Lecture 4 : Complete Reducibility

4.1 Complete Reducibility of Representations
4.2 Decomposition of the regular representation
4.3 Representations of the symmetric group S_3.

4.1 - Complete Reducibility of Representations

Let G be a finite group, and let ρ be a representation of G in a finite dimensional complex vector space.

▸ **Definition** - The representation ρ of G is called "completely reducible" if it can be expressed as a direct sum of irreducible representations.

▸ **Theorem** - Every representation of a finite group in a finite dimensional complex vector space is completely reducible.

▹ **proof** - By induction on the degree n of the representation.

Base of induction : if $n=1$, the representation (ρ, V) is 1-dimensional, hence irreducible, and there's nothing to prove.

Inductive hypothesis: Assume that every representation of degree less than K is completely reducible.

Inductive step: Let (ρ, V) be a representation of degree K. If ρ is irreducible there is nothing to prove. Otherwise, let $W \subseteq V$ be a proper non-trivial G-invariant subspac of V. By Mascke's Theorem, W has a G-invariant complement, so we can write

$$V = W \oplus W_0$$

with W and W_0 both G-stable.
The restriction of ρ to W and W_0 gives rise to representations of G of dimension less than k, so the inductive hypothesis hold: W and W_0 are a direct sum of irreducible representations. The same is true for $V = W \oplus W_0$. ∎

CORORLLARY ⇝ see the claim at the end of the lecture!

By the previous theorem, every representation of G is a direct sum of irreducible representation.
A natural question arises: is this decomposition unique?
The following remarks are intended to answer this question.

▷ Remark 1. The decomposition of V as a direct sum of irreducible subrepresentations is, in general, not unique.

Here's is a counterexample: let $V = \mathbb{C}^n$ and let
$\rho : G \longrightarrow GL(V) \cong GL(\mathbb{C}, n),\ g \mapsto I_n$
be the trivial representation of G of degree n. Assume $n > 1$.
For every choice of a basis $\mathcal{B} = \{\underline{x}_i\}_{i=1..n}$ of $\mathbb{C}^n$,
we can write:

$$\mathbb{C}^n = \mathbb{C}\underline{x}_1 \oplus \mathbb{C}\underline{x}_2 \oplus \dots \oplus \mathbb{C}\underline{x}_n$$

and this is a decomposition of the representation $(\rho, \mathbb{C}^n)$ as a direct sum of irreducible representations. [For all i, $\mathbb{C}\underline{x}_i$ is a G-stable subspace of $\mathbb{C}^n$, and the restriction of ρ to $\mathbb{C}\underline{x}_i$ is isomorphic to the trivial repr. of G of degree 1.] ∎

This example proves the non-uniqueness of the decomposition. In general, though, we are only interested in the equivalence class of the irreducible representations that appear in the decomposition. And, in this sense, the decomposition is indeed unique.

Remark 2 Let $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ be a decomposition of V as a direct sum of irreducible representations. Even if this decomposition is not unique, the number of W_i's that are isomorphic to a given irreducible representation of G does NOT depend on the choice of the decomposition.

Proof - Fix an irreducible representation U of G. For every $i = 1 \dots k$, we have:

$$\dim_{\mathbb{C}} \left[\mathrm{Hom}_G(U, W_i) \right] = \begin{cases} 1 & \text{if } U \cong W_i \\ 0 & \text{if } U \not\cong W_i. \end{cases}$$

It follows that

$$\dim_{\mathbb{C}} \left[\mathrm{Hom}_G(U, V) \right] = \# \text{ of } W_i\text{'s isomorphic to } U.$$

Because the space $\mathrm{Hom}_G(U,V)$ is independent of the choice of the decomposition of V in irreducible representations, the claim follows. $\boxed{}$

Corollary - Up to isomorphism, every representation of G admits a "unique" decomposition in the form

$$V \cong W_1^{\oplus a_1} \oplus W_2^{\oplus a_2} \oplus \dots \oplus W_k^{\oplus a_k}$$

where the W_i's are distinct irreducible representations of G -

This decomposition is unique in the sense that if

$$V \cong V_1^{\oplus b_1} \oplus V_2^{\oplus b_2} \oplus \dots \oplus V_\ell^{\oplus b_\ell}$$

Then $k = \ell$ and (after possibly permuting the V_i's) we have

$$\boxed{W_i \cong V_i} \text{ and } \boxed{a_i = b_i}, \text{ for all } i = 1..k.$$

▸ <u>Definition</u> - Let $V \cong W_1^{\oplus a_1} \oplus \dots \oplus W_k^{\oplus a_k}$ be the (unique) decomposition of V as a direct sum of distinct irreducible representations of G.

We call $\boxed{W_i^{\oplus a_i}}$ the isotypic component of W_i in V, and $\boxed{a_i}$ the multiplicity of W_i in V.

