LECTURE 7]

# THE NUMBER OF IRREDUCIBLE CHARACTERS

Our next issue is to compute the number of irreducible characters. So far we only know that this number is lower than or equal to the order of G.

Indeed the irreducible characters are orthonormal (hence l.i.) elements of $\mathbb{C}^G$, which is a vector space of dimension $|G|$.
[A basis of $\mathbb{C}^G$ consists of the functions $\{\delta_g\}_{g\in G}$, where, $\forall g\in G$, $\delta_g : G \to \mathbb{C}$, $s \mapsto \begin{cases} 1 & \text{if } s=g \\ 0 & \text{otherwise.} \end{cases}$]

We can say more. Consider the subspace of $\mathbb{C}^G$ consisting of all class functions:
$\mathbb{C}_{class}(G)$ ~~$R(G)$~~ $= \{ f \in \mathbb{C}^G : f(x) = f(y^{-1}xy) \ \ \forall x, y \in G \}$.
Then, because all characters are invariant on conjugacy classes, the irreducible characters also form an independent system in ~~$R(G)$~~ $\mathbb{C}_{class}(G)$.
What's the dimension of ~~$R(G)$~~ $\mathbb{C}_{class}(G)$.

**Lemma** - $\dim$ ~~$R(G)$~~ $\mathbb{C}_{class}(G) = k$, where $k$ is the number of conjugacy classes of G.

**proof** - Let $C_1, \dots, C_k$ be the conjugacy classes of G.
For all $i = 1..k$, define a function $e_i \in$ ~~$R(G)$~~ $\mathbb{C}_{class}(G)$ by:

$$e_i(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{e_1, e_2, \dots, e_k\}$ are l.i. elements of ~~$R(G)$~~ $\mathbb{C}_{class}(G)$, and they span ~~$R(G)$~~ $\mathbb{C}_{class}(G)$:

• $\sum_{i=1}^{k} a_i e_i = 0 \iff \sum_{i=1}^{k} a_i e_i(g) = 0 \quad \forall g \in G$

$G = \bigcup_{j=1}^{k} C_j$ $\quad \iff \sum_{i=1}^{k} a_i \underbrace{e_i(C_j)}_{=\delta_{ij}} = 0 \quad \forall j = 1..k$

$\iff a_j = 0 \quad \forall j = 1..k.$

• If $f \in R(G)$, and $x \in C_j$, then we can write:
$f(x) = f(C_j) = \sum_{i=1}^{k} f(C_i) \underbrace{e_i(x)}_{\substack{=1 \text{ if } i=j \\ =0 \text{ otherwise}}}. \quad [\forall j = 1..k]$

$\Rightarrow f = \sum_{i=1}^{k} f(C_i) e_i.$ ■

Corollary - The number of irreducible characters is smaller than or equal to the number of conjugacy classes.

Proof - the formula $\langle f, g \rangle = \frac{1}{|G|} \sum_{s \in G} \overline{f(s)} g(s)$ also gives an inner product in ~~$R(G)$~~ $\mathcal{C}(G)_{\text{class}}$.
The irreducible characters are orthonormal, hence linearly independent. Therefore
$\#$ irred. characters $\leq \dim(\cancel{R(G)}\ \mathcal{C}_{\text{class}}(G)) = \#$ conjugacy classes ■

Remarks - For abelian groups, we have proved that the number of irreducible characters is actually equal to the number of conjugacy classes.
This result generalizes to all groups.
To prove that the equality always holds we

need the following lemma:

**Lemma** For every <u>irreducible</u> representation $(\rho, V)$ of $G$, and every class function $f \in \cancel{R(G)}\,\mathbb{C}_{class}(G)$

$$\sum_{x\in G} \overline{f(x)}\,\rho(x) = \frac{|G|}{\dim V} \langle \chi_\rho, f \rangle \mathbb{1}_V .$$

<u>proof</u> - Because each $\rho(x)$ is a linear map from $V$ to $V$, also $T = \sum_{x\in G} \overline{f(x)}\,\rho(x)$ is an endomorphism of $V$.

the claim is that $T = \alpha \mathbb{1}_V$, with $\alpha = \frac{|G|}{\dim V}\langle \chi_\rho, f\rangle$.

