**Theorem 2** For every pair of partitions $\pi, \lambda \vdash n$

$$\chi_{S^\lambda}(\pi) = [x^{\lambda+\delta}]\left(P_\pi(x)\, a_\delta(x)\right).$$

**Proof** - Write $P_\pi(x)$ as a linear combination of Shur functions:

$$P_\pi(x) = \sum_{\lambda \vdash n} \omega_\lambda(\pi)\, s_\lambda(x).$$

Then $\omega_\lambda(\pi) = [x^{\lambda+\delta}]\left(P_\pi(x)\, a_\delta(x)\right)$ is the coefficient of $x^{\lambda+\delta}$ inside $P_\pi(x)\, a_\delta(x)$.

So we need to prove that $\omega_\lambda(\pi) = \chi_{S^\lambda}(\pi)$, for every pair $\lambda, \pi$ of partitions of $n$.

Because $s_\lambda(x) = \sum_{\mu \leq \lambda} K_{\lambda\mu}\, m_\mu$, we can write:

$$P_\pi(x) = \sum_{\lambda \vdash n} \omega_\lambda(\pi)\, s_\lambda(x) = \sum_{\lambda \vdash n} \omega_\lambda(\pi) \left(\sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu\right) =$$

$$= \sum_{\mu \vdash n} \left(\sum_{\lambda \geq \mu} K_{\lambda\mu}\, \omega_\lambda(\pi)\right) m_\mu .$$

Comparing this expression with:

$$P_\pi(x) = \sum_{\mu \vdash n} \chi_{M^\mu}(\pi)\, m_\mu$$

and using the fact that the monomial functions are a basis, we get that

$$\chi_{M^\mu}(\pi) = \sum_{\lambda \geq \mu} K_{\lambda\mu}\, \omega_\lambda(\pi) \qquad [\forall \mu \vdash n].$$

Using only these equations, we'll be able to prove that $\omega_\lambda(\pi) = \chi_{S^\lambda}(\pi)$, $\forall \lambda$.

Let

$$M^\mu = \bigoplus_{\lambda \vdash n} c_{\lambda\mu}\, S^\lambda$$

be the decomposition of $M^\mu$ as a direct sum of Specht modules. [Notice that we are not assuming that $c_{\lambda\mu} = K_{\lambda\mu}$. This will be a corollary of our theorem!!!].

Then

$$\chi_{M^\mu}(\pi) = \sum_{\lambda \vdash n} c_{\lambda\mu}\, \chi_{S^\lambda}(\pi).$$

So we have:

$$\begin{cases} \chi_{M^\mu}(\pi) = \displaystyle\sum_{\lambda \geq \mu} K_{\lambda\mu}\, \omega_\lambda(\pi) & \boxed{\mathrm{I}} \\ \chi_{M^\mu}(\pi) = \displaystyle\sum_{\lambda \vdash n} c_{\lambda\mu}\, \chi_{S^\lambda}(\pi). & \boxed{\mathrm{II}} \end{cases}$$

Order partitions in lexicographical ordering. The matrix $\{K_{\lambda\mu}\}$ is triangular, with one's on the diagonal.

$\Rightarrow \{K_{\lambda\mu}\}$ is invertible. We notice that the entries of $\{K_{\lambda\mu}\}$ are non negative integers, hence the entries of $\{K_{\lambda\mu}\}^{-1}$ are integers.

We obtain (from $\boxed{\mathrm{I}}$) that $\omega_\lambda(\pi)$ is a l.c. of $\chi_{M^\mu}(\pi)$ with integer coefficients.

Using $\boxed{\mathrm{II}}$ (and the fact that also the $c_{\mu}$'s are integers, because they are multiplicities of Specht modules in $M^\mu$), we deduce that $\omega_\lambda(\pi)$ is a l.c. of $\chi_{S^\lambda}(\pi)$'s with (not-necessarily $\geq 0$) integer coefficients. $[\forall \pi \vdash n]$

$\Rightarrow$ $\omega_\lambda$ is a l.c. of characters of Specht modules with (not-necessarily $\geq 0$) integer coefficients

$\Leftrightarrow$ $\omega_\lambda$ is a <u>virtual character</u> of $S_n$.

Write $\omega_\lambda = \sum a_\mu \chi_{S_\mu}$, with $a_\mu \in \mathbb{Z}$ $\forall \mu \vdash n$.

Then

$$\langle \omega_\lambda, \omega_\lambda \rangle = \langle \sum_{\mu_1} a_{\mu_1} \chi_{S_{\mu_1}}, \sum_{\mu_2} a_{\mu_2} \chi_{S_{\mu_2}} \rangle =$$

$$= \sum_{\mu_1,\mu_2} a_{\mu_1} a_{\mu_2} \underbrace{\langle \chi_{S_{\mu_1}}, \chi_{S_{\mu_2}} \rangle}_{\delta_{\mu_1\mu_2}}$$

$$= \sum_{\mu} a_\mu^2 .$$

Claim - $\underbrace{\langle w_\lambda, w_\theta \rangle}_{\text{inner product as class functions}} = \delta_{\lambda\theta}, \quad \forall \lambda, \theta \vdash n.$

Proof - By definition, $P_\pi = \sum_\lambda w_\lambda(\pi) s_\lambda$. So we can write:

$$\underbrace{\langle P_\pi, s_\mu \rangle}_{\substack{\text{inner product} \\ \text{in } \Lambda \\ \text{(symm functions)}}} = \sum_\lambda w_\lambda(\pi) \underbrace{\langle s_\lambda, s_\mu \rangle}_{\delta_{\lambda\mu}} = w_\mu(\pi) \Rightarrow \langle \frac{P_\pi}{\sqrt{z_\pi}}, s_\mu \rangle = \frac{w_\mu(\pi)}{\sqrt{z_\pi}}$$

