[LECTURE 6]

ORTHOGONALITY OF CHARACTERS

Let G be a finite group. Let $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$ be the set of all functions from G to $\mathbb{C}$. Then $\mathbb{C}^G$ is a finite dimensional vector space (of dimension equal to $|G|$), with "pointwise" operations

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in G$$

$$(\lambda f)(x) = \lambda f(x) \qquad \forall \lambda \in \mathbb{C}, \forall x \in G.$$

We define an inner product on $\mathbb{C}^G$ by:

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} \overline{f(x)}\, g(x).$$

The irreducible characters of G are orthonormal w.r.t. this inner product. Indeed, if ρ and ρ' are irreducible representations of G, then $\chi_\rho, \chi_{\rho'} \in \mathbb{C}^G$ and

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \begin{cases} 1 & \text{if } \rho \cong \rho' \\ 0 & \text{if } \rho \not\cong \rho'. \end{cases}$$

The proof of this important theorem will follow from the following lemma:

<u>Lemma</u> ("FIRST PROJECTION FORMULA").
Let (ρ, V) be any representation of G. The map

$$\varphi : V \longrightarrow V, \quad v \longmapsto \frac{1}{|G|} \sum_{g \in G} \rho(g) v$$

is a projection of V onto V^G.

[We have denoted by V^G the set

$$V^G = \{ v \in V : \rho(g) v = v \ \ \forall g \in G \}.$$ Notice that if V is reducible, V^G is the isotypical component of the trivial representation.

If V is irreducible and trivial, then $V^G = V$.
If V is irreducible and non-trivial, then $V^G = \{0\}$.]

<u>proof</u> - The map $\varphi = \frac{1}{|G|}\sum_{g\in G}\rho(g)$ is a projection on V^G if it satisfies these two properties:

(i) $\varphi|_{V^G} = \mathbb{1}_{V^G}$, i.e. $\varphi(v) = v \quad \forall v \in V^G$.

(ii) $\mathrm{Im}\,\varphi \subseteq V^G$ (hence $\mathrm{Im}\,\varphi = V^G$), i.e. $\rho(h)\varphi(v) = \varphi(v)$ $\forall h \in G,\ v\in V$.

Both conditions are very easy to prove.

(i) For $v \in V^G$, $\varphi(v) = \frac{1}{|G|}\sum_g \underbrace{\rho(g)v}_{=v} = \frac{1}{|G|}\sum_{g\in G} v = v$.

(ii) For $v\in V$, $h \in G$, $\rho(h)\varphi(v) = \rho(h)\left[\frac{1}{|G|}\sum_{g\in G}\rho(g)v\right] =$

$$= \frac{1}{|G|}\sum_{g\in G}\underbrace{\rho(h)\rho(g)}_{=\rho(hg)}v = \frac{1}{|G|}\sum_{g\in G}\rho(hg)v = \frac{1}{|G|}\sum_{s\in G}\rho(s)v = \varphi(v).$$ ▨

<u>Corollary 1</u> - For every representation (ρ, V) of G

$$\boxed{\dim(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)}$$

<u>proof</u> - Write a decomposition of V as a direct sum of isotypi[c] components of irreducible representations:

$$V \cong V^G \oplus W_1^{\oplus a_1} \oplus \dots \oplus W_k^{\oplus a_k}$$

(here V^G = the isotypic component of the trivial representation and $W_1 \dots W_k$ are distinct irreducible representations not isomorphic to the trivial).

Choose a basis of V that respects this decomposition.
Because φ is a projection on V^G, the matrix of φ w.r.t. this basis has a block structure $\left[\begin{array}{c|c} I_m & \\ \hline & 0 \end{array}\right]$, with $m = \dim V^G$.

