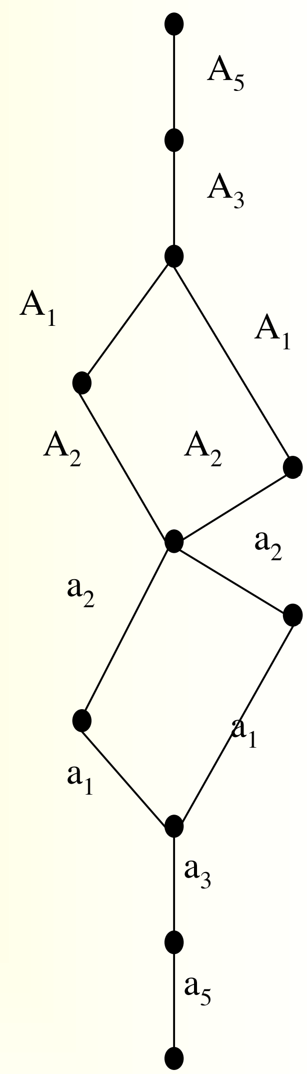
The Geometry of Conjugacy Classes of Nilpotent Matrices



Alessandra Pantano

Oliver Club Talk, Cornell
April 14, 2005

References: H. Kraft and C. Procesi, *Minimal Singularities in GLn*, Invent. Math. 62, 1981

David H. Collingwood, William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*

Introduction

- $\mathfrak{g}$ = complex **classical** Lie algebra
- $\mathcal{N}$= set of **nilpotent** matrices in $\mathfrak{g}$
- $\mathcal{G}$ = the adjoint group = G/Z(G)

$\mathcal{G}$ acts on $\mathcal{N}$ by conjugation

The orbits are conjugacy classes…

- ☆ Combinatorial description
- ☆ Formula for the dimension
- ☆ Geometric description of $\mathcal{N}$

Remarks

Why **NILPOTENT** ?

There are only **finitely many** c.c. of nilpotent matrices

Why should $\mathfrak{g}$ be **classical**?

If $\mathfrak{g} \subset \mathfrak{gl}(N)$, we can use the standard representation of $\mathfrak{g}$ on $\mathbb{C}^N$ to obtain a classification of c.c. via **partitions of N**

Outline of the talk

Part 1

Combinatorial description of nilpotent orbits

Part 2

Dimension of nilpotent orbits

Part 3

Partial ordering of nilpotent orbits

The 'easy' case: $\mathfrak{g} = \mathfrak{sl}(n)$

- $\mathfrak{g} = \mathfrak{sl}(n) = \{X \in M_n(\mathbb{C}) : tr(X) = 0\}$
- $SL(n) = \{A \in M_n(\mathbb{C}) : \det(A) = 1\}$
- $\mathcal{G} = PSL(n) = SL(n) / Z$
- $\mathcal{N}$ = all nilpotent matrices

The GL_n, SL_n, PSL_n-conjugacy classes coincide!

$$A \cdot X \cdot A^{-1} = \left(\frac{A}{\det(A)}\right) \cdot X \cdot \left(\frac{A}{\det(A)}\right)^{-1} = \left(\frac{A}{\sqrt[n]{\det(A)}}\right) \cdot X \cdot \left(\frac{A}{\sqrt[n]{\det(A)}}\right)^{-1}$$

We can use the theory of **Jordan forms**.

Partition-type Classification for $g=sl(n)$

Conjugacy classes of nilpotent n x n matrices

Normal Jordan Block Form

$$J = \left[\begin{array}{c|c|c|c} J_{p_1} & 0 & \dots & 0 \\ \hline 0 & J_{p_2} & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & J_{p_d} \end{array}\right]$$

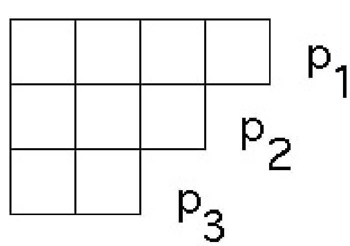
$$J_i = \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 1 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

Partitions of n

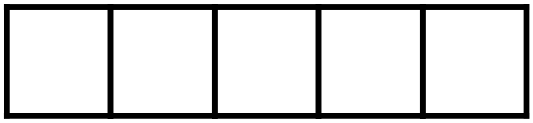
$$\pi = (p_1 \geq p_2 \geq \ldots \geq p_d)$$

$$\sum_{i=1}^{d} p_i = n$$

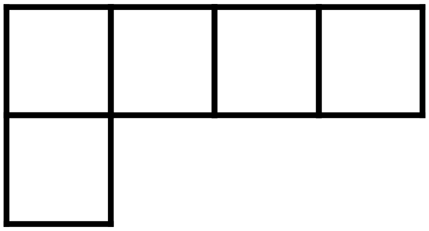
Young Diagrams



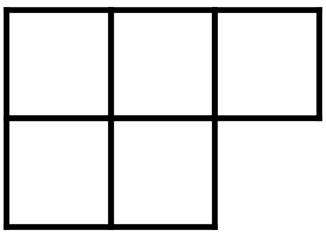
The example of $sl(5)$



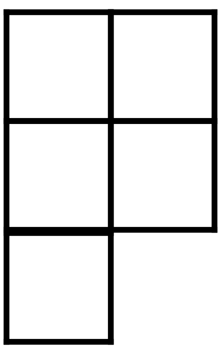
(5)



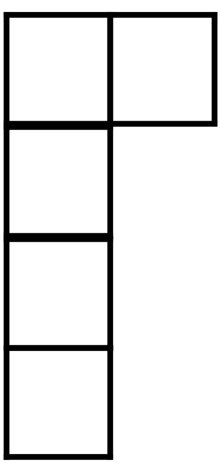
(4,1)



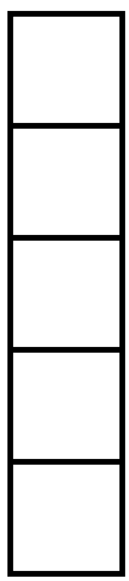
(3,2)



(2,2,1)



(2,1,1,1)



(1,1,1,1,1)

Partitions of 5 pametrize the the nilpotent conjugacy classes in $sl(5)$

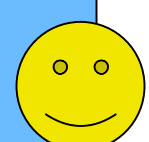
The ‘not so easy’ cases:
$g = so(2n), so(2n+1), sp(2n)$

Problems:

1. Conjugation by G is no longer equivalent to conjugation by GL_N

2. You can’t use Jordan forms to represent a conjugacy class, because a matrix in Jordan form does not belong to g

Nonetheless, we can still use partitions to parametrize the c.c.