Because $\rho$ is irreducible, in order to prove that $T$ is a scalar multiple of the identity, we just need to prove that $T$ is an intertwining operator from $(\rho, V)$ to itself: i.e. there's a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "T"] \\ V \arrow[r, "\rho(g)"'] & V \end{tikzcd}, \quad \forall g \in G$$

$$\cdot\ T\rho(g) = \left[\sum_{x\in G} \overline{f(x)}\,\rho(x)\right]\rho(g) = \sum_{x\in G} \overline{f(x)}\,\rho(xg) =$$

$$= \sum_{x\in G} \overline{f(g^{-1}xg)}\,\rho(xg) =$$

$f \in R(G)$

$$= \sum_{x\in G} \overline{f(g^{-1}xg)}\,\rho(gg^{-1}xg) =$$

$$= \rho(g)\left[\sum_{x\in G} \overline{f(\underbrace{g^{-1}xg}_{v})}\,\rho(\underbrace{g^{-1}xg}_{\overset{\shortparallel}{u}})\right] = \rho(g)\underbrace{\left[\sum_{u\in G} \overline{f(u)}\,\rho(u)\right]}_{=T} = \rho(g)T \checkmark$$

We obtain : $T = \alpha \mathbb{1}_V$, for some $\alpha \in \mathbb{C}$. [This is Shur's lemma].

Next, we evaluate $\alpha$ by taking the trace of $T$.

$$\alpha\,(\dim V) = \operatorname{trace}(T) = \sum_{x \in G} \overline{f(x)}\ \operatorname{trace}(\rho(x)) =$$

$$= \sum_{x \in G} \overline{f(x)}\ \chi_\rho(x) = |G| \langle f, \chi_\rho \rangle$$

$$\Rightarrow \quad \alpha = \frac{|G|}{\dim V} \langle f, \chi_\rho \rangle .$$

This completes the proof. $\blacksquare$

We are now ready to prove That The irreducible characters form a basis of ~~$R(G)$~~ $\mathbb{C}_{class}(G)$. [This will of course imply That The # of irred. characters equals The # of conjugacy classes].

<u>theorem</u> - The irreducible characters of $G$ are an orthonormal basis of ~~$R(G)$~~ $\mathbb{C}_{class}(G)$.

<u>proof</u> - We just need to show That

$\mathbb{C}_{class}(G)$ ~~$R(G)$~~ = Span (all distinct irred. Characters).

We procede by contradiction.

Suppose That ~~$R(G)$~~ $\mathbb{C}_{class}(G)$ $\supsetneq$ Span (irred. characters).

Then The exists an element $f \neq 0$ in ~~$R(G)$~~ $\mathbb{C}_{class}(G)$ which is perpendicular To ~~all~~ every irreducible

Let $(\rho, V)$ be the regular representation of $G$ (with basis $\{e_x\}_{x\in G}$). then $V = \bigoplus_{W_i \text{ irred.}} (W_i)^{\oplus \dim W_i}$ and
$\chi_V = \sum_{W_i \text{ irred}} (\dim W_i)\, \chi_{W_i}$.
Because $f$ is orthogonal to each irreducible character $\chi_{W_i}$, for all $x \in W_i$ we can write:

$$\sum_g \overline{f(g)}\, \rho(g) x \underset{\rho(g) = \rho_{W_i}(g)}{=} \left[\sum_g \overline{f(g)}\, \rho_{W_i}(g)\right] x \underset{W_i \text{ is irreducible and } f \in \mathbb{C}_{class}(G) \text{ so the lemma applies}}{=}$$

$$= \left[\left(\frac{\dim W_i}{|G|}\right)^{-1} \langle f, \chi_{W_i}\rangle\, \mathbb{1}_{W_i}\right] x =$$

$$= \frac{|G|}{\dim W_i} \underbrace{\langle f, \chi_{W_i}\rangle}_{=0} x = 0.$$

So the operator $\sum_{x\in G} \overline{f(x)}\, \rho(x) : V \to V$ is identically zero on $V$.
Applying this operator to the basis element $e_1$ ($1 \in G$ is the identity of $G$), we get:

$$0 = \sum_{x\in G} \overline{f(x)}\, \underbrace{\rho(x)\, e_1}_{= e_x} = \sum_{x \in G} \overline{f(x)}\, e_x .$$