So the trace of φ equals the dimension of V^G.
By definition of character, we can write:

$$\dim V^G = \mathrm{Trace}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \mathrm{trace}(\rho(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g). \quad ▨$$

<u>Corollary 2</u>- For every non-trivial <u>irreducible</u> representation ρ of G, we have:

$$\boxed{\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = 0}.$$

<u>Remark</u>: Because the trivial character takes the value 1 on each element of the group, we can re-write the previous formula as

$$0 = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{triv}(g)}\, \chi_\rho(g) = \langle \chi_{triv}, \chi_\rho \rangle.$$

This shows that every non trivial irreducible character of G is orthogonal to the trivial character. Hence we get the first example of orthogonality of irreducible characters.

The general case will follow by applying "Corollary 1" to the representation $\mathrm{Hom}(V_1, V_2)$.

<u>Theorem</u> [Orthogonality relations for characters]

Let (ρ_1, V_1) and (ρ_2, V_2) be <u>irreducible</u> representations of G.

Then
$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \begin{cases} 1 & \text{if } \rho_1 \simeq \rho_2 \\ 0 & \text{if } \rho_1 \not\simeq \rho_2. \end{cases}$$

<u>proof</u> - Consider the representation $V = \mathrm{Hom}(V_1, V_2)$ of G.
[Recall that $g \in G$ acts on $T: V_1 \to V_2$ by
$$\rho(g)T(v_1) = \rho_2(g)\,T(\rho_1(g^{-1})v_1)$$
for all $v_1 \in V_1$]. Then V^G is just the space of G-linear isomorphisms from V_1 to V_2.

Because $(\rho_1 V_1)$ and $(\rho_2 V_2)$ are both irreducible, it follows from Shur's lemma that $\dim(V^G) = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{if } \rho_1 \not\cong \rho_2. \end{cases}$

Next, we apply corollary 1 to (ρ, V):
$$\dim(V^G) = \frac{1}{|G|}\sum_{g\in G} \chi_\rho(g) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{\rho_1}(g)}\,\chi_{\rho_2}(g) = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle.$$

recall that $\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2$

so $\chi_\rho = \overline{\chi_{\rho_1}}\,\chi_{\rho_2}$

this concludes the proof. ∎

It follows from this Theorem that a representation is completely determined from its character:

let $V \cong W_1^{\oplus a_1} \oplus W_2^{\oplus a_2} \oplus \dots \oplus W_k^{\oplus a_k}$ be the decomposition of V as a direct sum of isotypic components of irreducible representations. For all $i = 1..k$, the multiplicity a_i of W_i in V equal to the inner product $\langle \chi_{W_i}, \chi_V \rangle$.

Indeed we can write

$$<\chi_{W_i}, \chi_V> = <\chi_{W_i}, \sum_{j=1}^{k} a_j \chi_{W_j}> = \sum_{i=1}^{k} a_j \underbrace{<\chi_{W_i}, \chi_{W_j}>}_{=\delta_{ij} \text{ because the } W_j\text{'s are not isomorphic...}} = a_i.$$

Also notice that

$$<\chi_V, \chi_V> = <\sum_{i=1}^{k} a_i \chi_{W_i}, \sum_{j=1}^{k} a_j \chi_{W_j}> = \sum_{i,j=1}^{k} a_i a_j \underbrace{<\chi_{W_i}, \chi_{W_j}>}_{=\delta_{ij}} =$$

$$= \sum_{i=1}^{k} a_i^2.$$

We obtain an irreducibility criterion for representations:

$$\boxed{V \text{ is irreducible if and only if } <\chi_V, \chi_V> = 1}$$

(because, in the previous notations, $V = W_1^{\oplus a_1} \oplus W_2^{\oplus a_2} \oplus \dots \oplus W_k^{\oplus a_k}$ is irreducible if and only if $k=1$ and $a_1 = 1$.)

This irreducibility criterion gives us hints on how to construct new characters from existing ones...

