[LECTURE 8]

THE NUMBER OF LINEAR CHARACTERS.

<u>theorem</u> - the number of the linear characters of G equals the index of the commutator subgroup $[G,G]$ in G

<u>proof</u> - We divide the proof in various steps:

1) $A = \dfrac{G}{[G,G]}$ is abelian

2) There is a bijective correspondence
$\{\text{irreducible repr.s of } A\} \leftrightarrow \{\text{irreducible reps } \rho \text{ of } G \text{ st. } \rho(G) \text{ is abelian}\}$

3) Every irreducible repr. ρ of G ST $\rho(G)$ is abelian has dimension 1.

From these three steps it will clearly follow that

$$\#\{\text{linear char.s of } G\} = \#\{\text{1-dim.l reprs of } G\}$$
$$= \#\{\text{irred. reprs } \rho \text{ of } G \text{ ST } \rho(G) \text{ is abelian}\}$$
$$= \#\{\text{irred reprs of } A\}$$
$$\underset{A \text{ abelian}}{=} |A| = [G:[G,G]] = \frac{|G|}{|[G,G]|}.$$

<u>OK</u>: let's start the proof!

① Let $\pi : G \to A = G/[G,G]$ be the usual projection.
Because π is surjective, for all $\tilde a, \tilde b \in A$ we can write:

$$\tilde a^{-1} \tilde b^{-1} \tilde a \tilde b = \pi(a)^{-1} \pi(b)^{-1} \pi(a)\pi(b) = \pi(\underbrace{a^{-1}b^{-1}ab}_{\text{in } [G,G]}) = \tilde 0 \quad \text{in } A = \frac{G}{[G,G]}.$$

↳ for some $a, b \in G$

This proves that A is abelian. ✓

② We show that the map

$$\theta : \left\{ \begin{matrix} \text{irred repr.s} \\ \text{of } A = G/[G,G] \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{irred repr.s } \rho \\ \text{of } G \text{ st } \rho(G) \text{ is abelian} \end{matrix} \right\}$$

$$\tilde\rho \longmapsto \rho = \tilde\rho \circ \pi$$

is a bijective correspondence.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\pi"'] & GL(V) \\ G/[G,G] \arrow[ur, "\tilde\rho"'] & \end{tikzcd}$$

First of all, we notice that θ is well defined because

$$\rho(G) = \tilde\rho \circ \pi(G) = \underset{\substack{\uparrow \\ \text{group homom}}}{\tilde\rho}(\underset{\substack{\uparrow \\ \text{abelian}}}{A})$$

is abelian, for every (irred.) representation $\tilde\rho$ of A. Moreover, $\rho = \tilde\rho \circ \pi$ is irreducible if ρ is such.

Suppose that r is a representation of G st $r(G)$ is abelian. For every $g \in G$ we can write:

$$\tilde r(g[G,G]) = r(g),$$

and obtain a well defined representation of

$A = \frac{G}{[G,G]}$. (if $g_1[G,G] = g_2[G,G]$, then $g_1^{-1}g_2 \in [G,G]$

so $\rho(g_1^{-1}g_2) = \rho(x^{-1}y^{-1}xy)$ for some $x, y \in G$.
But $\rho(G)$ is abelian, so $\rho(x^{-1}y^{-1}xy) = \rho(x)^{-1}\rho(y)^{-1}\rho(x)\rho(y)$
is the identity). It is clear that $\tilde{\chi}$ is uniquely determined by χ, so θ is bijective. ✓

③ Suppose that (V, ρ) is an irreducible repr. of G s.t. $\rho(G)$ is abelian. Fix $x \in G$.
For all $y \in G$, we can write:

$$\rho(x)\rho(y) = \rho(y)\rho(x).$$

↳ $\rho(G)$ abelian

$\Rightarrow$ $\rho(x)$ is an intertwining operator. Because ρ is irreducible, it follows from Shur's lemma that $\rho(x) = \lambda_x \mathbb{1}_V$ for some $\lambda_x \in \mathbb{C}$.
Then every one-dimensional subspace of V is G-stable. We reach a contradiction (unless $\dim V = 1$). ✓

This concludes the proof. ∎

Corollary. The number of linear characters of G divides the order of group.

CHARACTER TABLES

- Let $\chi_1, \chi_2, \dots, \chi_K$ be the irreducible characters of G and let $g_1, \dots, g_K$ be representative for the conjugacy classes of G.
The $K \times K$ matrix whose i-j entry is $\chi_i(g_j)$ is called the character table of G.

