CONJUGACY CLASSES IN D_{2n}

$\star$ n odd $\star$

$C_1 = \{1\} \longrightarrow$ CARDINALITY 1

$C_2 = \{a ; a^{-1}\} \longrightarrow$ " " 2

$C_3 = \{a^2 ; a^{-2}\} \longrightarrow$ " " 2

$\vdots$

$C_{\frac{n-1}{2}+1} = \{ a^{\frac{n-1}{2}}, a^{\frac{1-n}{2}} \} \longrightarrow$ " " 2

$a^{\frac{1-n}{2}} = a^{n+\left(\frac{1-n}{2}\right)} = a^{\frac{n+1}{2}}$

$C_{\frac{n-1}{2}+2} = \{a^k b : k = 0, 1, \dots, n-1\} \longleftarrow$ CARDINALITY n.

<u>Check</u>: $1 + \frac{n-1}{2} \cdot 2 + n = 2n$ ✓

Notice that $Z(D_{2n}) = \{1\}$ when n is odd, indeed the only conjugacy class of cardinality one is the one of 1.

$\star$ n even $\star$

$C_1 = \{1\} \longrightarrow$ cardinality 1

$C_2 = \{a, a^{-1}\} \longrightarrow$ " " 2

$C_3 = \{a^2, a^{-2}\} \longrightarrow$ " " 2

$\vdots$

$C_{\frac{n}{2}} = \{a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}\} \longrightarrow$ " " 2

$C_{\frac{n}{2}+1} = \{a^{n/2}\} \longrightarrow$ " " $\boxed{1}$

$C_{\frac{n}{2}+2} = \{ a^i b : i \underline{\underline{\text{even}}}, i = 0 .. n-1 \}$ ← cardinality $\frac{n}{2}$

$C_{\frac{n}{2}+3} = \{ a^i b : i \underline{\underline{\text{odd}}}, i = 0 .. n-1 \}$ ← // // //

<u>check</u>: $1 + 1 + \left(\frac{n}{2} - 1\right) \cdot 2 + \frac{n}{2} \cdot 2 = 2n$ ✓.

Notice that $Z(D_{2n}) = \{1, a^{n/2}\}$ when n is even.

$\boxed{\text{EXAMPLES}}$

$D_8 \rightsquigarrow$ conjugacy classes :

1

$\{a, a^{-1} = a^3\}$

$\{a^2\}$

$\{b, a^2 b\}$

$\{ab, a^3 b\}$

$D_{10} \rightsquigarrow$ conjugacy classes

1

$\{a, a^4 = a^{-1}\}$

$\{a^2, a^{-2} = a^3\}$

$\{a^k b : k = 0, 1, 2, 3, 4\}$.

CONJUGACY CLASSES IN S_n

Theor - the conjugacy class of an element $\sigma \in S_n$ consists of all permutation in S_n that have the same cyclic structure as σ.

Example : in S_3
- 1
- $\{(12), (13), (23)\}$
- $\{(123), (132)\}$.

in S_4 :
- 1^{S_4} ← cardinality 1
- $(12)^{S_4}$ ← cardinality $\frac{4\cdot 3}{2} = 6$
- $(123)^{S_4}$ ← cardinality $\frac{4\cdot 3\cdot 2}{3} = 8$
- $(1234)^{S_4}$ ← cardinality $\frac{4\cdot 3\cdot 2\cdot 1}{4} = 6$
- $(12)(34)^{S_4}$ ← cardinality $\frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 2\cdot 2!} = 3$.

: Computing the size of a conjugacy class *

$$|x^G| = \frac{|G|}{|Z_G(x)|} = \frac{4!}{\text{centralizer of } x \text{ in } G} = \frac{4!}{\{g \in S_4 : g^{-1}xg = x\}}.$$

$G = S_4$

Suppose $x = (i_1 \ldots i_k)$ and $g \in Z_{S_4}(x)$, then
$g^{-1}xg = (g(i_1) \ldots g(i_k)) = (i_1 \ldots i_k)$
so $\{g(i_1) \ldots, g(i_k)\} = \{i_1 \ldots i_k\}$. The ordering must be respected, you are only allowed to permute the indices cyclically.

Suppose $x = (i_1 \cdots i_k)(j_1 \cdots j_s) \cdots (r_1 \cdots r_t)$
and assume that every cycle has a different length. For each cycle, the previous argument applies.
then we obtain $k \cdot s \cdots t$ elements in the stabilizer.

Suppose $x = \underbrace{(i_1 i_2 \cdots i_k)(j_1 \cdots j_k) \cdots (l_1 \cdots l_k)}_{d_k}$ is a product

of d_k disjoint cycles (all) of length k.
then you are also allowed permutations that switch the ordered sets $\{i_1 \cdots i_k\}$, $\{j_1 \cdots j_k\}$, $\cdots$ $\{l_1 \cdots l_k\}$.
So you get $d_k!$ extra permutations in the stabilizer.

<u>conclusion</u>

If σ ~~contains~~ decomposes as a product of disjoint cycles of length $m_1 \cdots m_r$
then
$$|\text{Stabilizer of } \sigma| = m_1 m_2 \cdots m_r \, d_1! \, d_2! \cdots d_n!$$

where $d_j = \#$ of cycles of length j.