<u>Lemma</u> - The conjugate of an irreducible character is irreducible

[proof - $(\overline{\chi_V})$ is the character of V^*. To prove that V^* is irreducible, we evaluate the inner product of $\overline{\chi_V}$ with itself:

$$<\overline{\chi_V}, \overline{\chi_V}> = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \overline{\overline{\chi_V(g)}} = \overline{\left[\frac{1}{|G|} \sum_{g\in G} \chi_V(g), \overline{\chi_V(g)}\right]} =$$

$$= \overline{<\chi_V, \chi_V>} = \overline{1} = 1.]$$

<u>Lemma</u> - The product of an irreducible character by a linear

character is irreducible.

[proof - Suppose that V is irreducible and that U is one-dimensional. Set $W = V \otimes U$. Then $\chi_W = \chi_V \cdot \chi_U$.
Let's show that W is again irreducible:

$$\langle \chi_W, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)}\, \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \overline{\chi_U(g)}\, \chi_V(g) \chi_U(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2 \, |\chi_U(g)|^2 .$$

Notice that, because (U, ρ_U) is one-dimensional, $\chi_U(g) = \rho_U(g)$ must be an m^{th} root of unity (where m = order of g).
So $|\chi_U(g)|^2 = 1 \quad \forall g \in G$.
It follows that

$$\langle \chi_W, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2 = \langle \chi_V, \chi_V \rangle = 1. \quad \blacksquare]$$

Let's now discuss another corollary of the orthogonality of irreducible characters....
Suppose that

$$V \cong \left(W_1^{\oplus m_1}\right) \oplus \left(W_2^{\oplus m_2}\right) \oplus \cdots \oplus \left(W_k^{\oplus m_k}\right)$$

is the decomposition of V as a direct sum of isotypic components. Then

$$\langle \chi_V, \chi_V \rangle = \left\langle \sum_{j=1}^{k} m_j \chi_{W_j}, \sum_{i=1}^{k} m_i \chi_{W_i} \right\rangle =$$

$$= \sum_{i,j=1}^{k} m_j m_i \underbrace{\langle \chi_{W_j}, \chi_{W_i} \rangle}_{=\delta_{ij}} = \sum_{i=1}^{k} m_i^2 .$$

The relation $\boxed{\langle \chi_V, \chi_V \rangle = \sum_{i=1 \ldots m} m_i^2}$ can give a lot of insight on the number of irreducible constituents of a represen-

tation.

Examples:

- If $\langle \chi_V, \chi_V \rangle = 1 (= 1^2)$, then V is irreducible
- If $\langle \chi_V, \chi_V \rangle = \underset{1^2+1^2}{\overset{\shortparallel}{2}}$, then V is the sum of two distinct irreducible sub-representations (each occurring with multiplicity 1).
- If $\langle \chi_V, \chi_V \rangle = 3 = 1^2+1^2+1^2$, then V is the sum of 3 distinct irreducible constituents.
- If $\langle \chi_V, \chi_V \rangle = 4 = \begin{cases} 2^2 \\ 1^2+1^2+1^2+1^2 \end{cases}$ Then V is either the sum of two copies of the same irreducible representations or the sum of 4 distinct irreducible constituents. [And so on...]

The final observation of this lecture is the fact that if G has irreducible characters $\chi_1, \dots, \chi_k$, then it is possible to recover χ_k from $\chi_1, \dots, \chi_{k-1}$.

[Hence, when constructing the character table of G, you really need to work hard only on $k-1$ characters!]

The "inversion formula" is given by the following lemma:

<u>Lemma</u> - For all $g \neq e$, $\displaystyle\sum_{\substack{\text{all irreducible} \\ \text{characters}}} (\dim V)\, \chi_V(g) = 0.$

<u>proof</u> - $\sum_{\text{all irred}} (\dim V)\, \chi_V(g)$ is the value of the character of the regular rep. at $g \neq e$

Examples of Character Tables

1] * $G = D_6$ *

- $G = \langle a, b : a^3 = 1;\ b^2 = 1;\ b^{-1}ab = a^{-1} \rangle$
- conjugacy classes : $\{1\}$
  $\{a, a^{-1} = a^2\}$
  $\{b, ab, a^2 b\}$