TRICK

Because $\mathfrak{g} \subset \mathfrak{gl}(N)$ we can make use of the standard representation ρ of $\mathfrak{g}$ on $\mathbb{C}^N$

FACTS you need to know:

1. For each X nilpotent, there is a **standard triple** $\{X, Y, H\} \subset \mathfrak{g}$ with H semisimple (diagonalizable) and

$$< X, Y, H > \simeq sl(2, \mathbb{C})$$

2. Each f.d. representation of $sl(2, \mathbb{C})$ decomposes into a sum of irreducibles

3. Up to equivalence, $sl(2, \mathbb{C})$ has exactly **one irreducible** representation λ_k **in each dimension k**.

Partition associated with an orbit

Given X nilpotent, fix the standard triple $\{X, Y, H\}$

ρ = **standard repr. of $\mathfrak{g}$ on $\mathbb{C}^N$**

Restrict to $\mathfrak{sl}(2) = \langle X,Y,H \rangle$

$$\rho = \lambda_{n_1} + \lambda_{n_2} + \ldots + \lambda_{n_d}$$

Pick the sizes of the irred. summands

$$\pi = n_1 + n_2 + \ldots + n_d$$

π **is the partition associated to *X* !!!**

Partition-type Classification for $g = so(2n+1)$

Nilpotent c.c. are in 1-1 correspondence with partitions of (2n+1) in which <u>even parts appear with even multiplicity</u>

Nilpotent c.c. for $sl(5)$	Nilpotent c.c. for $so(5)$
(5)	(5)
(4,1)	--------------
(3,2)	--------------
(2,2,1)	(2,2,1)
(2,1,1)	--------------
(1,1,1,1,1)	(1,1,1,1,1)

Partition-type Classification for $g=sp(2n)$

Nilpotent c.c. are in 1-1 correspondence with partitions of (2n) in which <u>odd parts appear with even multiplicity</u>

Nilpotent c.c. for $sl(4)$	**Nilpotent c.c. for $sp(4)$**
(4)	(4)
(3,1)	---------------
(2,2)	(2,2)
(2,1,1)	(2,1,1)
(1,1,1,1)	(1,1,1,1)

Partition-type Classification for $\mathfrak{g}=so(2n)$

Nilpotent c.c. are parametrized by partitions of (2n) in which <u>even parts appear with even multiplicity</u>.

The correspondence is "almost" 1-1.

<u>Very even partitions</u> (i.e. partitions with only even parts, each appearing with even multiplicity) <u>correspond to two distinct nilpotent c.c.</u>, so they should be counted twice.

Partition-type Classification for $g = so(2n)$

- Even parts appear with even multiplicity
- Very even partitions represent two orbits

Nilpotent c.c. for $sl(4)$	**Nilpotent c.c. for $so(4)$**
(4)	---------------
(3,1)	(3,1)
(2,2)	(2,2),(2,2)
(2,1,1)	---------------
(1,1,1,1)	(1,1,1,1)

Remarks

Why do we get a parity condition on the partitions ???

For all $\mathfrak{g} \neq \mathfrak{sl}(n)$, nilpotent c.c. are parametrized by partitions with an even number of rows of even/odd length. Why? To treat all cases at once we need some notations:

ε	$= +1, -1$
$\langle\, ,\, \rangle_\varepsilon$	= a non degerate bilinear form of parity ε
$\mathfrak{g}_\varepsilon$	= the Lie subalgebra of $\mathfrak{sl}(n)$ preserving $\langle\, ,\rangle_\varepsilon$ $= \{X : \langle Xv, w\rangle_\varepsilon = -\langle v, Xw\rangle_\varepsilon$ for all v, w$\}$
I_ε	= the isotropy group of $\langle\, ,\rangle_\varepsilon$ $= \{x$ in GL_n: $\langle xv, xw\rangle_\varepsilon = \langle v, w\rangle_\varepsilon$ for all v, w$\}$

Let π be the partition associated to a conjugacy class and let n_k be the number of parts of π of length k.

We can construct a vector space of dim n_k with a non-degenerate bilinear form.
This form is <u>symplectic</u> for ε=1, k even and for ε = - 1, k odd. For such combination of ε and k, the dimension n_k of the vector space must be even.

The result is a parity condition on the number of rows with even/odd length:

$\mathfrak{g}=so(2n), so(2n+1) \Rightarrow \varepsilon=+1 \Rightarrow$ n_{even} is even

$\mathfrak{g}=sp(2n) \Rightarrow \varepsilon=-1 \Rightarrow$ n_{odd} is even

Remarks

Why is the correspondence not 1-1 in the case of $so(2n)$???

The set of partitions satisfying the proper parity condition is always in 1-1 correspondence with the set of nilpotent c.c. under the isotropy group.

If $g \neq so(2n)$, each c.c. under the isotropy group I_ε coincides with a c.c. under the adjoint group $\mathcal{G}$.

If $g = so(2n)$, then an I_ε- c.c. coincides with a $\mathcal{G}$- c.c. only if the partition is not very even.
When the partition is very even, then an I_ε- c.c. splits into two distinct $\mathcal{G}$- c.c..

Outline of the talk

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Combinatorial description of nilpotent orbits

Part 2

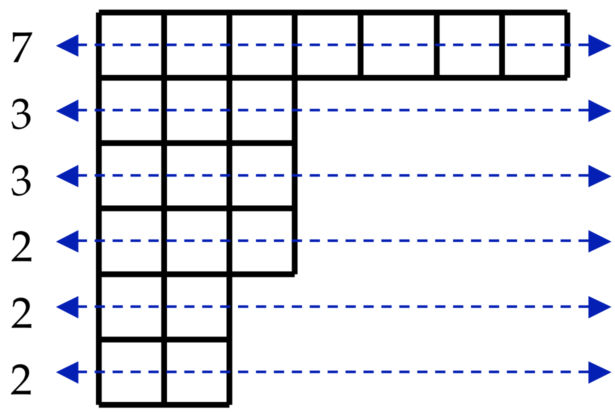
Dimension of nilpotent orbits

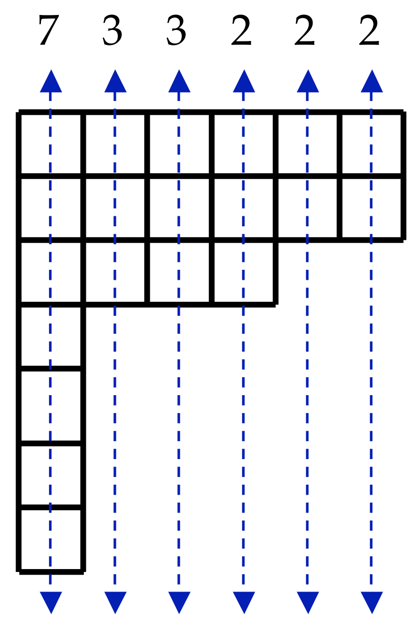
Part 3

Partial ordering of nilpotent orbits

Notations

Dual Partition:





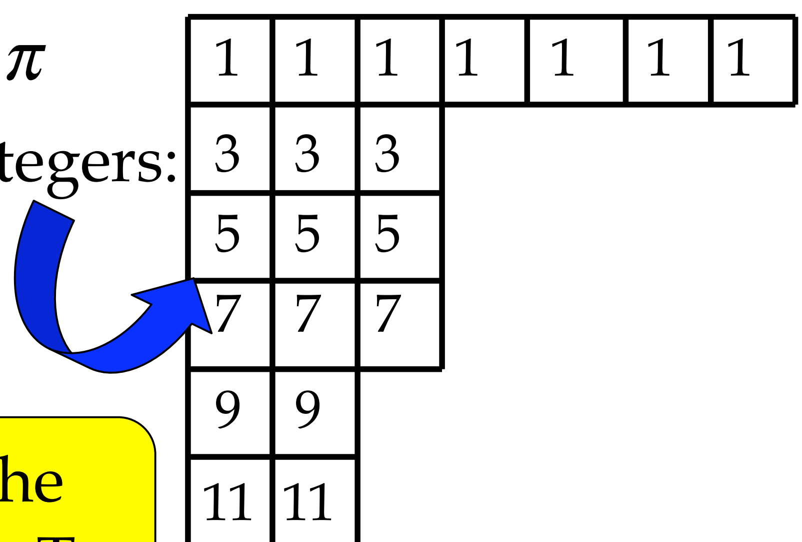
$\pi = (p_1 \geq p_2 \geq \ldots \geq p_d)$

$= (7,3,3,2,2,2)$

$\hat{\pi} = (\hat{p}_1 \geq \hat{p}_2 \geq \ldots \geq \hat{p}_d)$

$= (6,6,4,1,1,1,1)$

Let T_π be the Y.d. of π filled up with odd integers: Then



$\sum_{j \geq 1} (\hat{p}_j)^2$ = Sum of the entries in T_π

Dimension of a nilpotent orbit

	$\pi = (p_1 \ldots p_d)$
$sl(n)$	$n^2 - \sum_{j \geq 1} \hat{p}_i^2$
$so(2n)$	$2n^2 - n - \frac{1}{2} \sum_{i \geq 1} \hat{p}_i^2 + \frac{1}{2} \# \begin{pmatrix} Odd \\ Parts \end{pmatrix}$
$so(2n+1)$	$2n^2 + n - \frac{1}{2} \sum_{i \geq 1} \hat{p}_i^2 + \frac{1}{2} \# \begin{pmatrix} Odd \\ Parts \end{pmatrix}$
$sp(2n)$	$2n^2 + n - \frac{1}{2} \sum_{i \geq 1} \hat{p}_i^2 - \frac{1}{2} \# \begin{pmatrix} Odd \\ Parts \end{pmatrix}$

Examples of dimension of orbits

π = (4,4,2,1,1) is a partition of 12 with an even number of odd parts.
It represents **both** a c.c. in $\mathfrak{sl}(12)$ and a c.c. in $\mathfrak{sp}(12)$.

$\sum_{i\geq 1} \hat{p}_i^2 =$ sum of entries in

1	1	1	1
3	3	3	3
5	5		
7			
9			

= 42.

As a c.c in $\mathfrak{sl}(12)$:

$$\dim = n^2 - \sum_{i\geq 1} \hat{p}_i^2 = 102$$

As a c.c in $\mathfrak{sp}(12)$:

$$\dim = 2n^2 + n - \frac{1}{2}\sum_{i\geq 1} \hat{p}_i^2 - \#\frac{1}{2}\begin{pmatrix} odd \\ parts \end{pmatrix} = 56.$$

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Partial ordering

$\mathcal{N}$ **is an affine algebraic variety** in $\mathbb{C}^{\dim(g)}$

(being nilpotent is a polynomial condition).

Use the Zarinski topology.

Nilpotent orbits form a stratification of $\mathcal{N}$:

every nilpotent matrix is in *exactly one* conjugacy class (**stratum**), and the closure of a stratum is a union of strata.

Partial Ordering of Nilpotent orbits:

$$O_A \prec O_B \quad \longleftrightarrow \quad O_A \subseteq \overline{O_B}$$

Analytically:

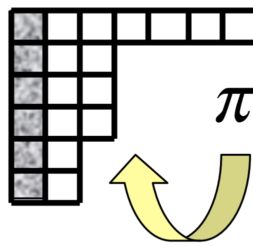
rank(A^k) $\leq$ rank(B^k) for all $k>0$.

Partial ordering in terms of partitions

$$O_A \prec O_B \Longleftrightarrow \text{rank}(A^k) \le \text{rank}(B^k), \forall k>0$$

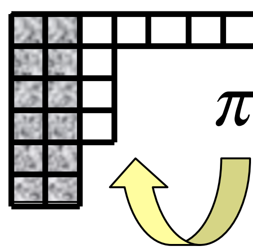
We need to relate $\text{rank}(A^k)$ to the partition π representing O_A…

Rank(A)= # boxes in



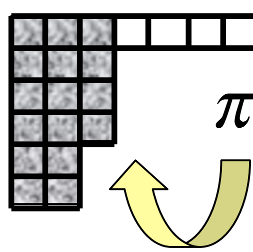
π- 1st column

Rank(A^2)= # boxes in



π- 2nd column

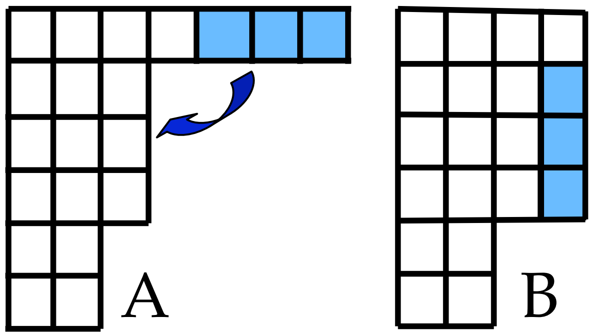
Rank(A^3)= # boxes in



π- 3rd column

…….

