CHARACTERS

[Le]t G be a finite group. Let ρ be a representation of G in a finite di[m]ensional complex vector space V.
[Fo]r each basis $\mathcal{B}$ of V, let $[\rho(g)]_{\mathcal{B}}$ be the matrix associated to $\rho(g)$ [w].r.t. $\mathcal{B}$.

[D]ef.- The character of (ρ, V) is the function
$$\chi_\rho : G \mapsto \mathbb{C},\ g \mapsto \text{trace}([\rho(g)]_{\mathcal{B}}).$$

[R]emark 1.- The definition of χ is independent of the choice of $\mathcal{B}$. Indeed, the trace of $[\rho(g)]_{\mathcal{B}}$ is the sum of the eigenvalues of $\rho(g)$. Alternatevely, if $\mathcal{B}'$ is another basis, then $[\rho(g)]_{\mathcal{B}'} = C\,[\rho(g)]_{\mathcal{B}}\,C^{-1}$ for some $C \in GL(n, \mathbb{C})$, and the trace is invariant under conjugation.

[R]emark 2. When V is one-dimensional, $\chi(g) \neq 0$ $\forall g \in G$ and $\chi : G \to \mathbb{C}^*$ is a group homomorphism. When $\dim V > 1$, a non-zero character is not a group homomorphism.

PROPERTIES OF CHARACTERS

$\boxed{1}$ Isomorphic representations have the same character:
if $\rho_1 \cong \rho_2$ then $\chi_{\rho_1} = \chi_{\rho_2}$.

$\boxed{2}$ Every character is a class function, i.e. is invariant on conjugacy classes: $\chi_\rho(g) = \chi_\rho(s g s^{-1})$ $\forall s, g \in G$

$\boxed{3}$ If g has order m in G, then $\chi_\rho(g)$ is a sum of m^{th}-roots of unity. In particular, if $m = 2$, then $\chi_\rho(g)$ is an integer.

4 Let $n = \dim V$ be the degree of the representation.
then
$$|\chi(g)| \leq n \qquad \forall g \in G$$
and
$$|\chi(g)| = n \quad \text{if and only if} \quad \rho(g) = c\mathbb{1}_V \text{ for some } c \in \mathbb{C}^*.$$

5 Let $n = \dim V$ be the degree of the representation.
Then
$$\chi(1) = n$$
and
$$\chi(g) = \chi(1) \quad \text{if and only if} \quad g \in \ker \rho \quad (\text{i.e. } \rho(g) = \mathbb{1}_V).$$

6 For all $g \in G$, $\chi(g^{-1}) = \overline{\chi(g)}$.

PROOFS

1 If $\rho_1 \cong \rho_2$, then there exists an invertible linear transformation $T: V_1 \to V_2$ such that $\rho_2(g) = T \circ \rho_1(g) \circ T^{-1}$.
Because the trace is invariant under conjugation, $\text{trace}(\rho_2(g)) = \text{trace}(T \circ \rho_1(g) \circ T^{-1}) = \text{trace}(\rho_1(g)) \quad \forall g \in G$.

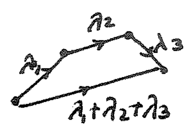
2 If g and s are in G, we have: $\chi(sgs^{-1}) = \text{trace}(\rho(sgs^{-1})) = \text{trace}(\rho(s)\rho(g)\rho(s)^{-1}) = \text{trace}(\rho(g)) = \chi(g)$.
Again, we have used the invariance of the trace under conjugation.

3 If $g^m = 1$, then $\rho(g)^m = \mathbb{1}_V$ so the eigenvalues of $\rho(g)$ are n^{th} root of unity. The claim follows from the fact that the trace is the sum of the eigenvalues.
Assume, in particular, that $m = 2$. Then the eigenvalues of every $\rho(g)$ can only be ± 1. It follows that the trace is an integer. We can say more: $\chi(g) \equiv \chi(1) \mod 2$.
Indeed, if $r = \#\{\text{eigenvalues of } \rho(g) \text{ that are } = +1\}$ and
$s = \#\{\text{eigenvalues of } \rho(g) \text{ that are } = -1\}$, then
$\chi(1) = \dim V = r + s$, and $\chi(g) = \text{trace}(\rho(g)) = r - s$.
So $\chi(g) = (r+s) - 2s = \chi(1) - 2s \underset{\text{mod } 2}{\equiv} \chi(1)$.

