EXERCISE: Complete the following character table. Give brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size		3	3	16	3	3		3
χ_1			-1			-1		-1
χ_2					1			
χ_3	1		1	ω	1	1		1
χ_4	1		1	ω^2	1	1		1
χ_5	3		$-1+2i$		-1	1		$-1-2i$
χ_6								
χ_7	3		1		-1	$-1-2i$		1
χ_8					-1	$-1+2i$		
order	1		4	3	2	4		4

oking for g
d g^{-1}"

STEP 1: Let $g \in \Gamma_6$. Because $\chi_7(g) \notin \mathbb{R}$, g is not conjugate to its inverse. Let's look for the conjugacy class of g^{-1}: since $\chi_7(g^{-1}) = \overline{\chi_7(g)} = -1+2i$, g^{-1} is not in $\Gamma_1, \Gamma_3, \Gamma_5$ or Γ_8. Also, g^{-1} does not belong to Γ_4 because the conjugacy class of g^{-1} has the same size as the conjugacy class of g. We conclude that $g^{-1} \in \Gamma_2$. It follows that $\chi_i(\Gamma_2) = \overline{\chi_i(\Gamma_6)}$ for all $i = 1..8$.
In particular, $\chi_1(\Gamma_2) = -1$; $\chi_3(\Gamma_2) = \chi_4(\Gamma_2) = \chi_5(\Gamma_2) = 1$; $\chi_7(\Gamma_2) = -1+2i$, and $\chi_8(\Gamma_2) = -1-2i$.
Moreover, the elements in Γ_2 have order 2.

A similar argument shows that if $g \in \Gamma_4$ then $g^{-1} \in \Gamma_7$.
Notice that ω must be a cubic root of unity (because elements of Γ_4 have order 3, χ_3 is a 1-dimensional character, and $\chi_3(\Gamma_4) = \omega$), and it cannot be equal to 1 [$\to$ If $\omega = 1$ then χ_3 and χ_4 differ by at most one entry; but $\chi_3 \perp \chi_4$ and $\|\chi_3\| = \|\chi_4\|$, so χ_3 and χ_4 actually becomes equal, which is a contradiction] therefore we get: $\chi_3(\Gamma_7) = \overline{\chi_3(\Gamma_4)} = \bar{\omega} = \omega^2$

and $\chi_4(\Gamma_7) = \overline{\chi_4(\Gamma_4)} = \overline{\omega^2} = \omega$. Moreover, Γ_7 has size 16 and the elements in Γ_7 have order 3.

oking for χ
$\bar{\chi}$"

STEP 2: If χ is an irreducible character with complex values, then $\bar{\chi}$ is a new irreducible character.
A quick glance at the character table shows that $\overline{\chi_5} = \chi_6$ and $\overline{\chi_7} = \chi_8$.

At this point we get:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size		3	3	16	3	3	16	3
χ_1		-1	-1			-1		-1
χ_2								
χ_3	1	1	1	ω	1	1	ω^2	1
χ_4	1	1	1	ω^2	1	1	ω	1
χ_5	3	1	$-1+2i$		-1	1		$-1-2i$
χ_6	3	1	$-1-2i$		-1	1		$-1+2i$
χ_7	3	$-1+2i$	1		-1	$-1-2i$		1
χ_8	3	$-1-2i$	1		-1	$-1+2i$		1
order	1	4	4	3	2	4	3	4

lentify $\Gamma = \{1\}$ and $\chi = \chi_{triv}$

STEP 3 - At this stage is clear that the identity element of G must be in Γ_1 (← the unique conjugacy class of order 1) and that χ_2 must be the trivial character. We obtain: $\chi_2(\Gamma_j) = 1$ $\forall j = 1..8$ and "size of $\Gamma_1 = 1$".

STEP 4 – All the sizes of the conjugacy classes are known, so we can find $|G| = \sum_{i=1}^{8} \#\Gamma_i = 48$.
Once the order of the group is know, we can also find the degree of the missing character:

$$|G| = \sum_{j=1}^{8} (\deg \chi_j)^2$$

$\Rightarrow \quad \chi_1(\Gamma_1) = 3.$

The first, second, third, sixth and eighth columns are now complete.