Moving down some boxes…

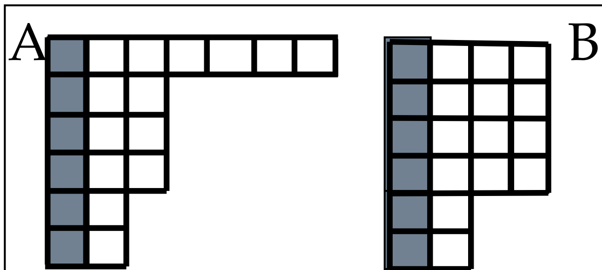


If B is obtained from A by moving down boxes, then

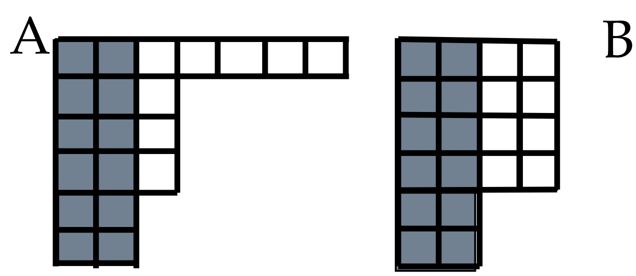
O_B is in the closure of O_A

i.e. $O_B \prec O_A$

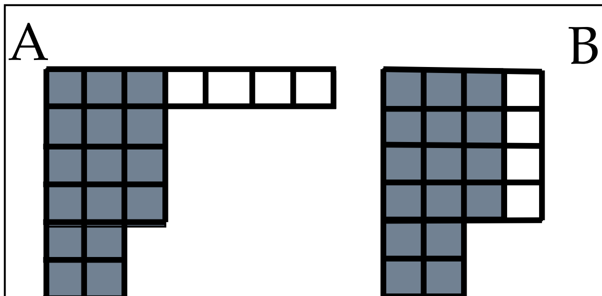
Let us compare the ranks of A^k and B^k :



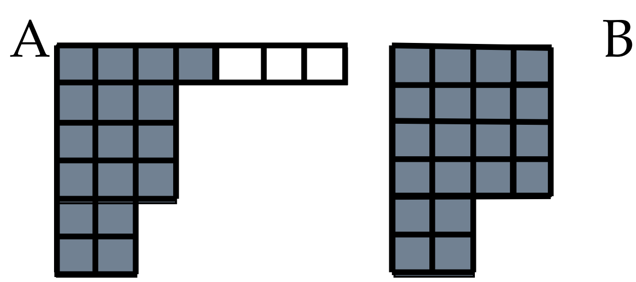
$\text{Rank}(A) = \text{Rank}(B)$



$\text{Rank}(A^2) = \text{Rank}(B^2)$



$\text{Rank}(A^3) = \text{Rank}(B^3)$



$\text{Rank}(A^4) > \text{Rank}(B^4)$

Minimal Degeneration

The closure $\overline{O}_A$ of a nilpotent orbit is a union of orbits. If $O_B \subseteq \overline{O}_A$ i.e. $O_B \prec O_A$, we say that **O_B is a degeneration of O_A**.

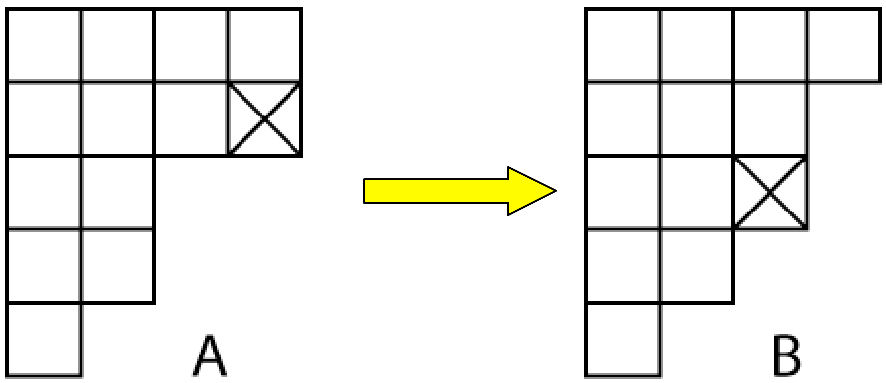
If O_B is also open in $\overline{O}_A$, we say that O_B is a **minimal degeneration**. In this case there is no orbit O_C such that

$$O_B \prec O_C \prec O_A$$

O_B and O_A are **adjacent orbits** w.r.t. the partial ordering.

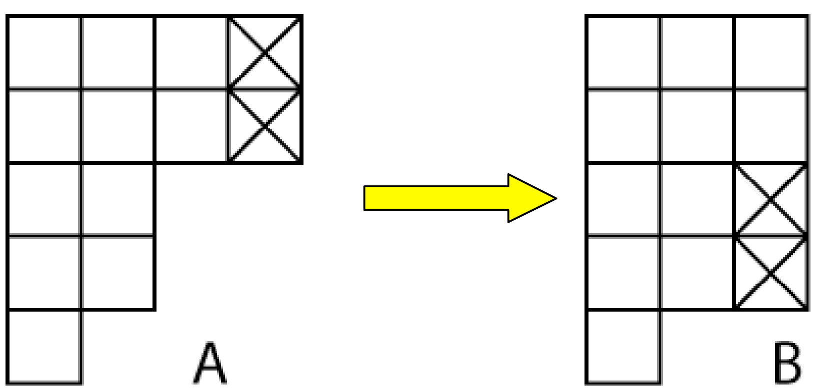
A degeneration is obtained by moving down some boxes…Careful!! The result must be again an acceptable partition.

Example of degeneration



This is a *minimal* degeneration in *sl*(13) but *not* a degeneration in *so*(13)

NOTE: In *so*(13) every even part must appear with even multiplicity

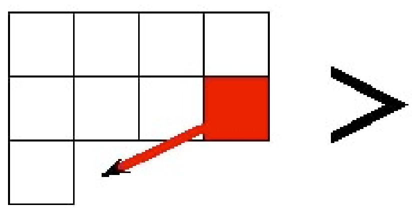


This is a degeneration in *sl*(13) and a *minimal* degeneration in *so*(13)

Minimal degenerations in *sl*(n)

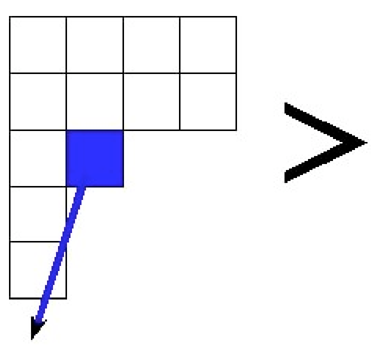
A minimal degeneration is obtained by ***moving down one box*** *with* ***two elementary operations****:*