But the set $\{e_x\}_{x\in G}$ forms a basis of $V$. Hence we need $f(x) = 0$ $\forall x \in G$.
We get a contradiction, because $f$ was assumed to be non-zero. ▨

<u>Corollary</u> - the number of conjugacy classes equals the number of irreducible characters of $G$.

# THE REPRESENTATION RING

All The results known about characters can be represeal in Terms of The representation ring of G...

Let G be a finite group. The representation ring $R(G)$ is defined as follows:

- as a group, $R(G)$ is The free abelian group generated by all The isomorphism classes of irreducible representations of G (an element of $R(G)$ is an <u>integral</u> linear combination of irreducible representations of G, and is called "<u>a virtual representation</u>");
- The ring structure is given by the Tensor product (defined on the generators of $R(G)$, and extended by linearity to $R(G)$). If V, W are irreducible representations, define $V \cdot W = \underbrace{a_1 U_1 \oplus \cdots \oplus a_r U_r}$.

suppose That The decomposition of $V \otimes W$ as a direct sum of irreducible summands is

$$V \otimes W = \oplus \left[ U_i^{\oplus a_i} \right]$$

It's easy to see That $R(G)$ is a commutative ring with identity:

$$[V \otimes W] \otimes U \cong V \otimes [W \otimes U]$$

$$V \otimes (W \oplus U) \cong [V \otimes W] \oplus [V \otimes U]$$

$V \otimes \text{triv} \cong V$.

---

Consider the map

$$\chi : R(G) \to \mathbb{C}_{class}(G)$$

defined on the generators by $\chi(V) = \chi_V$ and extended by linearity to $R(G)$.

$\chi$ is a ring homomorphism (the ring structure on $\mathbb{C}_{class}(G)$ is given by the usual sum and product of complex valued functions).

Because every representation of $G$ is determined by its character, $\chi$ is injective.

We can extend $\chi$ to a vector space homomorphism:

$$\chi_{\mathbb{C}} : R(G) \otimes \mathbb{C} \longrightarrow \mathbb{C}_{class}(G)$$

$$V \otimes \alpha \longmapsto \alpha \chi_V .$$

the fact that the characters of irreducible representation of $G$ form a basis of $\mathbb{C}_{class}(G)$ implies that $\chi_{\mathbb{C}}$ is a (complex vector space) isomorphism.

# A GENERAL PROJECTION FORMULA

Let $(\rho, V)$ be a representation of $G$.
Let $V = W_1^{\oplus a_1} \oplus \dots \oplus W_r^{\oplus a_r}$ be the decomposition of $V$ as a direct sum of isotypic components of irreducible representations of $G$. We know that the map

$$T_{triv} = \frac{1}{|G|} \sum_{g \in G} \rho(g) : V \to V$$

is a projection on the isotypic component of the trivial representation.

We can generalize this result: for each irreducible representation $W$ of $G$, the map

$$T_W = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)}\, \rho(g) : V \to V$$

is a projection on the isotypic component of $W$ in $V$.

PROOF - Write $V = W_1^{\oplus a_1} \oplus \dots \oplus W_j^{\oplus a_j}$. For all $x$ in the isotypic component of $W_i$, we can write:

$$T_W(x) = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)}\, \rho(g)\, x \underset{\uparrow}{=}$$

$\rho(g)|_{W_i^{\oplus a_i}} = \rho_{W_i}(g)$

$$\frac{\dim W}{|G|} \left[ \sum_{g \in G} \overline{\chi_W(g)}\, \rho_{W_i}(g) \right] x =$$