Properties

$\boxed{1}$ It's a square matrix, because the number of irreducible representations of G equals the number of conjugacy classes.

- $\boxed{2}$ Rows are indexed by the irreducible characters of G. Columns are indexed by the conjugacy classes of G.

$\boxed{3}$ Rows are orthonormal w.r.t. the inner product

$$\langle \chi_r, \chi_s \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_r(g)\, \chi_s(g) =$$

$$= \frac{1}{|G|} \sum_{j=1}^{K} \chi_r(g_j)\, \chi_s(g_j)\, c_j \, .$$

size of the conjugacy class of g_j ...

In other words, $\langle \chi_r, \chi_s \rangle = \delta_{rs}$.

- $\boxed{4}$ Columns are "orthogonal":

$$\sum_{i=1}^{K} \overline{\chi_i(g_r)}\, \chi_i(g_s) = \frac{|G|}{c_r}\, \delta_{rs} = \begin{cases} \frac{|G|}{c_r} & \text{if } r=s \\ 0 & \text{if } r \neq s \end{cases}$$

$\boxed{5}$ If g has order m and χ has degree d, then $\chi_d(g)$ is a sum of d m^{th} roots of unity.

$\boxed{6}$ $\chi_i(g)$ is real if and only if g is conjugate to g^{-1}.

Remark: A character table gives you a lot of information about the group and its representations ....

INFORMATION ABOUT THE REPRESENTATIONS:

- ρ is faithful if and only if $\ker \chi_\rho = \{1\}$.

We define: $\operatorname{Ker}(\chi_\rho) = \{ g \in G : \chi_\rho(g) = \chi_\rho(1) \}$.

- ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.

- if r is irreducible, r appears in ρ with multiplicity $\langle \chi_r, \chi_v \rangle$.

INFORMATION ABOUT THE GROUP:

- $|G| = \sum_{\text{all irred } \chi} \text{degree}(\chi)^2$.
- the number of (irreducible) linear characters of G equals the <u>index</u> of the commutator subgroup of G: $\#\text{ linear chars} = \dfrac{\# G}{\# [G,G]}$.

- G is abelian if and only if every character is linear.

- the size of a conjugacy class Γ can be computed using the formula:

$$\sum_{\text{all irred } \chi} \overline{\chi(\Gamma)}\,\chi(\Gamma) = \sum_{\text{all irred. } \chi} |\chi(\Gamma)|^2 = \frac{|G|}{\#\Gamma}.$$

- Once you know the sizes of the conjugacy classes, you can find the orders of the stabilizers:

$$|St(g)| = \frac{|G|}{\#\text{conj. class of } g}.$$

- the order of $St(g)$ often gives you a lot of information about the order of g ("order of g" | "order of $St(g)$").

- For every character χ, $\ker\chi = \{g \in G : \chi(g) = \chi(1)\}$ is a normal subgroup of G. The other normal subgroups $H \trianglelefteq G$ can be found using the facts that $|H| \mid |G|$ and H is a union of conjugacy classes.

- $[G,G]$ is the intersection of the kernel of all the <u>linear</u> representations. [A repr. is called linear if it has dimension 1].

- If $z \in Z(G)$, then $|\chi(z)| = \chi(1)$ $\forall \chi$. This information is useful to find $Z(G)$. [Moreover, $Z(G)$ is normal so it's a union of conjugacy classes]. Even easier: $g \in Z(G) \iff$ the conjugacy class of g has size 1.

Example A group G has character Table

	Γ_1	Γ_2	Γ_3	Γ_4
χ_1	1	1	1	1
χ_2	3	-1	0	0
χ_3	1	1	α	α^2
χ_4	1	1	α^2	α

Find

1. $|G|$
2. the sizes of the conjugacy classes
3. the orders of the elements in each conjugacy class
4. the normal subgroups of G.
5. $Z(G)$ and $[G,G]$.

1. $|G| = \sum_{V \text{ irred}} (\dim V)^2 = \sum_{i=1}^{4} \chi_i(\Gamma_1)^2 = 1+9+1+1 = 12.$

Γ_1 is the identity element

2. For all $j = 1, \dots, 4$, let c_j be the conjugacy class of Γ_j. then $\sum_{i=1}^{4} |\chi_i(j)|^2 = \frac{|G|}{c_j}$.

we find:

$$c_1 = \frac{12}{1+9+1+1} = 1$$

$$c_2 = \frac{12}{1+1+1+1} = 3$$

$$c_3 = \frac{12}{1+|\alpha|^2+|\alpha|^4} = \frac{12}{1+1+1} = 4 = c_4.$$

because χ_3 and χ_4 are linear character, $\chi_3(g)$ and $\chi_4(g)$ are roots of unity $\forall g \in G \Rightarrow |\chi_3(g)| = |\chi_4(g)| = 1 \ \forall g \in G$

3. For all $j = 1, \dots, 4$, let g_j be a representative of Γ_j. The stabilizer of g_j in G has order $\frac{|G|}{c_j}$.