4 We can write: $|\chi(g)| = |\text{trace of } \rho(g)| = \left|\sum_{i=1}^{n} \lambda_i\right| \leq$

let $\lambda_1 .. \lambda$ be the eigenvalues of $\rho(g)$

$\leq \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} 1 = n$
(every λ_i is an m-th root of unity, so $|\lambda_i| = 1$).



The equality $\left|\sum_{i=1}^{n} \lambda_i\right| = \sum_{i=1}^{n} |\lambda_i|$ only holds if the n complex numbers $\lambda_1, \lambda_2, .., \lambda_n$ have the same argument ($\Leftrightarrow$ they are collinear). If this is the case, then $\lambda_1 = \lambda_2 = .. = \lambda_n$ (because they also have the same modulus, 1).
Set $c = \lambda_1 = \lambda_2 = .. = \lambda_n$. Then $\rho(g)$ is a diagonalizable operator with n eigenvalues equal to c $\Rightarrow$ $\rho(g)$ is a scalar operator $\Rightarrow$ $\rho(g) = c\,\mathbb{1}_V$.
We have shown that if $|\chi(g)| = \overset{\dim V}{n}$, then $\rho(g)$ is a multiple of $\mathbb{1}_V$.
Viceversa, assume that $\rho(g) = c\,\mathbb{1}_V$. Then $\text{Trace}(\rho(g)) = c \dim V$.
Because c must be an m^{th} root of unity ($m = \text{order } g$), $|\text{trace}(\rho(g))| = |c| \cdot |\dim V| = \dim V = n$. So $|\chi(g)| = n$ iff $\rho(g) = c\,\mathbb{1}_V$, for some $c \in \mathbb{C}^*$.

$\boxed{5}$ If $g=1$, Then $\rho(g)=\mathbb{1}_V$ and $\chi(g) = \text{trace}(\mathbb{1}_V) = \dim V = n$.
Now assume that g is any element of G s.T. $\chi(g) = \chi(1) = n$.
Of course, $|\chi(g)| = n$, so $\rho(g)$ must be a multiple of the identity: $\rho(g) = c\,\mathbb{1}_V$, $c \in \mathbb{C}^*$.
Then $\text{Trace}(\rho(g)) = c \dim V = c\,\chi(1)$. This forces $c=1$ and proves the claim.

$\boxed{6}$ Choose a basis of V s.T. (every) $\rho(g)$ is a unitary operator. Then
$$\chi(g^{-1}) = \text{Trace}([\rho(g^{-1})]_{\mathcal{B}}) = \text{Trace}([\rho(g)]_{\mathcal{B}}^{-1}) = \text{trace}(\overline{[\rho(g)]_{\mathcal{B}}}^{T}) =$$
$$= \text{trace}(\overline{[\rho(g)]_{\mathcal{B}}}) = \overline{\text{trace}([\rho(g)]_{\mathcal{B}})} = \overline{\chi(g)} \,. \quad \square$$

Corollary If g is conjugate To g^{-1}, Then $\chi(g)$ is real.

OTHER REMARKS

- Notice That every element g of the symmetric group has this property. Indeed g^{-1} has The same cyclic structure as g. It follows That every character of The symmetric group is real.

- In order To decide whether a representation is faithful we just need to look at its character, indeed
$$\ker \rho = \{ g : \rho(g) = \mathbb{1}_V \} = \{ g \in G : \chi(g) = \chi(1) \}.$$
$\Rightarrow$ the representation is faithful if and only if "$\chi(g) = \chi(1) \Rightarrow g=1$".