First, we look for linear characters.
If $\dim V = 1$, then $\rho(a)$ must be a cubic root of 1 and $\rho(b)$ must be ± 1. Because of the relation $\rho(b)^{-1}\rho(a)\rho(b) = \rho(a)^{-1}$ $\iff \rho(a) = \rho(a^{-1}) \iff \rho(a) = \overline{\rho(a)}$, $\rho(a)$ must be real.
So there are only two linear characters :

	1 $\#1$	a $\#2$	b $\#3$
χ_1	1	1	1
χ_2	1	1	-1

We notice that D_6 is non-abelian. So there must be an irreducible representation of dimension ≥ 2.
But $\sum_{\text{all irred } W} (\dim W)^2 = |G| = 6$, so there is only one more irreducible representation of G, and it has dimension 2.
To find this character, we use the fact that $\sum_{i=1}^{3} (\dim \chi_i)\, \chi_i(g) = 0$ for all $g \neq 1$.

For $g = a$, we find:

$$1 \cdot \underbrace{\chi_1(a)}_{1} + 1 \cdot \underbrace{\chi_2(a)}_{1} + 2 \cdot \chi_3(a) = 0 \implies \chi_3(a) = -1.$$

For $g = b$, we find:

$$1 \cdot \underbrace{\chi_1(b)}_{1} + 1 \cdot \underbrace{\chi_2(b)}_{-1} + 2 \cdot \chi_3(b) = 0 \implies \chi_3(b) = 0$$

So we find:

	1 #1	a #2	b #3
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Let's do some checkings:

- $\langle \chi_3, \chi_3 \rangle = \frac{1}{6}\left[1 \cdot 2 \cdot 2 + 2 \cdot (-1)(-1) + 3 \cdot 0 \cdot 0\right] = 1$ ✓
- $\langle \chi_1, \chi_1 \rangle = \frac{1}{6}\left[1 + 2 + 3\right] = 1$ ✓
- $\langle \chi_2, \chi_2 \rangle = \frac{1}{6}\left[1 + 2 + 3(-1)^2\right] = 1$ ✓
- $\langle \chi_1, \chi_2 \rangle = \frac{1}{6}\left[1 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot (-1)\right] = 0$ ✓
- $\langle \chi_1, \chi_3 \rangle = \frac{1}{6}\left[1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot (-1) + 3 \cdot 1 \cdot 0\right] = 0$ ✓
- $\langle \chi_2, \chi_3 \rangle = \frac{1}{6}\left[1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot (-1) + 3 \cdot (-1) \cdot 0\right] = 0$ ✓.

Finally we notice that we could have found $\chi_3(a)$ (and $\chi_3(b)$) in many other ways.
For instance, by imposing the orthogonality between the first and second (third) column of the character table:

and $1 \cdot 1 + 1 \cdot 1 + 2 \cdot \chi_3(a) = 0 \longrightarrow \chi_3(a) = -1$

$1 \cdot 1 + 1 \cdot (-1) + 2 \cdot \chi_3(b) = 0 \longrightarrow \chi_3(b) = 0.$

★ $G = D_8$ ★

- $G = \langle a, b : a^4 = 1,\ b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$
- conjugacy classes: $\{1\}$
  $\{a, a^3\}$
  $\{a^2\}$
  $\{b, a^2b\}$
  $\{ab, a^3b\}$

First, we look for linear character.

If (ρ, V) is a 1-dimensional representation, then $\rho(b) = \pm 1$ and $\rho(a) \in \{1, -1, i, -i\}$. Because a is conjugate to its inverse, $\rho(a)$ must be real, so $\rho(a) = \pm 1$.