RED OPERATION



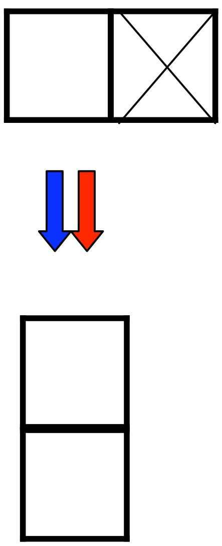
move a box down to the next ***row***

BLUE OPERATION

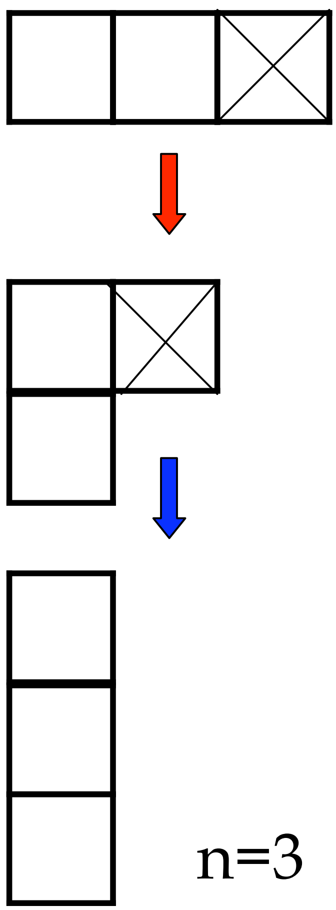


move a box down to the next ***column***

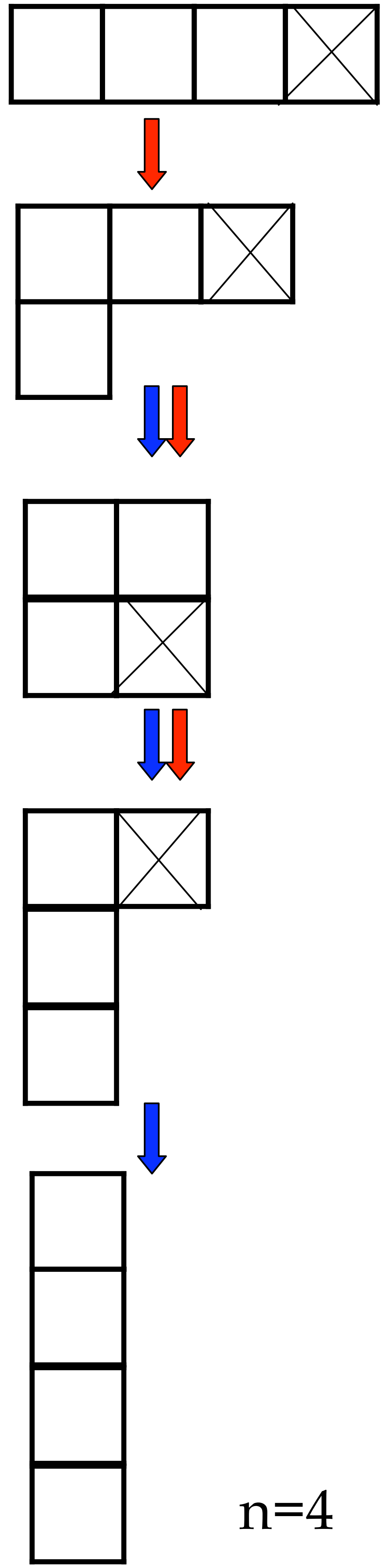
The diagram of minimal degenerations for *sl*(n)