<u>example</u>

$$x = (1\,2)(3\,4)(5\,6\,7)(8\,9\,10)(11\;12\;13)(14\;15\;16) \in S_{16}$$

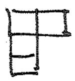
$$|Z_G(x)| = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \, (2!) \cdot (4!)$$

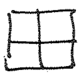
* example *

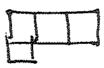
$|\text{conj. class}|$

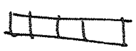
$S^4 \rightsquigarrow$

 $1+1+1+1 \rightsquigarrow |Z(x)| = 1\cdot 1\cdot 1\cdot 1\cdot 4! = 4! \rightsquigarrow 1$

 $2+1+1 \rightsquigarrow |Z(x)| = 2\cdot 1\cdot 1\cdot 2! = 4 \rightsquigarrow \frac{4!}{4} = 3! = 6$

 $2\cdot 2\cdot 2! \rightsquigarrow |Z(x)| = 8 \rightsquigarrow \frac{4!}{8} = \frac{4\cdot 3\cdot 2}{8} = 3$

 $3\cdot 1 \rightsquigarrow |Z(x)| = 3 \rightsquigarrow \frac{4!}{3} = \frac{4\cdot 3\cdot 2}{3} = 8$

 $4 \rightsquigarrow |Z(x)| = 4 \rightsquigarrow \frac{4!}{4} = 3! = 6$

CONJUGACY CLASSES IN A_n $(n>1)$

Let $A_n = \{\text{even permutations}\} \subseteq S_n$.
For each $x \in A_n$, let

- $x^{A_n} = \{g^{-1}xg : g \in A_n\}$ be the conj. class of x in A_n
- $x^{S_n} = \{g^{-1}xg : g \in S_n\}$ be the conj. class in S_n.

Then $x^{A_n} \subseteq x^{S_n}$.
Equality might not hold: for instance
$$(123)^{S_3} = \{(123); (132)\}$$
but
$$(123)^{A_3} = \{(123)\}.$$
The next proposition determines when x^{A_n} and x^{S_n} are equal, and what happens if equality fails...

<u>Proposition</u> - ① If x commutes with some <u>odd</u> permutation $\sigma \in S_n$, then $x^{A_n} = x^{S_n}$.

② If x does not commute with any <u>odd</u> permutation in S_n, then x^{S_n} splits into two A_n-conjugacy classes of equal size:
$$x^{S_n} = x^{A_n} \uplus \left[(12)^{-1}x(12)\right]^{A_n}.$$

<u>Proof</u> - ① Suppose $x\sigma = \sigma x$, for some $\sigma \in S_n$ ODD.
Let $y = g^{-1}xg \in x^{S_n}$. If g is even, then $y \in x^{A_n}$ and there's nothing to prove. If g is odd, we can write:

$$y = g^{-1}xg = g^{-1}(\underbrace{\sigma^{-1}x\sigma}_{})g = (g^{-1}\sigma^{-1})\,x\,(\sigma g) =$$

$= (\sigma g)^{-1} \; x \; (\sigma g) \in x^{A_n}$ because $\underset{\text{odd} \cdot \text{odd}}{\sigma g}$ is even. ✓

③ Suppose that x does not commute with any odd permutation in S_n. Then

$$Z_{A_n}(x) = \{ g \in A_n : gx = xg \} = Z_{S_n}(x) \left(= \{ g \in S_n : gx = xg \} \right).$$

Notice that

- $\displaystyle |Z_{A_n}(x)| = \frac{|A_n|}{|x^{A_n}|} = \frac{1}{2} \frac{|S_n|}{|x^{A_n}|}$

and

- $\displaystyle |Z_{S_n}(x)| = \frac{|S_n|}{|x^{S_n}|}$

so we can write:

$$|x^{S_n}| = \frac{|Z_{S_n}(x)|}{|S_n|} = \frac{|Z_{A_n}(x)|}{|S_n|} = 2 \, |x^{A_n}|.$$

Also, notice that every odd permutation can be written as:

$$g = (12)\tilde{g} \quad \text{with } \tilde{g} \text{ even}$$

(just choose $\tilde{g} = \underset{\text{odd}}{(12)} \cdot \underset{\text{odd}}{g}$).

So if $y = g^{-1} x g$, then we can write:

$$y = [(12)\tilde{g}]^{-1} x \, [(12)\tilde{g}] = \tilde{g}^{-1} (12)^{-1} x \, (12) \, \tilde{g} \in [(12)^{-1} x \, (12)]^{A_n}.$$

And if g is even, $y = g^{-1} x g \in x^{A_n}$.

$\Rightarrow$ $x^{S_n} = x^{A_n} \uplus [(12)^{-1} x \, (12)]^{A_n}$, as claimed. □

Example - Conjugacy classes in A_4.

Cyclic structures in S_4:

1	EVEN
(12)	ODD
(123)	EVEN
$(12)(34)$	EVEN
(1234)	ODD.

We are interested in x^{A_4}, for $x = (123)$ and $x = (12)(34)$.

Because $(12)\cdot(12)(34) = (34) = (12)(34)\cdot(12)$, $x = (12)(34)$ commutes with the odd permutation (12). Hence $x^{A_4} = x^{S_4}$.

Let's look at $x = (123)$. then x only commutes with $1, x$ and x^2. Indeed for all $g \in S_3$:

$$g^{-1}(123)g = (g(1)\ g(2)\ g(3))$$

and $(g(1)\ g(2)\ g(3)) = (123)$ if and only if one of the 3 possibilities occur:

(i) $g(1) = 1$; $g(2) = 2$; $g(3) = 3$

(ii) $g(1) = 2$; $g(2) = 3$; $g(3) = 1$

(iii) $g(1) = 3$; $g(2) = 1$; $g(3) = 2$.

So x^{S_4} splits into two conjugacy classes: x^{A_4} and $(12)x(12)^{A_4}$.

Notice that

$(12)x(12) = (12)(123)(12) = (132)$.

The conjugacy class of (123) in S_4 has cardinality $\frac{4\cdot3\cdot2}{3} = 8$.

So both x^{A_4} and $[(12)x(12)]^{A_4}$ have cardinality 4.