Lecture 3 : Mascke's theorem

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3.1 Indecomposable Representations

▸ Definition : A representation (ρ, V) of G is called "indecomposable" if it cannot be written as a direct sum of two proper non-trivial subrepresentations. Otherwise, we call (ρ, V) "decomposable".

▸ Remark : If ρ is irreducible, then ρ is also indecomposable. Indeed, if V does not have any proper non-trivial subspaces, then of course the desired decomposition does not exist.
Is the reverse true? Does reducible imply decomposable? Well, for finite dimensional representations of finite groups over complex vector spaces, it does.
This is not at all obvious, and it accounts to proving that every G-invariant subspace of V has a G-invariant complement in V (i.e. for all $W \subseteq V$ G-stable, there exists $W_c \subseteq V$ G-stable s.t. $V = W \oplus W_c$). This result is known as Mascke's theorem.

3.2 Mascke's Theorem

Theorem - Let $\rho : G \to GL(V)$ be a linear representation of G in V and let $W \subseteq V$ a vector subspace of V stable under G. Then there exists a complement W_0 of W in V which is also stable under G.

▸ <u>Corollary</u> Every reducible representation can be decomposed as the direct sum of two proper non-trivial subrepresentations.

▸ <u>Corollary</u> A representation of G is irreducible if and only if it is indecomposable.

The proof of Mascke's Theorem follows from this crucial lemma:

▸ <u>Lemma</u> - Let $\rho: G \to GL(V)$ be a representation of a finite group in a finite dimensional complex vector space. Then V has a G-invariant inner product, i.e. an inner product satisfying

$$\langle \rho(s) v, \rho(s) w \rangle = \langle v, w \rangle \quad \forall v, w \in V; \ \forall s \in G.$$

▸ <u>Remark</u> - This lemma is very important. It states that every finite dimensional complex representation of a finite group is "unitarizable" (i.e. can be made into a unitary representa-tion by choosing an invariant inner product).
As a consequence, we obtain that every $\rho(s)$ is diagonalizable and all the eigenvalues of $\rho(s)$ are complex numbers of absolute value one.

▸ <u>Proof of the lemma</u> Let $\langle | \rangle$ be any inner product on V (in general, $\langle | \rangle$ is not G-invariant). By "averaging on the group", we can transform $\langle | \rangle$ into a G-invariant inner product: for all v, w in V define

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g) v | \rho(g) w \rangle.$$

Then

(1) $\langle , \rangle : V \times V \to \mathbb{C}$ is linear on the first component, and conjugate linear on the second component ($\leftarrow$ use the linearity of $\rho(g)$).

(2) $\langle , \rangle : V \times V \to \mathbb{C}$ is positive definite

$\left(\langle v, v \rangle = \frac{1}{|G|} \sum_{g \in G} \underbrace{\langle \rho(g) v \mid \rho(g) v \rangle}_{\geq 0} \geq 0 \right.$ and

$\langle v, v \rangle = 0 \iff \langle \rho(g) v \mid \rho(g) v \rangle = 0 \ \forall g \in G \iff \rho(g) v = 0$

$\left. \forall g \in G \iff v = 0 \ \checkmark \right)$

(3) $\langle , \rangle$ is G-invariant $\left(\langle \rho(s) v, \rho(s) w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g) \rho(s) v \mid \rho(g) \rho(s) w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(gs) v \mid \rho(gs) w \rangle = \frac{1}{|G|} \sum_{t \in G} \langle \rho(t) v \mid \rho(t) w \rangle \right.$

$\left. = \langle v, w \rangle . \checkmark \right)$.

This shows that $\langle , \rangle$ is a G-invariant inner product on V, so it concludes the proof of the lemma. ∎

▸ __Remark__: If V is irreducible, then there exists $\overset{\text{up to a constant}}{\underline{\text{one and only one}}}$
G-invariant inner product on V. The unicity follows from Shur's
lemma: suppose that $\langle , \rangle$ and $(\mid)$ are both G-invariant. Let $\varphi : V \to V^*$, $v \to \langle v, \cdot \rangle$, and let
$\psi : V \to V^*$, $v \to (v \mid \cdot)$. Denote by T the composition $\psi^{-1} \circ \varphi$. Then T is an isomorphism of
V, and satisfies the condition $\langle v, w \rangle = (T(v) \mid w) \ \forall w \in V$. Because $(T(\rho(g)v) \mid w)$
$= \langle \rho(g) v, w \rangle = \langle v, \rho(g^{-1}) w \rangle = (T(v) \mid \rho(g^{-1}) w) = (\rho(g) T(v) \mid w) \ \forall w \in V$, $T(\rho(g) v) = \rho(g) T v \ \forall v \in V$.
$\Rightarrow$ T is a G-homom. from V to V. If V is irreducible, T must be a scalar multiple of the identity. ∎

Now we are ready to prove Mascke's Theorem.