$f = \chi_W \in \mathbb{C}_{class}(G)$ and $W_i$ is irreducible so the lemma applies

$$\frac{\dim W}{|G|} \left[ \frac{|G|}{\dim W_i} \langle \underbrace{\chi_W}_{"f"}, \chi_{W_i} \rangle \mathbb{1}_{W_i} \right] x =$$

$$\frac{\dim W}{\dim W_i} \langle \chi_W, \chi_{W_i} \rangle\, x =$$

both $W$ and $W_i$ are irreducible, so $\langle \chi_W, \chi_{W_i} \rangle = \begin{cases} 1 & \text{if } W \cong W_i \\ 0 & \text{if } W \not\cong W_i \end{cases}$

$$\begin{cases} x & \text{if } W \cong W_i \\ 0 & \text{if } W \not\cong W_i \end{cases}$$

This shows that $T_W$ is the projection on the isotypic of $W$ in $V$. ∎

# ORTHOGONALITY FORMULAS

# OF THE SECOND KIND

We already know that the characters of irreducible representations are orthonormal:

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_W(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \cong W. \end{cases}$$

Let $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ be the conjugacy classes of $G$.
For all $j = 1 \dots k$, set $c_j = \# \Gamma_j$.
Then the previous relation can be rewritten as:

$$\sum_{\Gamma_i} c_i\, \overline{\chi_V(\Gamma_i)}\, \chi_W(\Gamma_i) = \begin{cases} |G| & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

These are called "orthogonality relations of the first kind". Notice that the two characters are fixed, and the sum is over all conjugacy classes.

We now prove the "orthogonality relations of the second kind":

$$\sum_{\chi_i} \overline{\chi_i(\Gamma_1)}\, \chi_i(\Gamma_2) = \begin{cases} \dfrac{|G|}{c(\Gamma_1)} & \text{if } \Gamma_1 = \Gamma_2 \\ 0 & \text{if } \Gamma_1 \neq \Gamma_2. \end{cases}$$

[This time the two conjugacy classes are fixed, and the sum is over all the characters ... ].

proof - Fix a conjugacy class $\Gamma$, and let $c = \# \Gamma$. Define a class function by:

$$f_\Gamma(g) = \begin{cases} 1 & \text{if } g \in \Gamma \\ 0 & \text{if } g \notin \Gamma. \end{cases}$$

Equivalently, $f_\Gamma(\Gamma_2) = \begin{cases} 1 & \text{if } \Gamma_2 = \Gamma \\ 0 & \text{if } \Gamma_2 \neq \Gamma \end{cases}$

Because the characters of irreducible representations form a basis of $\mathbb{C}_{class}(G)$, we can write

$$f_\Gamma = \sum_{j=1}^{k} a_j \chi_j$$

with $a_j = \langle \chi_j, f_\Gamma \rangle$ [Indeed, $\langle \chi_j, f_\Gamma \rangle =$

$$= \langle \chi_j, \sum_{e=1}^{k} \cancel{a_j} \chi_e \rangle \underset{\substack{\uparrow \\ \langle,\rangle \text{ linear on the} \\ \text{second component}}}{=} \sum_{e=1}^{k} a_e \underbrace{\langle \chi_j, \chi_e \rangle}_{\delta_{je}} = a_j].$$

Let us compute $a_j$ more explicitely:

$$a_j = \langle \chi_j, f_\Gamma \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_j(g)} f_\Gamma(g) =$$

$$\underset{\substack{\nearrow \\ f_\Gamma(g) = \begin{cases} 1 \text{ if } g \in \Gamma \\ 0 \text{ o.w.} \end{cases}}}{=} \frac{1}{|G|} \sum_{g \in \Gamma} \overline{\chi_j(g)} = \frac{c}{|G|} \overline{\chi_j(\Gamma)}.$$

So we get: $f_\Gamma = \sum_{j=1}^{k} \frac{c}{|G|} \overline{\chi_j(\Gamma)} \chi_j = \frac{c}{|G|} \sum_{j=1}^{k} \overline{\chi_j(\Gamma)} \chi_j$