We get:

- $|St(g_1)| = 12$
- $|St(g_2)| = 4$
- $|St(g_3)| = 3 = |St(g_4)|$.

Because 3 is a prime number, both g_3 and g_4 must have order 3.

g_2 can have order 2 or 4: let's think! Since $|G| = 12$ and 2 is a prime number that divides $|G|$, G must contain an element of order 2. Such an element must sit in Γ_2 (because the elements of $\Gamma_1, \Gamma_3, \Gamma_4$ have orders 1, 3, 3 respectively). So any element of Γ_2 must have order 2.

4. For all $j = 1 \dots 4$, $\mathrm{Ker}\, \chi_j = \{ g \in G : \chi_j(g) = \chi_j(\Gamma_1) \}$ is a normal subgroup.

We find: $H_1 = G$, $H_2 = \{1\}$, $H_3 = H_4 = \Gamma_1 \sqcup \Gamma_2$ (of order 4)

Are there other normal subgroups?

If $H \trianglelefteq G$, then $|H| \,\big|\, |G|$ (so $|H| = 1, 2, 3, 4, 6, 12$) and H is a union of conjugacy classes.

The conjugacy classes have sizes 1, 3, 4, 4.

So we might only have: Γ_1, $\Gamma_1 \sqcup \Gamma_2$, $\Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3 \sqcup \Gamma_3$.

$\Rightarrow$ the only normal subgroups are H_1, H_2 and H_3.

5. $Z(G)$ is a normal subgroup of G, so it's H_1 or H_2 or H_3. Because G is not abelian (or it could not have a 3-dimensional irreducible character), $Z(G) \neq G$. We are left with 2 possibilities: $Z(G) = \{1\}$ and $Z(G) = \Gamma_1 \uplus \Gamma_2$.

If $Z(G) = \Gamma_1 \uplus \Gamma_2$, then $|\chi(\Gamma_2)| = \chi(1)$ $\forall \chi$.

Apply to χ_2 to get a contradiction.

$\Rightarrow$ $Z(G) = \{1\}$.

Now let's look at $[G,G]$. If m is the order of this group, then

$$\frac{12}{m} = \#\text{ linear chars} = 3$$

$\Rightarrow$ $m = 4$. The only normal subgroup of order 4 is $\Gamma_1 \uplus \Gamma_2 = H_3$, so $[G,G] = H_3$.

SOME "TRICKS" TO CONSTRUCT CHARACTER TABLES:

1. It's enough to find all the characters but one: the remaining character can be found using the relations

$$\sum_{\text{all irred.}} (\dim V)\, \chi_V(g) = 0, \quad \forall g \neq e$$

and

$$\sum_{\text{all irred}} (\dim V)\, \chi_V(e_G) = \sum_{\text{all irred}} (\dim V)^2 = |G|.$$

2. If χ is irreducible and χ_1 is one-dimensional, then $\chi \cdot \chi_1$ is irreducible.

3. Given χ_V, it is sometimes useful to construct

$$\chi_{\mathrm{Sym}^2 V}(g) = [\chi_V(g)^2 + \chi_V(g^2)]/2$$

and

$$\chi_{\mathrm{Alt}^2 V}(g) = [\chi_V(g)^2 - \chi_V(g^2)]/2.$$

They could be reducible. Find the multiplicity n_i of the various irreducible U_i by computing $\langle \chi_{U_i}, \chi_{\mathrm{Sym}^2 V} \rangle$ $(\mathrm{Alt}^2 V)$, then look at $\chi_{\mathrm{Sym}^2 V} - \sum n_i \chi_{U_i}$ $(\mathrm{Alt}^2 V)$. This might be an integral linear combination of new characters....

4. If $H \trianglelefteq G$ is a normal subgroup of G and

$\tilde{\rho}$ is a representation of G/H, then $\rho = \tilde{\rho} \circ \pi$ (with $\pi : G \to G/H$ the usual projection) is a representation of G.