STEP 5 – Use orthogonality of rows and columns to complete the table:

Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
3	-1	-1	0	0	-1	0	-1
1	1	1	1	1	1	1	1
1	1	1	ω	1	1	ω^2	1
1	1	1	ω^2	1	1	ω	1
3	1	$-1+2i$	0	-1	1	0	$-1-2i$
3	1	$-1-2i$	0	-1	1	0	$-1+2i$
3	$-1+2i$	1	0	-1	$-1-2i$	0	1
3	$-1-2i$	1	0	-1	$-1+2i$	0	1

The missing entry in the 5^{th} column can be found by orthogonality with the first column:
$\Rightarrow \quad \chi_1(\Gamma_5) = 3.$

Because $\langle \chi_1, \chi_1 \rangle = 1$, we then find: $|\chi_1(\Gamma_4)|^2 + |\chi_1(\Gamma_7)|^2 = 0$
$\frac{1}{|G|} \sum_i |\chi_1(\Gamma_i)|^2$

$\Rightarrow$

$\Rightarrow \chi_1(\Gamma_4) = \chi_1(\Gamma_7) = 0$.

Similarly, $\langle \chi_5, \chi_5 \rangle = 1 \Rightarrow \chi_5(\Gamma_4) = \chi_5(\Gamma_7) = 0$.
By conjugation, we also obtain: $\chi_6(\Gamma_4) = \chi_5(\Gamma_7) = 0$.

Finally, $\langle \chi_7, \chi_7 \rangle = 1 \Rightarrow \chi_7(\Gamma_4) = \chi_7(\Gamma_7) = 0$,
hence $\chi_8(\Gamma_4)$ and $\chi_8(\Gamma_7)$ are also zero.

The character table is now complete!

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	1	3	3	16	3	3	16	3
χ_1	3	-1	-1	0	3	-1	0	-1
χ_2	1	1	1	1	1	1	1	1
χ_3	1	1	1	ω	1	1	ω^2	1
χ_4	1	1	1	ω^2	1	1	ω	1
χ_5	3	1	$-1+2i$	0	-1	-1	0	$-1-2i$
χ_6	3	1	$-1-2i$	0	-1	-1	0	$-1+2i$
χ_7	3	$-1+2i$	1	0	-1	$-1-2i$	0	1
χ_8	3	$-1-2i$	1	0	-1	$-1+2i$	0	1
order	1	4	4	3	2	4	3	4

EXERCISE : Given The character Table

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	1	1
χ_3	-1	2	2	-1	-1	0	0	-1	2
χ_4	1	1	1	-1	1	-1	1	-1	-1
χ_5	1	1	1	-1	1	1	-1	-1	-1
χ_6	-1	2	2	1	-1	0	0	1	2
χ_7	-2	-2	2	0	2	0	0	0	0
χ_8	1	-2	2	$\sqrt{3}\,i$	-1	0	0	$-\sqrt{3}\,i$	0
χ_9	1	-2	2	$-\sqrt{3}\,i$	-1	0	0	$\sqrt{3}\,i$	0

determine :

- which conjugacy class is $\{1\}$
- what is $|G|$
- The sizes of The conjugacy classes
- the orders of The kernels of The irreducible repr.s
- the irreducible faithful representations
- all The normal subgroups of G
- $Z(G)$ and $[G,G]$.

The group G has 9 conjugacy classes. The conjugacy class corresponding to the identity is Γ_3 (← The only one with strictly positive integer entries).

The order of the group is 24:

$$|G| = \sum_{\text{all irred.}} (\dim V)^2 = \sum_{j=1}^{9} \chi_j(\underset{\{1\}}{\Gamma_3})^2 =$$

$$= 1+1+4+1+1+4+4+4+4 = 24.$$

Once we know the order of the group, we can find the sizes of the conjugacy classes using the formula

$$\sum_{j=1}^{8} |\chi_j(\Gamma_\ell)|^2 = \frac{|G|}{\text{size of } \Gamma_\ell}, \quad \forall \ell = 1..9.$$

If we denote by c_ℓ the size of conjugacy class Γ_ℓ, then

$c_1 = c_4 = c_5 = c_8 = c_9 = 2$

$c_2 = c_3 = 1$

$c_6 = c_7 = 6.$

As an immediate consequence of this calculation, we find that

$Z(G) = \Gamma_2 \sqcup \Gamma_3$ has order 2.