Because $f_\Gamma(g) = \begin{cases} 1 & \text{if } g \in \Gamma \\ 0 & \text{ow} \end{cases}$, we obtain:

$$\frac{c}{|G|} \sum_{j=1}^{k} \overline{\chi_j(\Gamma)}\, \chi_j(\Gamma') = \begin{cases} 1 & \text{if } \Gamma = \Gamma' \\ 0 & \text{ow} \end{cases}$$

$$\Longleftrightarrow \sum_{j=1}^{k} \overline{\chi_j(\Gamma)}\, \chi_j(\Gamma') = \begin{cases} \frac{|G|}{c} & \text{if } \Gamma = \Gamma' \\ 0 & \text{ow}. \end{cases}$$

$\blacksquare$

# CHARACTERS OF THE DIHEDRAL GROUP

**★ $D_{2n}$, n odd ★**

We know that $D_{2n}$ has conjugacy classes:

$$\{1\}$$
$$\{a, a^{n-1}\}$$
$$\{a^2, a^{n-2}\}$$
$$\vdots$$
$$\{a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}\}$$
$$\{a^j b : j = 0 \dots n-1\}.$$

The number of conjugacy classes equals $\frac{n-1}{2} + 2 = \frac{n+3}{2}$.
We must create an equal number of irreducible characters.

First, we look for linear characters.
Suppose that $\rho$ is a representation of $D_{2n} = \langle a, b : a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ of dimension 1. Then $\rho(a)$ is an $n^{th}$ root of unity, and $\rho(b) = \pm 1$.
We notice that $a$ is conjugate to $a^{-1}$, so $\chi_\rho(a) = \chi_\rho(a^{-1}) = \overline{\chi_\rho(a)} \Rightarrow \chi_\rho(a)$ is real.
But $\chi_\rho(a) = \rho(a)$ (because $\rho$ is 1-dimensional).
The only $n^{th}$ root of unity that is real is 1 (here n is odd). So $\rho(a)$ must be 1.
It follows that there are only two linear characters.
They are:

| | $a$ | $b$ | ..... |
|---|---|---|---|
| $\chi_1$ | 1 | +1 | ... |
| $\chi_2$ | 1 | −1 | ... |

We need other $\frac{n-1}{2}$ irreducible characters.
Because $D_{2n}$ is non-abelian, at least one of these characters has degree greater than 1.

Let's look for characters of degree 2.

A 2-dimensional representation of $D_{2n}$ is given by a group homomorphism

$$\rho : G \to GL(2,\mathbb{C})$$
$$a^i b^j \mapsto A^i B^j$$

where $A, B \in GL(\mathbb{C})$ satisfy : $A^n = I = B^2$; $B^{-1}AB = A^{-1}$.

We can choose : $A = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$ where $\omega$ is any $n^{\text{th}}$ root of unity

and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(indeed $B^{-1}AB = BAB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{bmatrix}}_{A^{-1}}$) ✓

For each choice of $\omega$ ($\leftarrow n^{\text{th}}$ root of unity), we obtain a 2-dimensional representation of $G$, that we denote by $\rho_\omega$.

When is $\rho_\omega$ irreducible ?

If $\rho_\omega$ splits in the direct sum of two one-dimensional subrepresentations : $\rho_\omega = U^1_\omega \oplus U^2_\omega$, then $U^1_\omega$ and $U^2_\omega$ are simoultaneous eigenspaces of $A$ and $B$.

The only eigenspaces of $B$ are $\operatorname{Span}\begin{vmatrix}1\\1\end{vmatrix}$ and $\operatorname{Span}\begin{vmatrix}-1\\1\end{vmatrix}$ and they are stable under the action of $A$ only if $\omega = \omega^{-1}$ ($\Leftrightarrow \omega = \bar{\omega} \Leftrightarrow \omega$ is real). Because $n$ is odd, this condition forces $\omega = 1$.

We have proved that the representation $\rho_{e^{\frac{2\pi i}{n}k}}$ is irreducible for all $k = 1, 2, \dots, n-1$.

Ok! We get $(n-1)$ irreducible characters, but only some of them are going to be distinct. [~~The~~ Recall that there are only $\frac{n-1}{2}$ irred. characters yet to find !!!].