[By Schur's lemma, $a_i = \dim_{\mathbb{C}} [\mathrm{Hom}_G(W_i, V)]$].

4.2 - <u>Decomposition of the regular representation</u>

Let G be a finite group. Let (ρ, V) be the regular representation of G, and let $V = U_1 \oplus U_2 \oplus \dots \oplus U_r$ be <u>any</u> decomposition of V as a direct sum of irreducible subrepresentations.

▸ <u>Theorem</u> - If (Γ, W) is <u>any</u> irreducible representation of G, then the number of irreducible summands U_i of the regular representations that are isomorphic to W is equal to the dimension of W.

Equivalently, $\dim_{\mathbb{C}} [\mathrm{Hom}_G(W, \underset{\text{reg. repr.}}{\underline{V}})] = \dim_{\mathbb{C}} W$.

▹ <u>Proof</u> - We construct a basis of $\mathrm{Hom}_G(W, V)$ consisting of exactly $n = \dim W$ G-homomorphisms.

Let $n = \dim W$, and let $w_1, w_2, \dots, w_n$ be a basis of W.

- For all $j = 1 \dots n$, consider the linear transformation

$$\begin{aligned} T_j : V &\longrightarrow W \\ \underline{e}_g &\longmapsto \Gamma(g) w_j \end{aligned}$$

(T_j is defined on a basis of V, and extended by linearity to V).

To prove that T_j is a G-homomorphism, we show that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho(s)} & V \\ T_j \downarrow & & \downarrow T_j \\ W & \xrightarrow{\Gamma(s)} & V \end{array}$$

is commutative, for all $s \in G$.

- $\bullet$ $T_j \rho(s) \underline{e}_g = T_j \underline{e}_{sg} = \Gamma(sg) w_j = \Gamma(s)\Gamma(g) w_j = \Gamma(s) T_j(\underline{e}_g)$

$(\forall g \in G) \Rightarrow T_j \rho(s) = \Gamma(s) T_j$. ✓

Next, we show that the G-homomorphisms $T_1, T_2, \dots, T_n$ are linear independent.

Suppose that $\sum_{j=0}^{n} c_j T_j = 0$ (← The 0 homomorphism from V to W

Apply $\sum_{j=0}^{n} c_j T_j$ to $\underline{e}_1$, 1 being the identity of G:

$$\Rightarrow 0 = \sum_{j=0}^{n} c_j T_j(\underline{e}_1) = \sum_{j=0}^{n} c_j \underbrace{\Gamma(1)}_{\Gamma(1) = \mathbb{1}_W} w_j = \sum_{j=0}^{n} c_j w_j.$$

- Because the basis vectors $w_1, \dots, w_n$ are l.i., we obtain that $c_1 = \dots = c_n = 0$

So $T_1, \dots, T_n$ are l.i. G-homomorphism of V in W.

Finally, we show that $T_1, \dots, T_n$ span the space $\mathrm{Hom}_G(V, W)$.

Let $S: V \to W$ be any G-homomorphism.
For all $g \in G$ we can write:

$$S(\underline{e}_g) = S(\rho(g)\underline{e}_1) = r(g)\underbrace{S(\underline{e}_1)}_{\text{some vector in } W:\ \text{say } S(\underline{e}_1) = \sum_{j=1}^{n} b_j w_j} =$$

$$= \sum_{j=1}^{n} b_j\, r(g) w_j =$$

$$= \sum_{j=1}^{n} b_j\, T_j(\underline{e}_g)$$

The constants $b_1, \dots, b_n$ are clearly independent of g

$$\Rightarrow \quad S = \sum_{j=1}^{n} b_j T_j.$$

This proves that $T_1, \dots, T_n$ are generators for $\mathrm{Hom}_G(V, W)$, hence they are a basis. ■

▶ Corollary - If $V_1, V_2, \dots, V_k$ form a complete set of not-isomorphic irreducible representations of G, then the decomposition of the regular representation in isotypic component is:

$$V \cong (V_1)^{\oplus \dim V_1} \oplus (V_2)^{\oplus \dim V_2} \oplus \dots \oplus (V_k)^{\oplus \dim V_k}.$$

In other words, (up to isomorphism) each irreducible representation of V appears in the decomposition of V in irreducible summands with a multiplicity equal to its dimension.

▶ Corollary - Let $V_1, V_2, \dots, V_k$ be a complete set of not-isomorphic irreducible representations of G. Then

$$\boxed{|G| = \sum_{j=1}^{k} (\dim V_j)^2}.$$

▶ An application - If G is a group of order 6, then the possible degrees of the not-isomorphic irreducible representations are :

case 1 → $\boxed{1, 1, 1, 1, 1, 1}$ or $\boxed{2, 1, 1}$ ← case 2.