▸ __Proof (Mascke's Theor.)__ - Let $W \subseteq V$ be a G-invariant subspace.
We will show the existance of a G-invariant complement for W in V.
Choose a G-invariant inner product $\langle , \rangle$ on V, and let W_0 be the orthogonal complement of W w.r.t. $\langle , \rangle$:

$$W_0 = W^{\perp} = \{ v \in V : \langle v, w \rangle = 0 \ \forall w \in W \}$$

It's clear that W_0 satisfies the condition: $V = W_0 \oplus W$.
So we just need to prove that W_0 is G-stable, i.e. that $\rho(s) v \in W_0$ for all $v \in W_0$ and all s in G.

This is easy to do, because:
$\langle \rho(s) v, w \rangle = \langle \rho(s) v, \rho(s) \rho(s^{-1}) w \rangle \overset{\text{invariance}}{=} \langle v, \underbrace{\rho(s^{-1}) w}_{\text{in } W} \rangle = 0 \ \forall w \in W$

This shows that every G-invariant subspace W of V has a G-invariant complement: we just need to take the orthogonal complement of W w.r.t. a G-invariant inner product! $\blacksquare$

3.3 Remarks on Mascke's Theorem

We have proved Mascke's's theorem under the following assumptions:

- G is finite
- V is a complex vector space
- The dimension of V is finite.

What happens if we weaken any of these hypothesis?

Claim 1 - Mascke's Theorem holds also for finite dimensional complex representations of an infinite group G, if G is compact.
An invariant inner product is again obtained by averaging, but you need to replace the sum over the group by the integral with respect to an invariant measure.
This is possible, because every compact group has an invariant measure (or "Haar measure"), i.e. a measure satisfying

$$\int_G f(t)\,dt = \int_G f(ts)\,dt = \int_G f(st)\,dt \quad \forall s \in G.$$

If you impose a normalizing condition, like "Total mass $= \int_G dt = 1$", then G has a unique invariant measure.
[V could be a finite dimensional complex vector space, or, more generally, a Hilbert space].

Claim 2 will come soon... (it's just misplaced)

Claim 3 Mascke's Theorem holds for representations of a finite group in a finite dimensional vector space defined over any other field of characteristic 0, and also over a field of characteristic p, when p does not divides the order of the group. If $p \mid |G|$, then Mascke's Theorem fails.

To prove This claim, we give an alternative proof of Maschke's theorem That does not require The existence of an inner product.

- Let $W \subseteq V$ be a G-stable subspace of V. Let W' be an arbitrary complement of W in The vector space V. In general, W' will not be G-invariant. Let p be The projection of V onto W

$$p : V = W \oplus W' \longrightarrow W$$
$$v = w + w' \longmapsto w.$$

For all $s \in G$, consider The conjugate of p by $\rho(s)$:

$$\rho(s) \circ p \circ \rho(s^{-1}) : V \longrightarrow W \quad \left(V \xrightarrow[\sim]{\rho(s^{-1})} V \xrightarrow{p} W \xrightarrow[\sim]{\rho(s)|_W} W \right)$$

and form The average of these conjugates over The group:

$$p_0 : V \longrightarrow W, \quad p_0 \overset{def}{\equiv} \frac{1}{|G|} \sum_{s \in G} \rho(s) \circ p \circ \rho(s^{-1}).$$

Because, by assumption, The field has either characteristic 0 or it has characteristic p with $p \nmid |G|$, The quantity $\frac{1}{|G|}$ is well defined.