In particular, it follows that the element in Γ_2 has order 2

It is also convenient to keep track of the sizes h_e of the stabilizer of an element in Γ_e $\left(h_e = \frac{|G|}{c_e}\right)$:

$h_1 = h_4 = h_5 = h_8 = h_9 = 12$

$h_2 = h_3 = 24$

$h_6 = h_7 = 4.$

Because $h_6 = h_7 = 4$, an element in Γ_6 or Γ_7 can only have order 2 or 4.

Next, we look at the kernels of the irreducible representations. [Recall that these are normal subgroups of G]

$\operatorname{Ker} \chi_1 = G \longrightarrow$ SIZE 24

$\operatorname{Ker} \chi_2 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_8 \cup \Gamma_9 \rightarrow$ SIZE 12

$\operatorname{Ker} \chi_3 = \operatorname{Ker} \chi_6 = \Gamma_2 \cup \Gamma_3 \cup \Gamma_9 \longrightarrow$ SIZE 4

$\operatorname{Ker} \chi_4 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5 \cup \Gamma_7 \longrightarrow$ SIZE 12

$\operatorname{Ker} \chi_5 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5 \cup \Gamma_6 \longrightarrow$ SIZE 12

$\operatorname{Ker} \chi_7 = \Gamma_3 \cup \Gamma_5 \longrightarrow$ SIZE 3

$\operatorname{Ker} \chi$ = $\operatorname{Ker} \chi$ = Γ → SIZE 1

This shows that :

- χ_8 and χ_9 are faithful. there are no other irreducible faithful characters.

- Elements in Γ_5 have order 3.

- Elements in Γ_9 have order 2 or 4.

What can we say about the order of the elements in $\Gamma_1, \Gamma_4, \Gamma_8$?
Because they lie in a group of order 12, it could only be 1, 2, 3, 4, 6 or 12.
Clearly, the order is not 1 (bc none of these elements is the identity of G).

Because $\chi_4(\Gamma_4) = -1$ and degree$(\chi_4) = 1$, $o(\Gamma_4) \neq 3$
[Same for Γ_8].
Because $\chi_3(\Gamma_4) = -1$ and degree$(\chi_4) = 2$, $o(\Gamma_4) \neq 2, 4$
[Same for Γ_8].
$\Rightarrow$ $o(\Gamma_4) = 6$ or 12, $o(\Gamma_8) = 6$ or 12.

A similar argument applies to Γ_1 : $\chi_3(\Gamma_1) = -1$
$\Rightarrow$ $o(\Gamma_1) \neq 2, 4$; $\chi_8(\Gamma_1) = 1$ and $\deg \chi_8 = 2$ $\Rightarrow o(\Gamma_1) \neq 3$
$\Rightarrow$ $o(\Gamma_1) = 6$ or 12.

We obtain the following table:

conjugacy class:	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
size	2	1	1	2	2	6	6	2	2
order	6 or 12	2	1	6 or 12	3	2 or 4	2 or 4	6 or 12	2 or 4

Next, we look at normal subgroups.

We already know that $H_1 = G$, $H_2 = \{0\}$, $H_3 = \Gamma_3 \cup \Gamma_5$, $H_4 = \Gamma_3 \cup \Gamma_2 \cup \Gamma_9$, $H_5 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \sqcup \Gamma_5 \sqcup \Gamma_8 \sqcup \Gamma_9$, $H_6 = \Gamma_1 \cup \Gamma_2 \sqcup \Gamma_3 \sqcup \Gamma_5 \sqcup \Gamma_6$ and $H_7 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \sqcup \Gamma_5 \sqcup \Gamma_7$ are normal subgroups, because they are kernel of representations.

Are there any other normal subgroups?

Assume that $H \trianglelefteq G$. then $\#H \mid \#G$ and H is a union of conjugacy classes.

If $\#H = 24$, Then $H G = H_1$.

If $\#H = 12$, then possible candidates are:

$$\Gamma_3 \cup \Gamma_2 \cup \Gamma_5 \cup \left(\begin{array}{c}\text{a union of other}\\ \text{conjugacy classes,}\\ \text{with total cardinality 8}\end{array}\right)$$

Γ_3: you need the identity

Γ_2: you need another class with odd cardinality

Γ_5: you need an element of order 3

So we could have

$\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup$ —— $\Gamma_1 \sqcup \Gamma_4 \sqcup \Gamma_8 \sqcup \Gamma_9$

or

$$\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_6 \sqcup \begin{cases}\Gamma_1\\ \Gamma_4\\ \Gamma_8\\ \Gamma_9\end{cases}$$

or $\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_7 \sqcup \begin{cases} \Gamma_1 \\ \Gamma_4 \\ \Gamma_8 \\ \Gamma_9 . \end{cases}$