Indeed the only two ways to decompose 6 as a sum of squares is $6 = 1+1+1+1+1+1$ and $6 = 4+1+1$.

If G is an abelian group of order 6, then all the representations of G are 1-dimensional [so case (1) certainly occurs]. As an example, we consider $G = C_6$ (the cyclic group of order 6). The 6 not-isomorphic irreducible (1-dimensional) representations of C_6 are :

$$\lambda_j : C_6 \mapsto \mathbb{C}^*, \quad a \longmapsto e^{\frac{2\pi i}{6} \cdot j} \qquad j = 0, 1, 2, 3, 4, 5 .$$

We have denoted by "a" a generator of C_6.

We will show in the next section that if $G = S_3$ (the symmetric group on 3 letters, also of order 6), then G has 2 inequivalent irreducible representations of degree 1 and 1 irreducible representation of degree 2. [So case (2) also occurs].

4.3 Representations of the symmetric group S_3

Let $G = S_3$ be the symmetric group on 3 letters.
In this section, we classify the irreducible representations of S_3 and we discuss how to decompose any representation of S_3 as a direct sum of isotypic components.

By the previous remarks, we expect at least two irreducible 1-dimensional representations (i.e. two distinct homomorphisms $S_3 \to \mathbb{C}^*$).

It's easy to "guess" two group homomorphisms from S_3 to $\mathbb{C}^*$:

- $\boxed{\rho_1 : S_3 \to \mathbb{C}^*, \sigma \mapsto 1} \Longrightarrow$ trivial representation, denoted by $\boxed{U}$
- $\boxed{\rho_2 : S_3 \to \mathbb{C}^*, \sigma \mapsto sg(\sigma)} \Longrightarrow$ sign representation, denoted by $\boxed{U'}$.

How can we construct the other representations?
S_3 has a natural action on the set $\{1, 2, 3\}$. Let's look at the permutation representation associated to this action

vector space: $\mathbb{C}\underline{e}_1 \oplus \mathbb{C}\underline{e}_2 \oplus \mathbb{C}\underline{e}_3 \cong \mathbb{C}^3$

action of G: $\sigma \cdot \underline{e}_i = \underline{e}_{\sigma(i)}$.

We can of course identify V with $\mathbb{C}^3$, then $\sigma \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ x_{\sigma^{-1}(3)} \end{bmatrix}$.

the group $G=S_3$ cannot have any irreducible representation of degree 3, indeed $3^2 > |G| = 6$.

So the permutation representation of G in $\mathbb{C}^3$ must be reducible. Let's look for the G-invariant subspaces:

The line $\left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix} : a \in \mathbb{C} \right\} = \operatorname{Span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the unique 1-dimensional G-invariant subspace. The restriction of the permutation representation to this line is isomorphic to the trivial representation.

The plane $\{ \underline{x} \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0 \} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$ is the unique 2-dimensional G-invariant subspace.

The restriction of the permutation representation to this plane must be irreducible (because this plane is not the direct sum of two copies of the above line ...).

Let us denote by $\boxed{V}$ this 2-dimensional irreducible representation of G. We call $\boxed{V}$ the <u>standard representation of $G = S_3$</u>.

▶ <u>Proposition</u> - Up to isomorphism, there are only 3 irreducible representations of S_3:

- U = the trivial representation (degree 1)
- U' = the sign representation (degree 1)
- V = the standard representation (degree 2)

▶ <u>Corollary</u> - The regular representation of S_3 is isomorphic to

$$U \oplus U' \oplus [V]^{\oplus 2}.$$

Next, we discuss a method to decompose any (reducible) representation of S_3 as a direct sum of isotypic components.

<u>Remark</u>: S_3 is generated by $\sigma=(12)$ and $\tau=(123)$. Let's look at the eigenvalues of σ and τ in the irreducible representations of S_3:

	eigenvalues of $\sigma=(12)$	eigenvalues of $\tau=(123)$
U = trivial repr.	$+1$	$+1$
U' = sign repr.	-1	$+1$
V = standard repr.	$+1, -1$	$\omega,\ \omega^{-1}=\bar{\omega}$

where $\omega = e^{2\pi i/3} = \frac{1+\sqrt{3}\,i}{2}$.