The next question to solve is therefore:
" When is $\rho_{e^{\frac{2\pi i k_1}{n}}}$ equivalent to $\rho_{e^{\frac{2\pi i k_2}{n}}}$ ??

Suppose that there exists an interwining operator $T$ from $\rho_{e^{\frac{2\pi i k_1}{n}}}$ to $\rho_{e^{\frac{2\pi i k_2}{n}}}$.

Then, in particular, the matrices $\rho_{\underbrace{e^{2\pi i k_1/n}}_{\omega_1}}(a) = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_1^{-1} \end{bmatrix}$ and $\rho_{\underbrace{e^{2\pi i k_2/n}}_{''\omega_2}}(a) = \begin{bmatrix} \omega_2 & 0 \\ 0 & \omega_2^{-1} \end{bmatrix}$ are conjugate, and have the same eigenvalues.
We obtain that $\omega_2$ must be either equal to $\omega_1$ or equal to $\omega_1$. (4 necessary condition for equivalance)
$\Longrightarrow$ $\rho_{\omega_1}$ can only be equivalent to $\rho_{\omega_1^{-1}}$ (and to itself)

The next step is to show that $\rho_{\omega_1}$ and $\rho_{\omega^{-1}}$ are indeed conjugate equivalent, but this is very easy because $T: \mathbb{C}^2 \to \mathbb{C}^2$, $x \mapsto \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ intertwises the two representations.

We deduce that $\forall j = 1, \dots, n-1$, $\rho_{\omega_j}$ is only conjugate

We obtain $\frac{n-1}{2}$ irreducible representations of degree 2.
In addition to the two linear characters, these form a complete set of (inequivalent) irreducible representations of $D_{2n}$. Indeed $2 + \frac{n-1}{2} = \#$ conjugacy classes in $D_{2n}$.

★ <u>n even</u> ★

Assume $n = 2m$. Let's describe the irreducible characters of $D_{2n} = D_{4m}$.
As usual, we first look for linear characters.
If degree$(\rho) = 1$, then $\chi_\rho(a) = \rho(a)$ must be a real $n^{th}$ root of unity; but $n$ is even, so $\rho(a)$ is allowed to be $\pm 1$.
We obtain 4 linear characters:

| | $a$ | $b$ | ... |
|---|---|---|---|
| $\chi_1$ | $+1$ | $+1$ | ... |
| $\chi_2$ | $+1$ | $-1$ | ... |
| $\chi_3$ | $-1$ | $+1$ | ... |
| $\chi_4$ | $-1$ | $-1$ | ... |

Recall that the conjugacy classes of $D_{2n} = D_{4m}$ are

$\{1\}$
$\{a, a^{n-1}\}$
$\vdots$
$\{a^{m-1}, a^{m+1}\}$
$\{a^m\}$
$\{a^{2j} b : j = 0 \dots m\}$
$\{a^{2j+1} b : j = 0 \dots m-1\}$

▸ There are $m + 3 = \left(\frac{n}{2} - 1\right) + 4$ distinct conjugacy classes!

Just like before, we can look at the $2m = n$ two-dimensional representations $\{\rho_{w_j}\}_{j=0 \dots n-1}$, where $w_j = e^{\frac{2\pi i}{n} j} = e^{\frac{\pi i}{m} j}$.

We notice that if $j = 0$ or $m$, then $\rho(a) = A = \pm I$, so $\rho(a)$ stabilizes the eigenspaces of $\rho(b)$ and $\rho$ is reducible.
For $j \neq 0, m$ ($j = 0 \dots 2m-1$) then $\rho$ is irreducible.
[↑ same proof as before, because $w_j \neq \overline{w_j}$ if $w_j \neq \pm 1$].

So we obtain $n-2$ irreducible representations. The same proof used before shows that $\rho_{w_j}$ is equivalent to $\rho_{w_{n-j}}$ $\forall j = 0 \dots n-1, j \neq 0, m$.
Therefore we get

$$\frac{n-2}{2} = \frac{n}{2} - 1$$

inequivalent irreducible representations.

Because we have reached the number of conjugacy classes, there cannot be any other representation of $D_{2n}$.