SOME EXAMPLES

Let $X = \{x_1, \dots, x_m\}$ be a finite set, and let $(g, x_i) \mapsto g \cdot x_i$ be an action of G on X.
Consider the permutation representation associated to this action: $V = \bigoplus_{i=1..m} \mathbb{C} e_{x_i}$, and $\rho(g)\, e_{x_i} = e_{g \cdot x_i}$ $\forall i = 1..m$.

Let's compute the character.

$$\chi(g) = \text{trace} \underbrace{\left([\rho(g)]_{\mathcal{B} = \{e_{x_1}, \dots, e_{x_m}\}} \right)}_{\text{call it } A} = \sum_{i=1}^{m} a_{ii}.$$

We notice that:

- if g fixes x_i, then $\rho(g) e_{x_i} = e_{g \cdot x_i} = e_{x_i}$ and $a_{ii} = 1$.
- if g does not fixes x_i, then $\rho(g) e_{x_i} = e_{x_j}$ $(j \neq i)$ and $a_{ii} = 0$.

Therefore:

$$\chi(g) = \#\{x \in X : g \cdot x = x\} = \#\text{ points in } X \text{ fixed by } g.$$

$$\boxed{\chi(g) = \#\text{ points in } X \text{ fixed by } g}$$

As a special case, let's consider the regular representation of G and the permutation representation of S_n.

$\star$ regular representation $\star$

G acts on $X = G$ by left multiplication. If $g \neq 1$ there are no fixed points; if $g = 1$ everything is fixed. Hence:

$$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1. \end{cases}$$

* permutation representation of S_n *

S_n acts on $X = \{1, 2, \dots, n\}$.
Let $\sigma \in S_n$. Write down a minimal decomposition of σ as a product of disjoint cycles, and let $\{i_1, i_2, \dots, i_k\}$ be the indeces involved. then σ fixes every index in

$$\{1, 2, \dots, n\} - \{i_1, i_2, \dots, i_k\}.$$

For instance, if $n = 4$, Then

- (1234) has no fixed points
- (123) has 1 fixed point $(=\{4\})$
- $(12)(34)$ has no fixed points
- (12) has 2 fixed points $(=\{3, 4\})$
- 1 has 4 fixed points $(=\{1, 2, 3, 4\})$.

Hence $\chi(\sigma) = \begin{cases} 4 & \text{if } \sigma = 1 \\ 2 & \text{if } \sigma \text{ is a Transposition} \\ 1 & \text{if } \sigma \text{ is a 3-cycle} \\ 0 & \text{if } \sigma \text{ is a 4-cycle or The product of Two disjoint Transpositions.} \end{cases}$

BUILDING NEW CHARACTERS

For all representations (ρ_i, V_i) of G :

1. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$

2. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

3. $\chi_{V_1^*} = \overline{\chi_{V_1}}$

4. $\chi_{Hom(V_1, V_2)} = \overline{\chi_{V_1}} \cdot \chi_{V_2}$

5. $\chi_{Alt^2(V_1)}(g) = \frac{1}{2}\left[\chi_{V_1}(g)^2 - \chi_{V_1}(g^2)\right]$, $\forall g \in G$

6. $\chi_{Sym^2(V_1)}(g) = \frac{1}{2}\left[\chi_{V_1}(g)^2 + \chi_{V_1}(g^2)\right]$, $\forall g \in G$

PROOFS

1. Let $\mathcal{B} = \{ \underbrace{v_1 \dots v_n}_{\mathcal{B}_1} \ \underbrace{w_1 \dots w_m}_{\mathcal{B}_2} \}$. Then

$$\chi_{V_1 \oplus V_2}(s) = \text{trace}\left((\rho_1 \oplus \rho_2)(s)\right)_{\mathcal{B}} = \text{trace}\begin{pmatrix} [\rho_1(s)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_2(s)]_{\mathcal{B}} \end{pmatrix} =$$

$$= \chi_{V_1}(s) + \chi_{V_2}(s) \ \checkmark$$

2. Let $\mathcal{B} = \{ \underbrace{v_1 \otimes w_1, v_1 \otimes w_2, \dots, v_1 \otimes w_m}_{v_1 \otimes \mathcal{B}_2}, v_2 \otimes w_1, \dots, v_n \otimes w_m \}$.