n=2

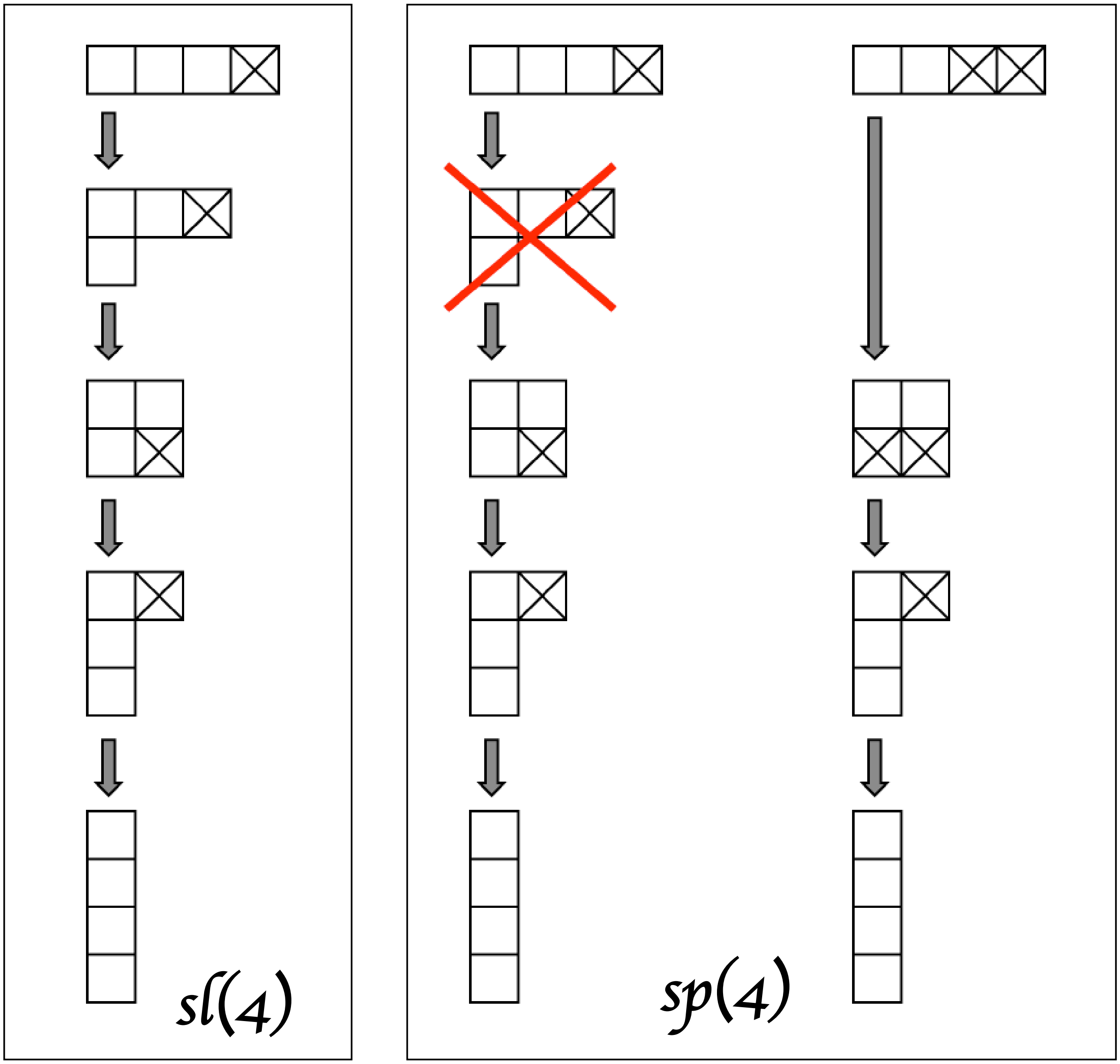


n=3



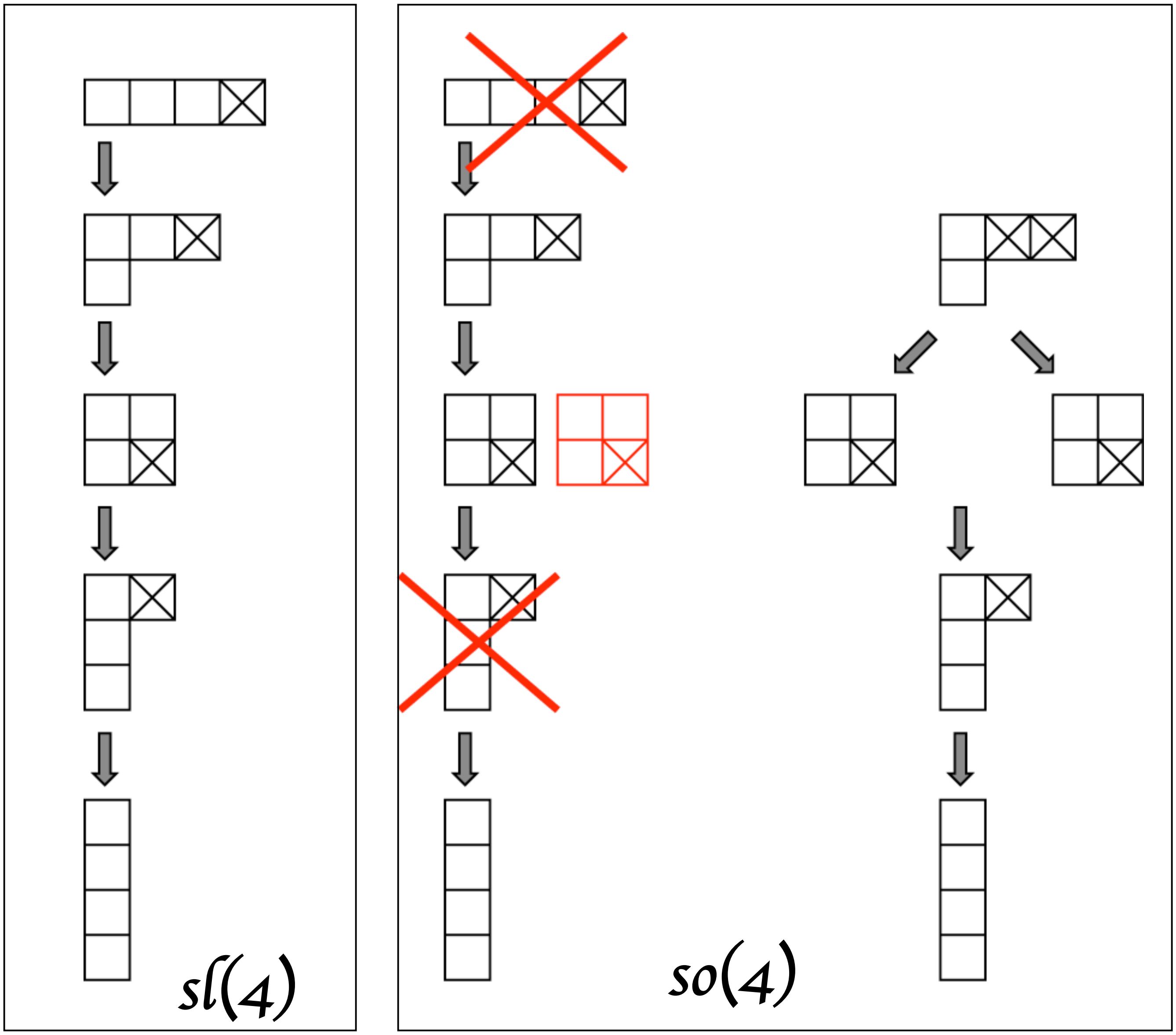
n=4

The diagram of minimal degenerations for *sp*(4)



In *sp*(4), every odd part must appear with even multiplicity

The diagram of minimal degenerations for *so*(4)



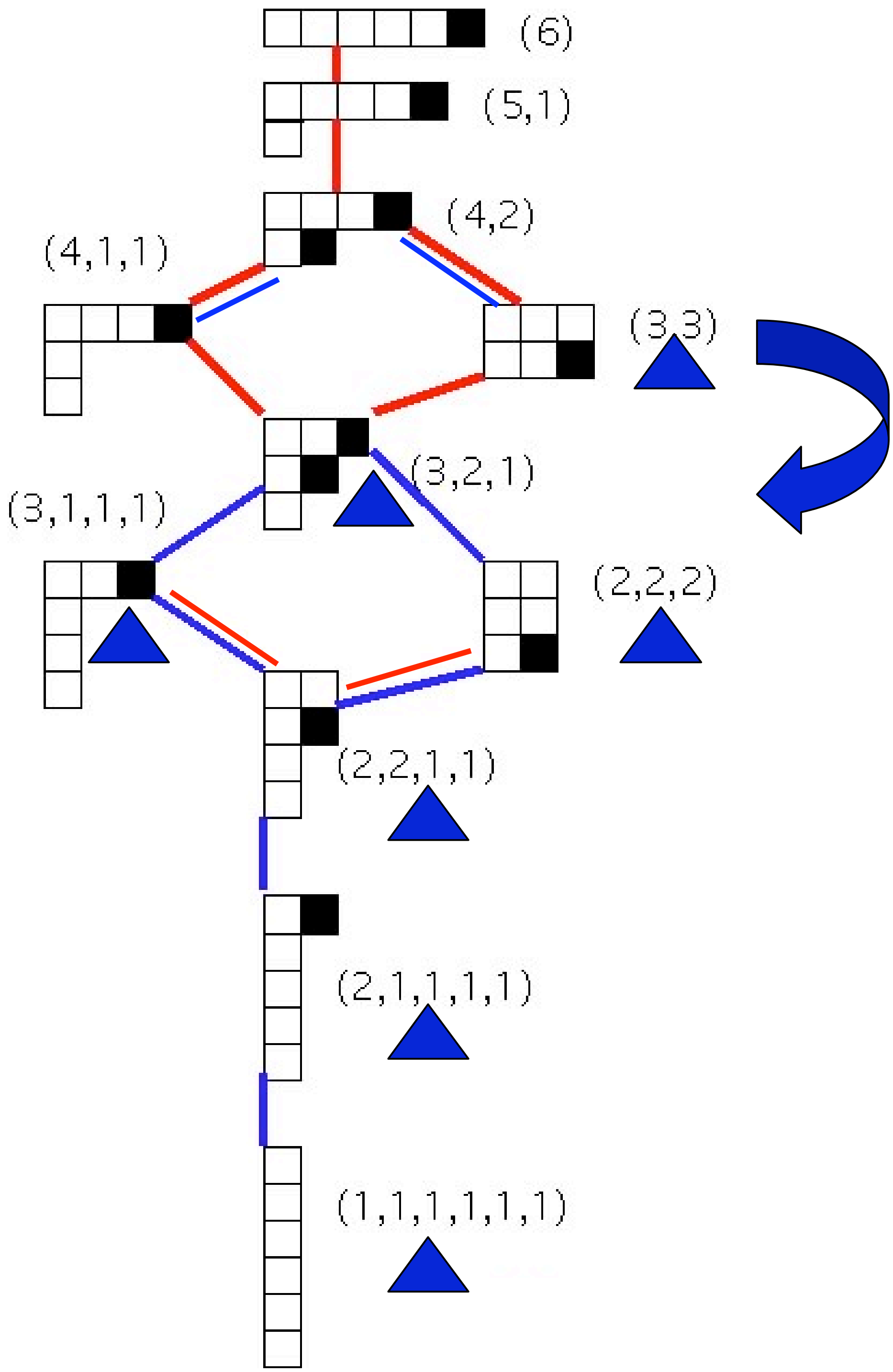
In *so*(4), every even part must appear with even multiplicity. Very even partitions represent two orbits.

