1 + \nu_2, 0 \le \nu_3 < \frac{1}{2}, 3\nu_1 + \nu_2 + \nu_3 < \frac{3}{2} < 3\nu_1 + 2\nu_2 - \nu_3\}$, and $\{2\nu_1 + \nu_2 < 1 < 3\nu_1 + \nu_2, 0 \le \nu_3 < \frac{1}{2}, 3\nu_1 + 2\nu_2 - \nu_3 < \frac{3}{2} < 3\nu_1 + \nu_2 + \nu_3\}$.
$A_2 + 2A_1$. Seven regions: $\{0 \le \nu_1 < 1, \nu_3 + \nu_4 < 1, 3\nu_1 + \nu_2 + \nu_3 + \nu_4 < 3\}$, $\{0 \le \nu_1 < 1, \nu_3 + \nu_4 < 1, 3\nu_1 + \nu_2 - \nu_3 + \nu_4 < 3 < 3\nu_1 - \nu_2 + \nu_3 + \nu_4\}$, $\{0 \le \nu_1 < 1, \nu_3 + \nu_4 < 1, 3\nu_1 - \nu_2 - \nu_3 + \nu_4 > 3\}$, $\{0 \le \nu_1 < 1, \nu_3 + \nu_4 < 1, 3\nu_1 + \nu_2 + \nu_3 - \nu_4 > 3\}$, $\{0 \le \nu_1 < 1, \nu_2 + \nu_4 > 1, \nu_2 + \nu_3 < 1, \nu_4 < 1, 3\nu_1 + \nu_2 + \nu_3 + \nu_4 < 3\}$, $\{0 \le \nu_1 < 1, \nu_2 + \nu_4 > 1, \nu_2 + \nu_3 < 1, \nu_4 < 1, 3\nu_1 - \nu_2 - \nu_3 + \nu_4 > 3\}$, and $\{0 \le \nu_1 < 1, \nu_2 + \nu_4 > 1, \nu_2 + \nu_3 < 1, \nu_4 < 1, 3\nu_1 + \nu_2 + \nu_3 - \nu_4 > 3\}$.
$4A_1$. Two regions: $\{0 \le \nu_1 \le \nu - 2 \le \nu_3 \le \nu_4 < \frac{1}{2}\}$ and $\{\nu_1 + \nu_4 < 1, \nu_2 + \nu_3 < 1, \nu_2 + \nu_4 > 1, -\nu_1 + \nu_3 + \nu_4 < \frac{3}{2} < \nu_1 + \nu_3 + \nu_4\}$.

References

[Ar] J. Arthur, *Unipotent automorphic representations: Conjectures*, Orbites unipotentes et répresentations II, Astérisque 171-172 (1989), 13–71.
[Ba1] D. Barbasch, *The spherical unitary dual of split classical real and p-adic groups,* preprint, `www.math.cornell.edu/~barbasch`.
[Ba2] ———, *Relevant and petite K−types for split groups*, Functional analysis VIII, 35–71, Various Publ. Ser. (Aarhus), 47, Aarhus Univ., Aarhus, 2004.
[Ba3] ———, *The unitary dual of complex classical groups*, Invent. Math. **96** (1989), 103–176.
[Ba4] ———, *Spherical unitary dual for unitary groups*, Proceedings of conference in honor of J. Carmona, Birkhäuser, 2003, 21-60.
[Bar] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) 48, (1947). 568–640.
[BC1] D. Barbasch, D. Ciubotaru, *Whittaker unitary dual of affine graded Hecke algebras of type E*, preprint, `arxiv.org`.
[BC2] ———, *The spherical unitary dual for split groups of exceptional type*, preprint.
[BC3] D. Barbasch, D. Ciubotaru, *Spherical unitary principal series* Pure and Applied Mathematics Quarterly, volume 1, number 4, 2005, Armand Borel special issue vol III, 2005.
[BM1] D. Barbasch, A. Moy, *A unitarity criterion for p-adic groups* Invent. Math., vol. 98, 1989, 19-38.
[BM2] ———, *Reduction to real infinitesimal character in affine Hecke algebras*, Journal of the AMS, vol 6, Number 3, 1993, 611-635.
[BM3] ———, *Unitary spherical spectrum for p-adic classical groups*, Representations of Lie groups, Lie algebras and their quantum analogues, Acta Appl. Math. 44 (1996), no. 1-2, 3–37.
[BM4] ———, *Whittaker models with an Iwahori fixed vector*, Representation theory and analysis on homogeneous spaces (New Brunswick, NJ, 1993), 101–105, Contemp. Math., 177, Amer. Math. Soc., Providence, RI, 1994.
[BP] D. Barbasch, A. Pantano, *Petite K-types for nonspherical minimal principal series*, preprint.
[Bo] A. Borel, *Admissible representations of a semisimple group over a local field with fixed vectors under an Iwahori subgroup*, Invent. Math., vol 35, 1976, 233-259.
[BW] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups* American Mathematical Society, Providence, RI, 1999.
[BK] C. Bushnell, P. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) 77 (1998), no. 3, 582–634.
[Car] R. Carter, *Finite groups of Lie type*, Wiley-Interscience, New York, 1985.