We obtain 4 linear characters:

	1 #1	a #2	a^2 #1	b #2	ab #2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	-1	1
χ_4	1	-1	1	1	-1

Because D_8 is non-abelian, there must be a non-linea character. From the condition $\sum_{\text{all irred}} (\dim V)^2 = 1$, we actual conclude that there is only another character (of degree

We can find this character by orthogonality of the j^{th} column with the first column:

- $1 + 1 - 1 - 1 + 2\,\chi_5(a) = 0 \implies \chi_5(a) = 0$
- $1 + 1 + 1 + 1 + 2\,\chi_5(a^2) = 0 \implies \chi_5(a^2) = -2$
- $1 - 1 - 1 + 1 + 2\,\chi_5(b) = 0 \implies \chi_5(b) = 0$
- $1 - 1 + 1 - 1 + 2\,\chi_5(ab) = 0 \implies \chi_5(ab) = 0.$

We obtain :

	1	a	a^2	b	ab
	#1	#2	#1	#2	#2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	-1	1
χ_4	1	-1	1	1	-1
χ_5	2	0	-2	0	0

Check : $\langle \chi_5, \chi_5 \rangle = \frac{1}{8}\left[2^2\cdot 1 + (-2)^2\cdot 1\right] = \frac{1}{8}[4+4] = 1$ ✓

$\langle \chi_1, \chi_5 \rangle = \frac{1}{8}\left[2-2\right] = 0$ ✓

Similarly, $\langle \chi_j, \chi_5 \rangle = \frac{1}{8}[2-2] = 0$ for $j = 2,3,4$. ✓

$\boxed{3}$ There exists a group G of order 10 which has precisely four conjugacy classes and has irreducible characters χ_1, χ_2 as follows :

	g_1	g_2	g_3	g_4
	#1	#2	#2	#5
χ_1	1	1	1	1
χ_2	2	α	β	0

with $\alpha = \frac{-1+\sqrt{5}}{2}$, and $\beta = \frac{-1-\sqrt{5}}{2}$.

Complete the character table.

From the character table we see that $g_1 = 1$.
We have : $\sum_{i \geq 3} \chi_i(1)^2 = |G| - 1 - 4 = 10 - 1 - 4 = 5$. So
G has at least one more linear character.

Let's call it χ_3.
The conjugacy class of g_2 and g_3 has cardinality 2, hence their centralizer has order 5.
Because 5 is prime, we deduce that both g_2 and g_3 have order 5.
So $\chi_3(g_2)$ and $\chi_3(g_3)$ are fifth roots of unity.
We notice that g_2 is conjugate to its inverse, or $g_2^{-1} = g_K$ for some $K \neq 2$ and $\chi_2(g_K) = \chi_2(g_2^{-1}) = \overline{\chi_2(g_2)} = \bar{\alpha} = \alpha$ (contradiction, because χ_2 takes the value α only once).
Because g_2 and g_2^{-1} lie in the same conjugacy class, $\chi_3(g_2)$ must be real, but the only real fifth root of unity is one. Hence $\chi_3(g_2) = 1$.
Similarly, $\chi_3(g_3) = 1$.
We find:

	g_1 #1	g_2 #2	g_3 #2	g_4 #5
χ_1	1	1	1	1
χ_2	2	α	β	0
χ_3	1	1	1	?

To find $\chi_3(g_4)$ we impose that $\langle \chi_3, \chi_1 \rangle = 0$.

$\Rightarrow \ 0 = \langle \chi_3, \chi_1 \rangle = \frac{1}{10}\left[1 + 2 + 2 + 5\chi_3(g_4)\right]$

$\Rightarrow \ \chi_3(g_4) = -1$.

Next, we look at $\chi_2 \otimes \chi_3$ (because χ_2 is irreducible and χ_3 is 1-dimensional, $\chi_2 \otimes \chi_3$ is irreducible).

	g_1	g_2	g_3	g_4
χ_2	2	α	β	0
χ_3	1	1	1	-1
$\chi_2 \otimes \chi_3$	2	α	β	0

It does not help! $\chi_2 \otimes \chi_3 = \chi_2$.