Because the inverse of an element in Γ_4 is in Γ_8, Γ_4 and Γ_8 cannot exist separately. This cuts out the possibilities to:

(i) $\Gamma_3 \cup \Gamma_2 \cup \Gamma_5 \longrightarrow \cup\, \Gamma_1 \sqcup \Gamma_4 \sqcup \Gamma_8 \sqcup \Gamma_9$

(ii) $\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_6 \sqcup \Gamma_1$

(iii) $\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_7 \sqcup \Gamma_1$

(iv) $\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_6 \sqcup \Gamma_9$

(v) $\Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_7 \sqcup \Gamma_9$.

Notice that (i) $\longleftrightarrow H_5$, (ii) $\longleftrightarrow H_6$ and (iii) $\longleftrightarrow H_7$, so the first 3 possibilities actually give normal subgroups. Let's show that (iv) and (v) fail to give a normal subgroup.

If $H = \Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_6 \sqcup \Gamma_9$ is a normal subgroup of G, then $H \cap H_6 = \Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_6$ is also a normal subgroup of G. Contradiction! (because $H \cap H_6$ has order 10 and $10 \nmid 24 = |G|$).

Similarly, if $H = \Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_5 \sqcup \Gamma_7 \sqcup \Gamma_9$, we reach a contradiction by looking at $H \cap H_7$.

$\Rightarrow$ the only normal subgroups of order 12 are H_5, H_6 and H_7. Their intersections are also normal subgroups of G. So we find that $H_8 \overset{def}{\equiv} \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3 \sqcup \Gamma_5$ is a normal subgroup of G (of order 6).

Are There other subgroups of order 6?

If $H \trianglelefteq G$ and $\# H = 6$ Then H must be of The form:

$$H = \Gamma_3 \cup \Gamma_2 \cup \Gamma_5 + \begin{cases} \Gamma_1 \\ \Gamma_4 \\ \Gamma_8 \\ \Gamma_9 . \end{cases}$$

identity — another conjugacy class with odd cardinality... — elements of order 3

We Know That Γ_4 and Γ_8 cannot exist ~~separately~~, so we just need To argue about

$$H = \Gamma_3 \cup \Gamma_2 \cup \Gamma_5 \cup \Gamma_9 .$$

If $H \trianglelefteq G$, then $\Gamma_3 \cup \Gamma_2 \cup \Gamma_5 = H \cap H_6 \trianglelefteq G$ (of order 4). We reach a contradiction because a sub group of order 4 cannot contain any element of order 3.

$\Rightarrow$ $H_8 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$ is The only normal subgroup of order 6.

Now we look for normal subgroups of order 8.

If $H \trianglelefteq G$ and $\# H = 8$, Then H can only be of The form:

$$H = \Gamma_3 \cup \Gamma_2 \cup \begin{cases} \Gamma_6 \\ \Gamma_7 \end{cases}$$

(because no elements of order 3 or 6 or 12 are allowed.

Suppose that $H = \Gamma_3 \sqcup \Gamma_2 \sqcup \left< \begin{matrix}\Gamma_6 \\ \Gamma_7\end{matrix}\right.$ is a normal subgroup of G. Then G/H is a well defined group of order 3, and it's necessarily abelian (so it has exactly 3 irreducible characters).

Every (linear) character of G/H lifts to a (linear) character of G with the property that H acts trivially. A quick look at the character table shows that Γ_6 and Γ_7 act trivially only in Two linear characters, so we reach a contradiction.

$\Rightarrow$ G has no normal subgroups of order 8.

Finally, we consider normal subgroups of size 2, 3 and 4.