Because the symmetric functions $\left\{ \frac{P_\pi}{\sqrt{z_\pi}} \right\}$ are a basis, we can write:

$$s_\mu = \sum_\pi \langle s_\mu, \frac{P_\pi}{\sqrt{z_\pi}} \rangle \frac{P_\pi}{\sqrt{z_\pi}} = \sum_\pi w_\mu(\pi) \frac{P_\pi}{z_\pi}.$$

Therefore:

$$\delta_{\lambda\theta} = \langle s_\lambda, s_\theta \rangle = \langle \sum_{\pi_1} w_\lambda(\pi_1) \frac{P_{\pi_1}}{z_{\pi_1}}, \sum_{\pi_2} w_\theta(\pi_2) \frac{P_{\pi_2}}{z_{\pi_2}} \rangle =$$

Shur functions are orthonormal

$$= \sum_{\pi_1, \pi_2} \frac{w_\lambda(\pi_1) w_\theta(\pi_2)}{\sqrt{z_{\pi_1}} \sqrt{z_{\pi_2}}} \underbrace{\langle \frac{P_{\pi_1}}{\sqrt{z_{\pi_1}}}, \frac{P_{\pi_2}}{\sqrt{z_{\pi_2}}} \rangle}_{\delta_{\pi_1, \pi_2}} =$$

$$= \sum_\pi \frac{w_\lambda(\pi) w_\theta(\pi)}{z_\pi} = \frac{1}{n!} \sum_\pi \frac{n!}{z_\pi} w_\lambda(\pi) w_\theta(\pi) =$$

$\boxed{\text{size of the conj. class } \pi}$

$= \langle w_\lambda, w_\theta \rangle$ (as class functions). ▨

Then we can write:

$$1 = \langle \omega_\lambda, \omega_\lambda \rangle = \sum_\mu \underset{\uparrow \text{Integer}}{a_\mu}^2$$

$\Rightarrow$ There exists one and only one $\mu_0$ s.t. $a_{\mu_0} = \pm 1$.
For any other partition $\mu$, $a_\mu = 0$.

$$\Rightarrow \omega_\lambda = \sum a_\mu \chi_{S^\mu} = \pm \chi_{\mu_0}.$$

We want to prove that $\omega_\lambda = + \chi_\lambda$, $\forall \lambda \vdash n$.
Order the partitions of $n$ in <u>inverse</u> lexicographical order (so that $(n)$ is the smallest partitions).
Denote by $\leq_i$ the inverse order, to distinguish it from $\leq$.
We procede by induction on the partitions.

<u>Base of induction</u>: $\lambda = (n)$
If $\lambda = (n)$, then $M^\lambda$ = trivial representation $= S^\lambda$.
So we can write:

$$\chi_{S^{(n)}}(\pi) = \chi_{M^{(n)}}(\pi) = \sum_{\lambda \geq (n)} K_{\lambda(n)} \, \omega_\lambda(\pi) =$$

The only partition $\geq (n)$ is $(n)$ itself

$$= \underbrace{K_{(n)(n)}}_{1} \, \omega_{(n)}(\pi) = \omega_{(n)}(\pi).$$

This is true $\forall \pi \vdash n$, so $\omega_{(n)} = \chi_{S^{(n)}}$, as claimed.

<u>Inductive hypothesis</u>: Suppose the result is true

for all $\mu \leq_i \lambda$ (ie $\forall \mu > \lambda$).

Then we can write:

$$\sum_{\mu \vdash n} c_{\mu\lambda} \chi^{\mu}_{S}(\pi) = \chi_{M^\lambda}(\pi) = \sum_{\mu \geq \lambda} K_{\mu\lambda} w_\mu(\pi) =$$

$\boxed{II}$ $\boxed{I}$

$$= \underbrace{K_{\lambda\lambda}}_{1} w_\lambda(\pi) + \sum_{\mu > \lambda} K_{\mu\lambda} \chi_\mu(\pi) =$$

inductive hypothesis

$$= w_\lambda(\pi) + \sum_{\mu > \lambda} K_{\mu\lambda} \chi_\mu(\pi).$$

Hence

$$\chi_{S^\lambda}(\pi) - w_\lambda(\pi) + \sum_{\mu > \lambda} (c_{\mu\lambda} - K_{\mu\lambda}) \chi_{S^\mu}(\pi) + \sum_{\mu < \lambda} c_{\mu\lambda} \chi_{S^\mu}(\pi) = 0$$

The characters of the Specht modules are orthonormal, hence l. i.. Notice that $c_{\lambda\lambda} > 0$ (because $M^\lambda$ contains $S^\lambda$)

$\Rightarrow$ we need $w_\lambda(\pi) = c_{\lambda\lambda} \chi_{S_\lambda} + \dots$ (to annihilate the coeff. of $S^\lambda$ in the above expression).

But we know that $w_\lambda(\pi) = \pm \chi_\theta$ for some $\theta$.

$\Rightarrow$ the only possibility is that $w_\lambda(\pi) = + S_\lambda$ (because $c_{\lambda\lambda} > 0$). This completes the induction, and also the proof of the Theorem. ∎

## Corollaries -

① $$P_\pi(x_1 \dots x_n) = \sum_{\lambda \vdash n} \chi_{\boxed{S^\lambda}}(\pi)\, \boxed{s_\lambda}(x_1 \dots x_n)$$

Shur function

Specht module

② $$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^\lambda$$