Remarks

What do we gain from the diagram of minimal degenerations?

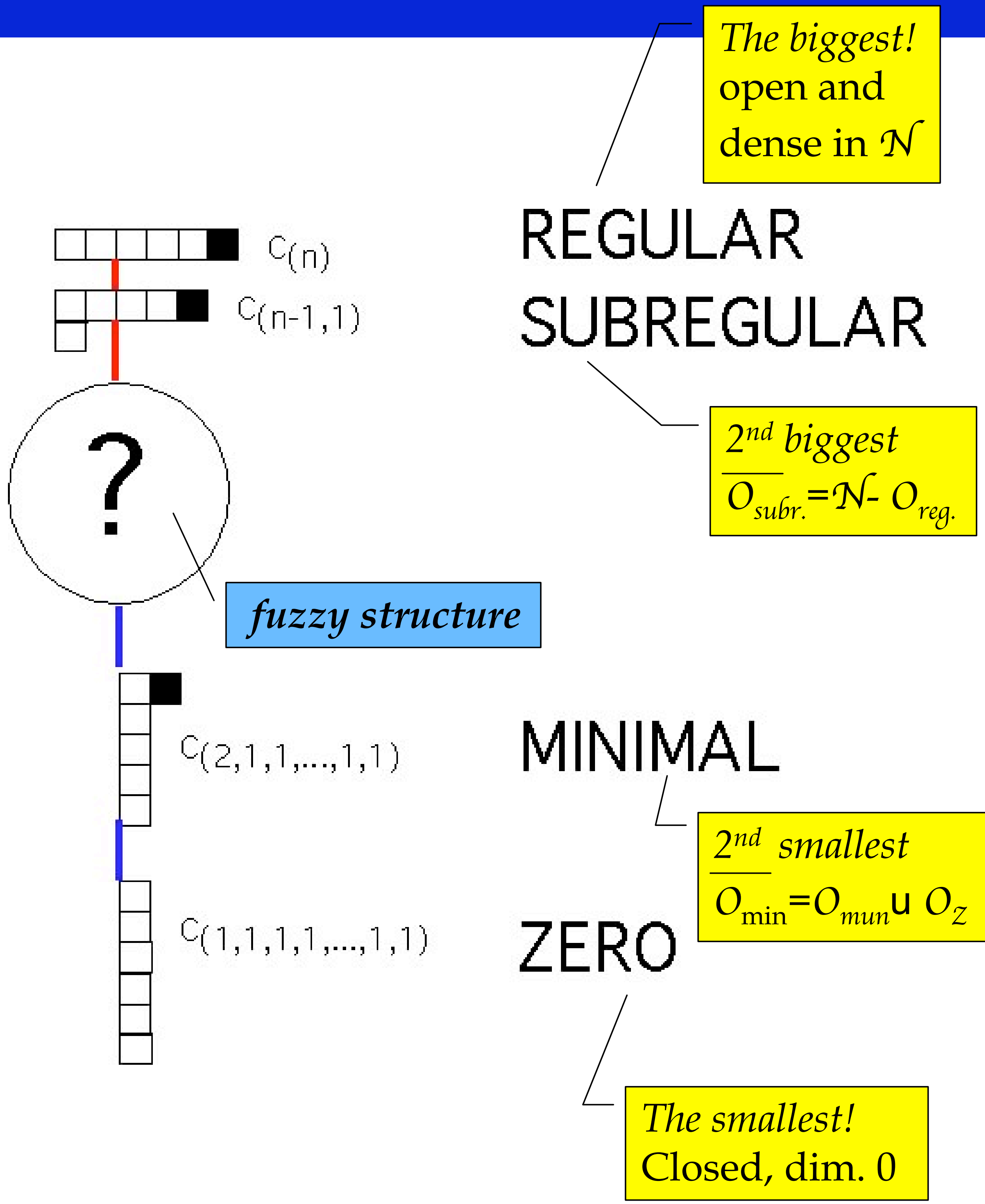
- a complete list of the nilpotent orbits
- an algorithm to compute the closure of a nilpotent orbit
- an algorithm to compute the dimension of a nilpotent orbit

The closure of an orbit in *sl*(6)

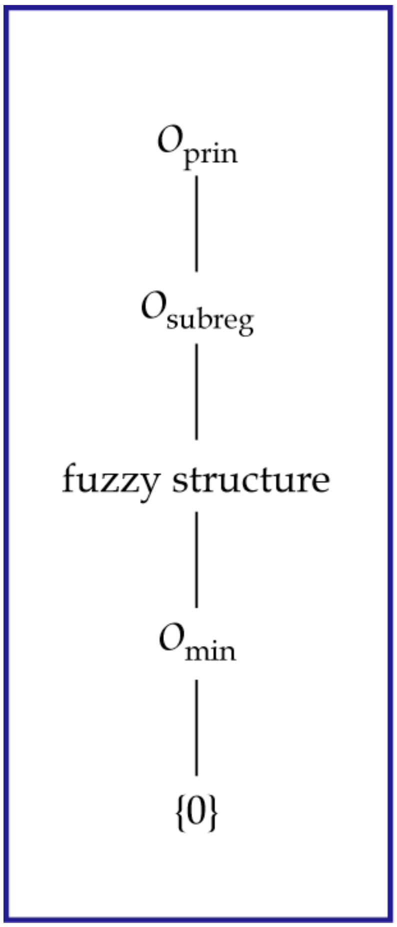


The closure of (3,3) consists of all the partitions sitting below (3,3) in this diagram.