- Notice That p_0 is again a projection on W. Indeed:
  - p_0 maps V onto W (here we use The fact That W is G-stable)
  - $p_0(w) = w$, $\forall w \in W$ (again, we use The stability of W:

$$\rho(s) \circ p \circ \underbrace{\rho(s^{-1}) w}_{\text{in } W} \underset{\underset{p|_W = 1_W}{\uparrow}}{=} \rho(s) \circ \rho(s^{-1}) w = w \quad \forall s \in G \Rightarrow p_0(w) = w.)$$

Let W_0 be the Kernel of The projection p_0. Then $V = W \oplus W_0$, i.e. W_0 is a complement of W in V.
We just need to prove That W_0 is G-invariant. If $v \in W_0 = Ker(p_0)$ and $t \in G$ then we can write:

$$p_0(\rho(t) v) = \frac{1}{|G|} \sum_{s \in G} \rho(s) \circ p \circ \underbrace{\rho(s^{-1}) \circ \rho(t)}_{\rho(s^{-1}t) = \rho(t^{-1}s)^{-1}} v =$$

- $$= \frac{1}{|G|} \sum_{s \in G} \rho(t) \rho(t^{-1}s)\, p\, \rho(t^{-1}s)^{-1} v = \frac{1}{|G|} \rho(t) \sum_{s \in G} \rho(t^{-1}s)\, p\, \rho(t^{-1}s)^{-1} v$$

$$= \rho(t)\left(p_0(v)\right) = \rho(t)\, \underline{0} = \underline{0} \Longrightarrow \rho(t) v \in W_0 = Ker(p_0).$$

This shows that W_0 is a G-invariant subspace of V, and concludes the proof.

$\Rightarrow$ Mascke's lemma holds if V is a f.d. vector space over any field of characteristic 0 or p, with $p \nmid |G|$.

Now assume that $p \mid |G|$, and let's give a counterexample to prove Mascke's Theorem fails in this case.

Take $G = \mathbb{Z}_2$, $\mathbb{K}$ = any field of characteristic 2 and $V = \mathbb{K}^2$.

Define a representation of G in V by

$$\rho(\bar{0}) = \mathbb{1}_V$$

$$\rho(\bar{1}) : V \to V, \quad \begin{vmatrix} x \\ y \end{vmatrix} \mapsto \begin{vmatrix} x+y \\ y \end{vmatrix}$$

ρ is well defined, because $\forall x, y \in \mathbb{K}$:

$$\rho(\bar{1})\,\rho(\bar{1}) \begin{vmatrix} x \\ y \end{vmatrix} = \rho(\bar{1}) \begin{vmatrix} x+y \\ y \end{vmatrix} = \begin{vmatrix} x+2y \\ y \end{vmatrix} \underset{\text{char } \mathbb{K} = 2}{=} \begin{vmatrix} x \\ y \end{vmatrix} = \mathbb{1}_V \begin{vmatrix} x \\ y \end{vmatrix} = \rho(\bar{0}) \begin{vmatrix} x \\ y \end{vmatrix}$$

Now consider the subspace $W = \left\{ \begin{vmatrix} x \\ 0 \end{vmatrix} : x \in \mathbb{K} \right\}$ of V.
W is clearly G-stable, and it's not hard to prove that it is the only G-stable subspace of V. It follows that W has no G-invariant complement, and we obtain a counterexample to Mascke's theorem. ■

▸ Claim 2 - Mascke's Theorem is, in general, false for infinite groups that are not compact.

To motivate this claim, we give a counterexample.
Let $G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$.

Because $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix}$, G is isomorphic to $\mathbb{Z}$.

Consider the representation of G on $\mathbb{C}^2$, defined by:

$$\rho : G \mapsto GL(\mathbb{C}^2) \cong GL(2, \mathbb{C})$$
$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

the subspace $W = \mathrm{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \subseteq \mathbb{C}^2$ is G-stable. [Indeed for all $n \in \mathbb{Z}$, $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.]

W is actually the only 1-dimensional G-stable subspace of $\mathbb{C}^2$.
(check that $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ if and only if $c = 1$ and $y_0 = 0$).

It follows that W does not have a G-invariant complement in $\mathbb{C}^2$, and Mascke's theorem fails.

[Now insert Claim 3].

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3.4 A cryterion for the irreducibility of representations

▶ <u>theorem</u> - Let (ϱ, V) be a finite dimensional complex representation of a finite group G. Then (ϱ, V) is irreducible if and only if $\dim_{\mathbb{C}}[\mathrm{Hom}_G(V,V)] = 1$, i.e. if and only if every G-homomorphism from V to V is a multiple of the identity.