What else can we say? Let's look at $\chi_2 \otimes \chi_2$:

	g_1 #1	g_2 #2	g_3 #2	g_4 #5
$\chi_2 \otimes \chi_2$	4	α^2	β^2	0

We notice that:

- $\langle \chi_2 \otimes \chi_2, \chi_2 \otimes \chi_2 \rangle = \frac{1}{10}\left[16 + 2(\alpha^2)^2 + 2(\beta^2)^2 + 5 \cdot 0\right] =$

$$= \frac{1}{10}\left[16 + 2\left(\frac{-1+\sqrt5}{2}\right)^4 + 2\left(\frac{-1-\sqrt5}{2}\right)^4\right] =$$

$$= \frac{1}{10}\left[16 + 2\left(\frac{1+5-2\sqrt5}{4}\right)^2 + 2\left(\frac{1+2\sqrt5+5}{4}\right)^2\right] =$$

$$= \frac{1}{10}\left[16 + \frac{(3-\sqrt5)^2}{2} + \frac{(3+\sqrt5)^2}{2}\right] = \frac{1}{10}\left[16+9+5\right] = 3.$$

The only way to write 3 as a sum of 3 square is $3 = 1+1+1$. So $\chi_2 \otimes \chi_2$ consists of a sum of 3 constituents (of dimension 1, 1 and 2).

- $\langle \chi_2 \otimes \chi_2, \chi_1 \rangle = \frac{1}{10}\left[4 + 2\alpha^2 + 2\beta^2 + 0\right] =$

$$= \frac{1}{10}\left[4 + \frac{3-\sqrt5}{2} + \frac{3+\sqrt5}{2}\right] = 1$$

$\Rightarrow \chi_2 \otimes \chi_2$ contains 1 copy of χ_1.

$\langle \chi_2 \otimes \chi_2, \chi_3 \rangle = \frac{1}{10}\left[4 + 2\alpha^2 + 2\beta^2 + 0\right] = 1 \Rightarrow \chi_2 \otimes \chi_2$ also contains a copy of χ_3.

Notice that

$\chi_2 \otimes \chi_2 - \chi_1 - \chi_3 \rightsquigarrow$

$4-2=2$	$\alpha^2-2 = \frac{-1-\sqrt{5}}{2} = \beta$	$\beta^2-2 = \frac{-1+\sqrt{5}}{2} = \alpha$	$0-1+1 = 0$

is a new irreducible character of G ($\leftarrow$ we know it's irreducible, and it's different from χ_2).

This completes our character table.

	g_1 #1	g_2 #2	g_3 #2	g_4 #5
χ_1	1	1	1	1
χ_2	2	α	β	0
χ_3	1	1	1	-1
χ_4	2	β	α	0

.

Suggested problems for lectures 5 and 6

[1] Use characters to show that

(a) the permutation representation of S_5 contains one copy of the trivial representation

(b) The standard representation of S_5 is irreducible.

[2] Let $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, and let ρ_1, ρ_2 be the representations of G for which

$$\rho_1(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \qquad \rho_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_2(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_2(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$.

(a) Find the characters of ρ_1 and ρ_2.

(b) Decide if ρ_1 and ρ_2 are irreducible and/or faithful.

(c) Write down the complete character table of D_{12}.

[3] Suppose that χ is an irreducible character of G and that $z \in Z(G)$ has order m. Prove that there exists an m^{th} root of unity $\lambda \in \mathbb{C}$ s.t. $\chi(zg) = \lambda\, \chi(g)$ $\forall g \in G$.

[4] Prove that if χ is faithful and irreducible, then
$$Z(G) = \{ g : |\chi(g)| = \chi(1) \}.$$

[5] Suppose that χ is a non-zero non-trivial character of G, and that $\chi(g)$ is a non negative real number for all g in G. Prove that χ is reducible.