The general picture for *sl(n)*



The general picture for *so(2n), so(2n+1), sp(2n)*



so(2n)
prin: [2n-1 , 1]
subreg: [2n-3 , 3]
min: $[2^2 , 1^{2n-4}]$

so(2n+1)
prin: [2n+1]
subreg: $[2n-1 , 1^2]$
min: $[2^2 , 1^{2n-3}]$

sp(2n)
prin: [2n]
subreg: [2n-2 , 2]
min: $[2 , 1^{2n-2}]$

An algorithm to compute the dimension of an orbit in *sl*(n)

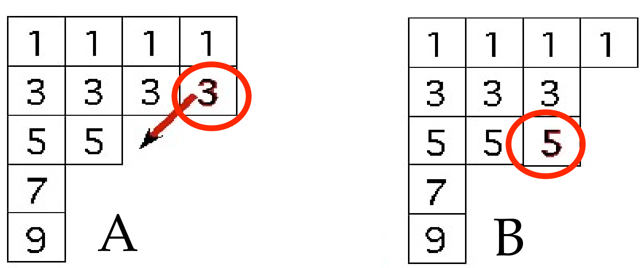
We use the formula

$$\dim(O) = n^2 - \sum_{i\geq 1} \hat{p}_i^2 = n^2 - \#\text{ of entries in}$$

1	1	1	1
3	3	3	3
5	5		
7			
9			

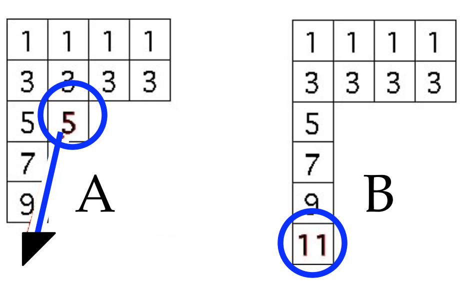
to compare the dim.s of ***adjacent*** *orbits.*

Red operation: move a box to the next row



$d_A - d_B = 2$

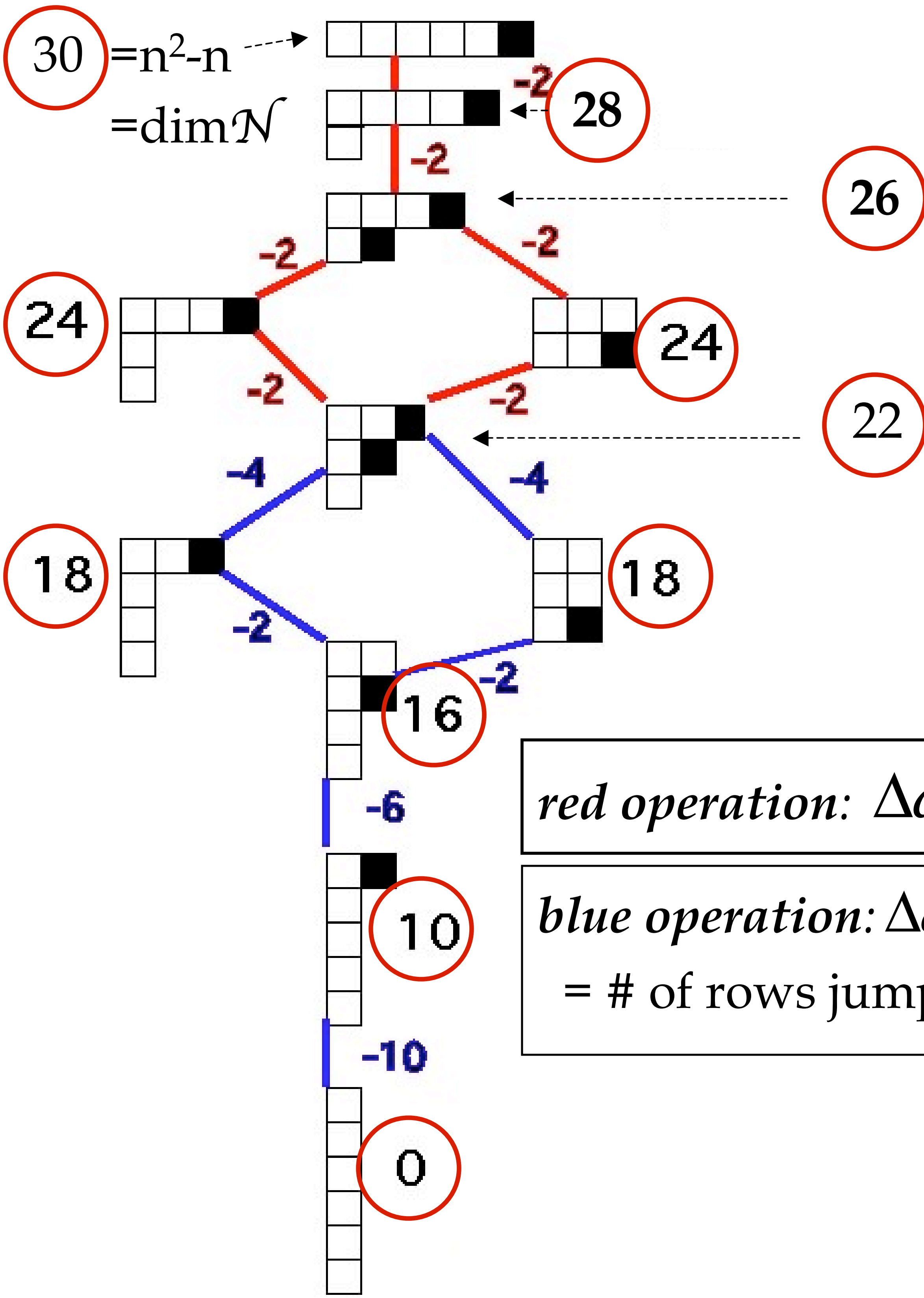
Blue operation: move a box to the next column



$d_A - d_B = 11 - 5 = 6 =$

= 2 (# of rows jumped)

Dimension of Nilpotent Orbits in $sl(6)$



red operation: $\Delta d = 2$

blue operation: $\Delta d =$ = # of rows jumped