Let $W = U^{\oplus a} \oplus (U')^{\oplus b} \oplus (V)^{\oplus c}$ be any representation of S_3. Then

$$\begin{array}{l}\#\text{ of occurrences of } \omega = e^{2\pi i/3} \\ \text{as an eigenvalue of } (123)\end{array} = c$$

$$\begin{array}{l}\#\text{ of occurrences of } +1 \\ \text{as an eigenvalue of } (12)\end{array} = a+c$$

$$\begin{array}{l}\#\text{ of occurrences of } -1 \\ \text{as an eigenvalue of } (12)\end{array} = b+c.$$

This easy remark tells us how to decompose any representation of S_3 as a direct sum of isotypic components

$\longrightarrow$

Proposition: If W is any representation of S_3, then
$$W \cong U^{\oplus a} \oplus (U')^{\oplus b} \oplus V^{\oplus c}$$
with

- $c = \#$ of occurrences of $\omega = e^{2\pi i/3}$ as an eigenvalue of (123) on W
- $a = (\#$ of occurrences of $+1$ as an eig. of (12) on $W) - c$
- $b = (\#$ of occurrences of -1 as an eig. of (12) on $W) - b$.

- **Application** - Let's decompose $V \otimes V$ as a direct sum of irreducible representations of S_3.

By the previous remark, we need to understand the eigenvalues of (123) and (12) on $V \otimes V$.

[Remark: If v_{λ_i} is an eigenvector of eigenvalue λ_i ($i=1,2$), then $v_{\lambda_1} \otimes v_{\lambda_2}$ is an eigenvector of eigenvalue $\lambda_1 \lambda_2$].

We notice that

- (123) has eigenvalues ω and ω^{-1} on V. So it has eigenvalues $\omega \cdot \omega = \omega^2 = \omega^{-1}$; $\omega \cdot \omega^{-1} = 1$; $\omega^{-1} \cdot \omega = 1$; $\omega^{-1} \omega^{-1} = \omega^{-2} = \omega$ on $V \otimes V$. In particular, ω occurs once as eigenvalue of (123) on $V \otimes V$.

- (12) has eigenvalues $+1$ and -1 on V. So it has eigenvalues $(+1)\cdot(+1) = 1$, $(+1)(-1) = -1$; $(-1)\cdot(+1) = -1$, $(-1)(-1) = 1$ on $V \otimes V$. In particular, both $(+1)$ and (-1) occur twice as eigenvalue of (12) on $V \otimes V$.

$c = 1$
$a = 2-1 = 1$
$b = 2-1 = 1$.

Hence we obtain:
$$\boxed{V \otimes V = U \oplus U' \oplus V}.$$

▸ Claim : If every representation of G is 1-dimensional, then G is abelian.

▸ Proof - Let V be the regular representation of G. Decompose V as a direct sum of irreducible representations : $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$.

Then, each V_i is 1-dimensional, and we can write $V_i = \mathrm{Span}_{\mathbb{C}}(w_i)$. By construction, each w_i is an eigenvector of $\rho(g)$, (for all $g \in G$)

Set $\mathcal{B} = \{w_1, \ldots, w_r\}$. With respect to this basis, the matrix of each $\rho(g)$ is diagonal.

Hence $\rho(x)$ and $\rho(y)$ commute, $\forall x, y \in G$.

We obtain that

$$\rho(x^{-1}y^{-1}xy) = \rho(x^{-1})\rho(y^{-1})\rho(x)\rho(y) = \mathbb{1}_V.$$

Because the regular representation is faithful, this condition forces

$$x^{-1}y^{-1}xy = 1$$

ie $xy = yx$ $\forall x, y \in G$. We have ~~showed~~ proved that G is abelian. $\boxtimes$

Lecture 4 : Suggested Problems

1) Find the degrees of the irreducible representations of D_8.

2) Let G be a finite group, and let V be the regular representation of G. Find a basis for $\mathrm{Hom}_G(V,V)$.

3) Let $V_1,\dots,V_k$ be a complete set of non-isomorphic irreducible representations of G, and let V and W be arbitrary representations. Assume that $\forall i=1..k$,

$$d_i = \dim[\mathrm{Hom}_G(V,V_i)] \quad \text{and} \quad e_i = \dim[\mathrm{Hom}_G(W,V_i)].$$

Show that

$$\dim[\mathrm{Hom}_G(V,W)] = \sum_{i=1}^{k} d_i e_i.$$

Summary of lecture 4

- If G is a finite group, every finite dimensional complex representation of G is completely reducible.
- Every representation of G has a "unique" decomposition as a direct sum of isotypic components: $V \cong W_1^{\oplus a_1} \oplus W_2^{\oplus a_2} \oplus \dots \oplus W_k^{\oplus a_k}$ where $W_1,\dots,W_k$ are non-isomorphic irreducible representations and $a_j = \dim_{\mathbb{C}}[\mathrm{Hom}_G(V, W_j)]$ $\forall j=1..k$.
- If V is the regular representation, then $V \cong \bigoplus_{\substack{\text{all irred.}\\ \text{non-isom representations}}} (W_j)^{\oplus \dim W_j}$
- If $W_1,\dots,W_k$ is a list of non-isomorphic irreducible representations of G, then $|G| = \sum_{j=1}^{k} (\dim W_j)^2$
- G is abelian if and only if every irreducible repr. is one-dimensional