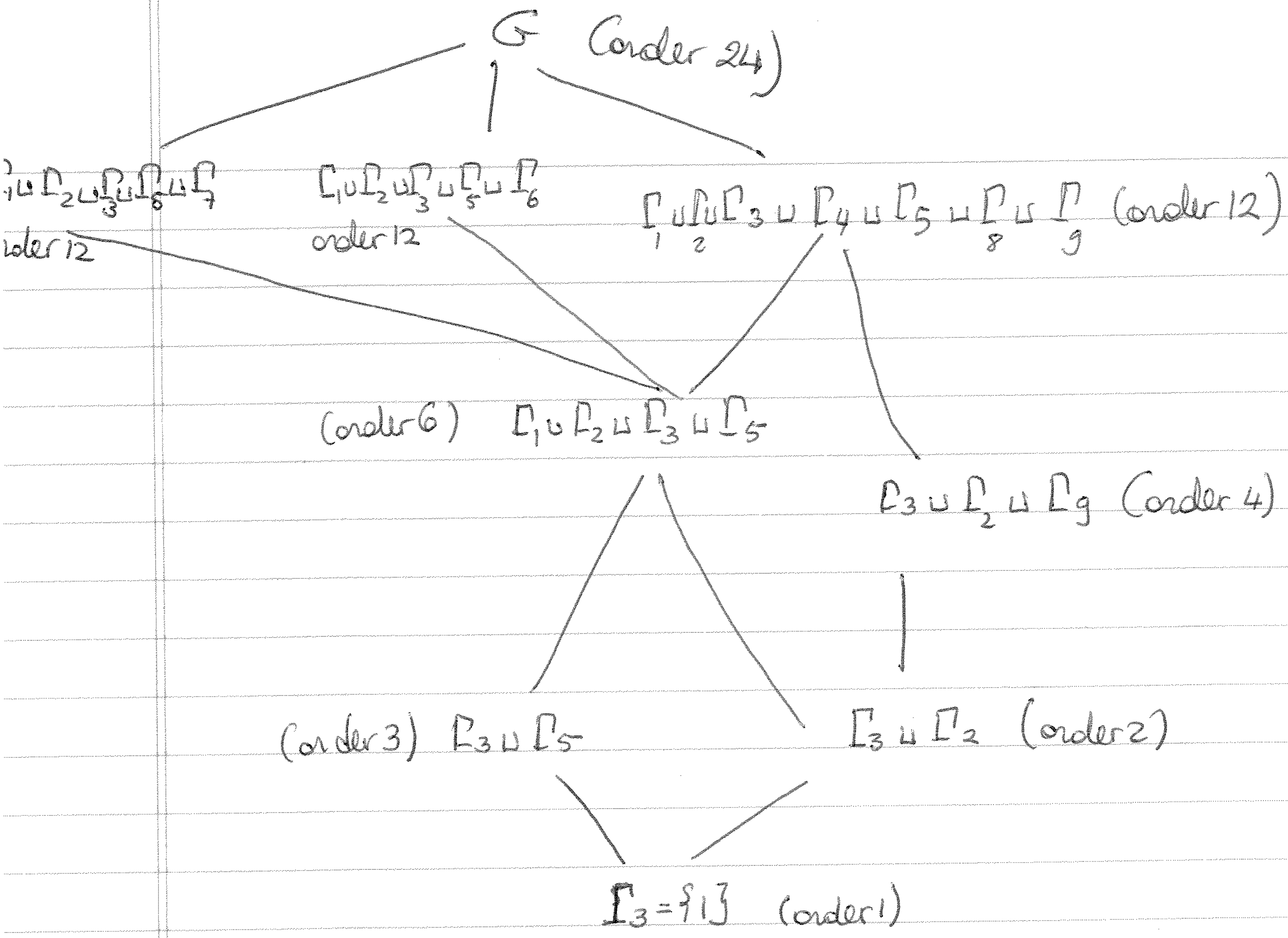
If $H \trianglelefteq G$ and $\# H = 2$, then $H = \Gamma_2 \sqcup \Gamma_3$. This is actually a normal subgroup, because we can recognize $Z(G)$.

If $H \trianglelefteq G$ and $\# H = 3$, then the only possibility is $\Gamma_3 \sqcup \Gamma_5$, which is indeed a normal subgroup ($= H_3$).

If $H \trianglelefteq G$ and $\# H = 4$, then H could only be

$$H = \Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_9$$ (because H must contain an element of order 2 and cannot contain elements of order ≥ 6).

$H = \Gamma_3 \sqcup \Gamma_2 \sqcup \Gamma_9$ is actually a subgroup ($= H_4$).



Notice that $Z(G) = \Gamma_2 \cup \Gamma_3$ (union of all the classes of size 1), and $[G,G] = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$ (order 6). We obtain $[G,G]$ as the intersection of the kernels of the linear characters.

Notice that $Z(G) \cong C_2$ and $[G,G] \cong C_6$ ($[G,G]$ has order 6 and is not isomorphic to S_3 because it has a normal subgroup of order 3).