[Cas1] W. Casselman, *The unramified principal series of p-adic groups I*, Comp. Math., vol. 40, 1980, 387–406.
[Cas2] ——, *Introduction to the theory of admissible representations of p-adic groups* (draft: 1 May 1995), www.math.ubc.ca/∼cass/research.html
[Ci] D. Ciubotaru, *The Iwahori spherical unitary dual of the split group of type F4*, Represent. Theory, vol 9, 2005, 94–137.
[CM] D. Collingwood, W. McGovern, *Nilpotent orbits in complex semisimple Lie algebras*, Van Nostrand Reinhold Company, New York, NY, 1993.
[HM] R. Howe, A. Moy, *Hecke algebra isomorphisms for* $\mathrm{GL}(n)$ *over a p-adic field*, J. Algebra 131 (1990), no. 2, 388–424.
[IM] N. Iwahori, Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Publ. Math. Inst. Hautes Étud. Sci. 25 (1965), 5–48.
[JW] K. Johnson, N. Wallach, *Composition series and intertwining operators for spherical principal series*, Bull. Am. Math. Soc. 78 (1972), 1053–1059.
[KL] D. Kazhdan, G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math., vol 87, 1987 , 153-215.
[Kn] A.W. Knapp, *Representation theory of semisimple groups. An overview based on examples*, Princeton University Press, Princeton, NJ, 2001.
[KZ] A.W. Knapp, G. Zuckerman, *Classification of irreducible tempered representations of semisimple Lie groups*, Ann. of Math. 116 (1982), 389–501.
[Ko] B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. 75 (1969), 627–642.
[Lu1] G. Lusztig, *Affine Hecke algebras and their graded version*, Jour. AMS, vol 2, 1989, 599-635.
[Lu2] ——, *Cuspidal local systems and graded Hecke algebras II*, CMS conf. Proc. 16, Banff, Alberta, 1995, 217–275.
[Mi] D. Miličić, *Asymptotic behaviour of matrix coefficients of admissible representations*, Duke Math. J. 44 (1977), no. 1, 59–88.
[Mu] G. Muić, *The unitary dual of p-adic* G_2, Duke Math. J. 90 (1997), no. 3, 465–493.
[Pa] A. Pantano, *Weyl group representations and signatures of intertwining operators*, Ph.D. thesis, Princeton University, 2004.
[Re] M. Reeder, *Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations*, Represent. Theory, vol. 6 (2002), 101–126.
[Ro] A. Roche, *Types and Hecke algebras for principal series representations of split p-adic groups*, Ann. Scient. Ec. Norm. Sup., vol. 31, 1998, 361–413.
[Sa] P.J. Sally, Jr., *Unitary and uniformly bounded representations of the two by two unimodular group over local fields*, Amer. J. Math. 90 1968 406–443.
[Sp] T.A. Springer, *A construction of representations of Weyl groups*, Invent. Math. 44 (1978), no. 3, 279–293.
[Ta] M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Ann. Scient. Ec. Norm. Sup., vol 19 , 1986, 335-382.
[Vo1] D.A. Vogan, Jr., *The unitary dual of* $GL(n)$ *over an Archimedean field*, Invent. Math. 83 (1986), 449–505.
[Vo2] ——, *The unitary dual of* G_2, Invent. Math. 116 (1994), No. 1-3, 677–791.
[Vo3] ——, *Representations of real reductive Lie groups*, Progress in Mathematics, Vol. 15, Birkhäuser, 1981.
[Vo4] ——, *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. 48 (1978), no. 1, 75–98.
[Ze] A. Zelevinski, *Induced representations of reductive p-adic groups II. On irreducible representations of GL(n)*, Ann. Scient. Ecole Norm. Sup., 13 (1980), 165-210.

(D. Barbasch) Dept. of Mathematics, Cornell University, Ithaca, NY 14850
E-mail address: barbasch@math.cornell.edu

(D. Ciubotaru) Dept. of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: ciubo@math.utah.edu

(A. Pantano) Dept. of Mathematics, University of California, Irvine, CA
E-mail address: apantano@math.uci.edu