▶ <u>proof</u> - By Shur's lemma, we know that $\dim_{\mathbb{C}}[\mathrm{Hom}_G(V,V)] = 1$ for every irreducible representation (ϱ, V) of G.
Viceversa, let's assume that $\dim_{\mathbb{C}}[\mathrm{Hom}_G(V,V)] = 1$, and let's prove by contradiction that (ϱ, V) is irreducible.
If $W \subseteq V$ is a <u>proper non-trivial</u> G-invariant subspace, then - by Moscke's theorem - we can write $V = W \oplus W_0$, with W_0 also G-invariant.
Denote by p the projection of V onto W, regarded as a linear transformation from V to V.
Then $p \in \mathrm{Hom}_G(V,V)$. Indeed $\forall v = w + w_0$ in $V = W \oplus W_0$, we have:

$$p\,\varrho(s)(w + w_0) = p(\underbrace{\varrho(s)w}_{\text{in } W} + \underbrace{\varrho(s)w_0}_{\text{in } W_0}) = \varrho(s)w = \varrho(s)\,p(w + w_0).$$

Because, by assumption, $\dim_{\mathbb{C}}(\mathrm{Hom}_G(V,V)) = 1$, p must be a multiple of the identity of V.
This is clearly a contradiction: $p \neq 0$ (because $\mathrm{Im}\,p = W \neq \{0\}$) and p is not an isomorphism (because $\mathrm{Ker}\,p = W_0 \neq \{0\}$).
So (ϱ, V) must be irreducible. ∎

This theorem is quite important. It says that in order to show that a representation (ϱ, V) is irreducible, we just need to prove that every G-homomorphism $V \to V$ is constant.

▸ An application: Show that the representation

$$\rho : D_8 = \langle a, b : a^4 = 1 = b^2,\ b^{-1}ab = a^{-1} \rangle \longrightarrow GL(2, \mathbb{C})$$

$$a \longmapsto A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$b \longmapsto B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the dihedral group into $\mathbb{C}^2$ is irreducible.

By the previous Theorem we only need to prove that every G homomorphism $T : \mathbb{C}^2 \to \mathbb{C}^2$, $\underline{x} \mapsto C\underline{x}$ is constant.

We notice that T is a G-homomorphism if and only if the diagram

$$\begin{array}{ccc} \mathbb{C}^2 = V & \xrightarrow{\rho(s)} & V = \mathbb{C}^2 \\ T \downarrow & & \downarrow T \\ \mathbb{C}^2 = V & \xrightarrow[\rho(s)]{} & V = \mathbb{C}^2 \end{array}$$

commutes $\forall s \in G$.

It's of course sufficient to consider the cases $s = a$ and $s = b$ (because a and b generate D_8).

Hence a G-homomorphism is given by a 2×2 matrix C satisfying

$$\begin{cases} AC = CA \\ BC = CB. \end{cases}$$

An easy linear algebra exercise proves that the only solutions to this problem are the matrices of the form $C = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. Hence the space of G-homomorphism is 1-dimensional, and ρ is irreducible. □

Suggested Problems [Lecture 3]

① Let $G = C_3 = \langle x : x^3 = 1 \rangle$ and let $V = \mathbb{C}^2$.
Choose a basis $\{v_1, v_2\}$ of V and define a representation of G in V by:

$$\begin{cases} \rho(x) v_1 = v_2 \\ \rho(x) v_2 = -v_1 - v_2. \end{cases}$$

(ρ is well defined, because $\rho(x) \rightsquigarrow \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \equiv A$ and $A^3 = I$.)

Decompose V as a direct sum of irreducible representations.

② Let G be a finite group, and let $\rho : G \to GL(2, \mathbb{C})$ be a representation of G. Suppose that there are elements $x, y \in G$ st $\rho(x)$ and $\rho(y)$ do not commute. Prove that ρ is irreducible.

Summary of Lecture 3

Let G be a finite group, and let (ρ, V) be a finite dimensional complex representation of G. Then

- 1 We can define an inner product on V s.t. ρ becomes unitary. If ρ is irreducible, this inner product is unique up to a constant.
- 2 ρ is irreducible if and only if ρ is indecomposable. Equivalently, every G-stable subspace of V has a G-stable complement.
- 3 (ρ, V) is irreducible if and only if $\dim_{\mathbb{C}} [\mathrm{Hom}_G(V, V)] = 1$.