Also set $A = [\rho_1(s)]_{\mathcal{B}_1}$ and $B = [\rho_2(s)]_{\mathcal{B}_2}$. Then

$$\chi_{V_1 \otimes V_2}(s) = \text{trace}(A \otimes B) = \text{trace} \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} =$$

$$= \sum_{i=1}^{n} [a_{ii} \text{ trace } B] = [\text{trace } A][\text{trace } B] = \chi_{V_1}(s) \cdot \chi_{V_2}(s). \checkmark$$

$\boxed{3}$ Let $\mathcal{B}_1^* = \{v_1^*, \dots, v_n^*\}$. Then for all $s \in G$:

$$\chi_{V_1^*}(g) = \text{trace}\,[\rho_1^*(s)]_{\mathcal{B}^*} = \text{trace}([\rho_1(s^{-1})]_{\mathcal{B}}^T) = \text{trace}([\rho_1(s^{-1})]_{\mathcal{B}}) =$$

$$= \text{trace}([\rho_1(s)]_{\mathcal{B}}^{-1}) = \text{trace}\left(\overline{[\rho_1(s)]_{\mathcal{B}}}^{\,T}\right) = \overline{\text{trace}\,[\rho_1(s)]_{\mathcal{B}}} = \overline{\chi_{V_1}(s)}.$$

you can assume that $[\rho_1(s)]_{\mathcal{B}}$ is unitary

$\boxed{4}$ Recall that $\text{Hom}(V, W) \cong V^* \otimes W$.
the claim follows from $\boxed{2}$ and $\boxed{3}$.

$\boxed{5}$ Let $\lambda_1 \dots \lambda_n$ be the eigenvalues of $\rho_1(s)$ on V.
then $\{\lambda_i \lambda_j : i < j\}$ are the eigenvalues of $\rho(s)$ on $V_1 \wedge V_1$
We get:

$$\chi_{\text{Alt}^2(V)}(g) = \sum_{\substack{i=1\dots n \\ i<j}} \lambda_i \lambda_j = \left[\frac{(\sum_{i=1}^{n} \lambda_i)^2 - \sum_{j=1}^{n} (\lambda_j^2)}{2} \right] =$$

$$\sum_{i<j} \lambda_i \lambda_j = \frac{(\sum \lambda_i)^2 - \lambda_1^2 - \lambda_2^2 \dots - \lambda_n^2}{2}$$

$$\frac{1}{2}[\text{trace}(\rho(g))]^2 - \frac{1}{2}\text{trace}(\rho(s^2)) = \frac{1}{2}[\chi(s)^2 - \chi_1(s^2)].$$

to find $\chi_{Sym^2(V)}(s)$ we use the fact that

$$V \otimes V = Sym^2(V) \oplus Alt^2(V).$$

By property $\boxed{II}$, $\chi_{Sym^2(V)}(s) = \chi_{V\otimes V}(s) - \chi_{Alt^2(V)}(s) =$

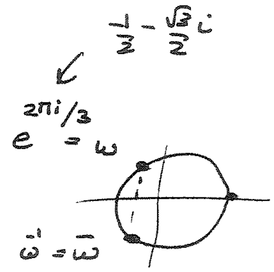
$$= [\chi_V(s)]^2 - \frac{1}{2}[\chi_V(s)^2 - \chi_V(s^2)] =$$

$$= \frac{1}{2}[\chi_V(s)]^2 + \frac{1}{2}\chi_V(s^2), \quad \forall s \in G. \checkmark$$

EXAMPLES

$\boxed{S_3}$

	1	(23) (13) (12)	(132) (123)
trivial U	1	1	1
sign U'	1	-1	1
standard V	2	0	$-1 = \omega + \bar{\omega}$



$(U')^*$	1	-1	1
$V \otimes V$	4	0	1
$U \oplus V$	3	1	-1
$Sym^2(V)$	3	1	0
$Alt^2(V)$	1	-1	1

} the sum equals $V \otimes V$ ✓