Notice that $\chi_\rho = \chi_{\tilde{\rho}}(gH)$, $\forall g \in G$.

[This construction is particularly useful when G/H is abelian, because the characters of an abelian group are very easy to write down...].

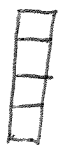
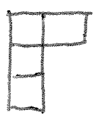
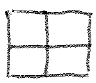
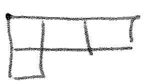
$\boxed{5}$ If χ_r is a linear character, then $\chi_r(g)$ is a root of unity, $\forall g \in G$. Because $\chi_r(g) = r(g)$ and r is a group homomorphism, it is enough to determine the value of χ_r on the generators of G. Also notice that if g has even order and g is conjugate to g^{-1}, then $\chi_r(g)$ can only be ± 1.

If g has odd order, and g is conjugate to g^{-1}, then $\chi_r(g)$ must be 1.

$\boxed{6}$ If χ is an irreducible character, and χ is not real, then $\overline{\chi}$ is a new irreducible character.

Character Table of S_4

First, we determine the sizes of the conjugacy classes.

FERRES DIAGRAM	REPRESENTATIVE	ORDER OF THE CENTRALIZER	CARDINALITY OF THE CONJUGACY CLASS
	1 = The identity element of S_4	$1\cdot1\cdot1\cdot1\cdot4! = 4!$	$\frac{4!}{4!} = 1$
	(12)	$2\cdot1\cdot1\cdot2! = 4$	$\frac{4!}{4} = 6$
	$(12)(34)$	$2\cdot2\cdot2! = 8$	$\frac{4!}{8} = 3$
	(123)	$3\cdot1 = 3$	$\frac{4!}{3} = 8$
	(1234)	4	$\frac{4!}{4} = 6$

check: $1+6+3+8+6 =$
$= 24 = 4!$ ✓

Next, we start building the character table.
It's known that S_4 has 2 irreducible representations of dimension 1 (the trivial and the sign), and an irreducible representation of dimension 3 (the standard).
The linear characters are easy to write down, the character of the standard representation can be obtained as

$$\chi_{\text{standard}} = \chi_{\text{permut.}} - \chi_{\text{trivial}}.$$

We obtain:

→

	#1	#6	#3	#8	#6
	1	(12)	(12)(34)	(123)	(1234)
$\chi_{triv.}$	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
$\chi_{perm.}$	4	2	0	1	0
$\chi_{stand.} = \chi_{perm.} - \chi_{tr.}$	3	1	-1	0	-1

Recall that $\chi_{perm.}(\sigma) =$ = # elements in $\{1,2,3,4\}$ that are fixed by σ

Remark: $\langle \chi_{stand}, \chi_{stand} \rangle = \frac{1}{24}\left[9+6+3+6\right] = 1$ ✓

Because the standard representation is irreducible, and the sign representation is 1-dimensional, the product $\chi_{stan.} \cdot \chi_{sign}$ is an irreducible character.

$\chi_{stan.} \cdot \chi_{sign}$	3	-1	-1	0	+1

So far we have constructed 4 irreducible characters. Because S_4 has 5 conjugacy classes, there is only one irreducible character left. We can find it using the formulas:

i. $\sum_{\substack{\text{all irred.} \\ \text{characters}}} (\dim V)\, \chi_V(g) = 0 \quad , \; g \neq 1$

and

ii. $\sum_{\substack{\text{all irred} \\ \text{characters}}} (\dim V)^2 = |G| = 24.$

Call W the remaining irreducible representation. Then:

$$[\dim(W)]^2 = 24 - 1^2 - 1^2 - 3^2 - 3^2 = 4 \Rightarrow \dim(W) = 2.$$

$$\chi_W((12)) = -\frac{1}{2}\left[1 - 1 + 3 - 3\right] = 0$$

$$\chi_W((12)(34)) = -\frac{1}{2}\left[1 + 1 - 3 - 3\right] = 2$$

$$\chi_W((123)) = -\frac{1}{2}\left[1 + 1\right] = -1$$

$$\chi_W((1234)) = -\frac{1}{2}\left[1 - 1 - 3 + 3\right] = 0.$$

The complete character Table of S_4 is therefore given by:

	#1	#6	#3	#8	#6
	1	(12)	(12)(34)	(123)	(1234)
U = trivial	1	1	1	1	1
U' = sign	1	-1	1	1	-1
V = standard	3	1	-1	0	-1
V' = sign ⊗ standard	3	-1	-1	0	1
W	2	0	2	-1	0