Theorem 1 For every pair of partitions $\lambda, \pi \vdash n$

$$\chi_{M^\lambda}(\pi) = [x^\lambda]\, P_\pi(x)$$

<u>proof</u>- Write $\pi = (1^{\ell_1} 2^{\ell_2} \cdots n^{\ell_n})$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$.
[We use different notations for the two partitions --- the reason will be clear soon!]
By definition of power sum symmetric function,

$$P_\pi(x_1,\dots,x_n) = \prod_{j=1}^{n} (x_1^j + x_2^j + \cdots + x_n^j)^{\ell_j}.$$

This is a symmetric function of degree $\sum_{j=1}^{n} j\ell_j = n$.
We need the coefficient of x^λ in $P_\pi(x)$. [The question makes sense because $\lambda \vdash n$ and x^λ is a monomial of degree n (= degree of $P_\pi(x)$)].

We notice that every monomial in $P_\pi(x)$ is a product of a monomial from $(x_1 + x_2 + \cdots + x_n)^{\ell_1}$, a monomial from $(x_1^2 + x_2^2 + \cdots + x_n^2)^{\ell_2}$, ..., and a monomial from $(x_1^n + x_2^n + \cdots + x_n^n)^{\ell_n}$. So we need to understand the monomials of

$$(x_1^j + x_2^j + \cdots + x_n^j)^{\ell_j}, \quad \forall j = 1 \dots n.$$

Let's start from the easiest case: $j=1$.

A monomial in

$$(x_1+\dots+x_n)^{l_1} = \underbrace{(x_1+x_2+\dots+x_n)\,(x_1+x_2+\dots x_n)\cdots(x_1+x_2+\dots+x_n)}_{l_1 \text{ times}} \qquad \circledast$$

has the form

$$x_1^{l_1(1)}\, x_2^{l_1(2)} \cdots x_n^{l_1(n)}$$

with $l_1(1)+l_1(2)+\dots+l_1(n) = l_1$, and $l_1(j) \in \mathbb{Z}_{\geq 0}$ $\forall j=1\dots n$.

In other words, $\{l_1(j)\}_{j=1\dots n}$ is a composition of l_1 in at most n parts.

The monomial $x_1^{l_1(1)}\, x_2^{l_1(2)} \cdots x_n^{l_1(n)}$ appears in $(x_1+\dots+x_n)^{l_1}$ with coefficient

$$\underbrace{\frac{l_1!}{l_1(1)!\,\cancel{[l_1-l_1(1)]!}}}\quad \underbrace{\frac{\cancel{[l_1-l_1(1)]!}}{l_1(2)!\,\cancel{[l_1-l_1(1)-l_1(2)]!}}}\quad \frac{\cancel{[l_1-l_1(1)-l_1(2)]!}}{l_1(3)!\,\cancel{[l_1-l_1(1)-l_1(2)-l_1(3)]!}}$$

choose $l_1(1)$ factors in $\circledast$ from which you'll extract x_1

out of the remaining $l_1-l_1(1)$ factors in $\circledast$, choose $l_1(2)$ factors from which you'll exact x_2

$$\dots \frac{\cancel{[l_1-\sum_{j=1}^{n-1} l_1(j)]!}}{l_1(n)!\,[l_1-\underbrace{\textstyle\sum_{j=1}^{n} l_1(j)}_{=0}]!} =$$

$$= \frac{l_1\,!}{\left[l_1(1)!\, l_1(2)! \cdots l_1(n)!\right] \underbrace{0!}_{1}} = \frac{l_1\,!}{l_1(1)!\, l_1(2)! \cdots l_1(n)!} =$$

$$= \frac{l_1\,!}{\prod_{j=1}^{n} l_1(j)!}.$$

Similarly, $(x_1^2 + \cdots + x_n^2)^{l_2}$ is a linear combination of monomials

$$x_1^{2 l_2(1)}\, x_2^{2 l_2(2)} \cdots x_n^{2 l_2(n)}$$

with $\sum_{j=1}^{n} 2 l_2(j) = 2 l_2 \left(\Longleftrightarrow \sum_{j=1}^{n} l_2(j) = l_2 \right)$, and $l_2(j) \in \mathbb{Z}_{\geq 0}$ $\forall j = 1 \ldots n$. Such a monomial appears with coefficient $\frac{l_2\,!}{\prod_{j=1}^{n} l_2(j)!}$.

More generally,

$$\left(x_1^k + x_2^k + \cdots + x_n^k\right)^{l_k}$$

is a sum of monomials $x_1^{k l_k(1)}\, x_2^{k l_k(2)} \cdots x_n^{k l_k(n)}$ (with $\{l_k(j)\}_{j=1 \ldots n}$ a composition of l_k), each appearing with coefficient $\frac{l_k\,!}{\prod_{j=1}^{n} l_k(j)!}$.

[Here $k = 1, 2, \ldots, n$].

We conclude that a monomial in $P_\pi(x_1 \ldots x_n)$ has the form

$$\prod_{k=1}^{n}\left(\prod_{j=1}^{n} x_j^{k\, \ell_k(j)}\right) = \prod_{j=1}^{n} x_j^{\sum_{k=1}^{n} k\, \ell_k(j)}$$

with $\{\ell_k(1), \ell_k(2), \ldots, \ell_k(n)\}$ a composition of ℓ_k $\forall k = 1 \ldots n$ (i.e. $\sum_{j=1}^{n} \ell_k(n) = \ell_k$).

Such a monomial appears in $P_\pi(x)$ with coefficient

$$\prod_{k=1}^{n}\left[\frac{\ell_k!}{\prod_{j=1}^{n} \ell_k(j)!}\right] = \frac{\prod_{k=1}^{n} \ell_k!}{\prod_{k,j=1\ldots n} \ell_k(j)!}.$$

In particular, the monomial x^λ appears in P_π with coefficient

$$\sum_{\substack{\{\ell_1(j)\}_{j=1\ldots n} \text{ composition of } \ell_1 \\ \{\ell_2(j)\}_{j=1\ldots n} \text{ composition of } \ell_2 \\ \vdots \\ \{\ell_n(j)\}_{j=1\ldots n} \text{ composition of } \ell_n \\ \text{such that} \\ \sum_{k=1}^{n} k\,\ell_k(j) = \lambda_j,\ \forall j = 1\ldots n}} \left[\frac{\prod_{k=1}^{n} \ell_k!}{\prod_{k,j=1\ldots n} \ell_k(j)!}\right] = \mathrm{I}$$

$$= \sum_{\substack{\text{sequences } \{\ell_k(j)\}_{j=1\ldots n} \ (k=1\ldots n) \\ \text{satisfying} \\ \sum_k k\,\ell_k(j) = \lambda_j \ ; \ \sum_j \ell_k(j) = \ell_k}} \left[\frac{\prod_{k=1}^{n} \ell_k \, !}{\prod_{k,j=1\ldots n} \ell_k(j)!} \right]$$

We need to show that this messy number is equal to the character of the permutation module M^λ on the conjugacy class π.

Write $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. then

$$M^\lambda = \mathrm{Ind}_{S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_n}}^{S_n} (\text{trivial}).$$

We can compute the character of M^λ using the formulas for induced characters...

Set :
- $G = S_n$
- $H = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_n}$
- C = conjugacy class of S_n parametrized by π
- $H \cap C = D_1 \uplus D_2 \uplus \ldots \uplus D_r$ $\Rightarrow$ decomposition of $C \cap H$ in conjugacy classes of H.

Then

$$\chi_{M^\lambda}(\pi) = \chi_{\mathrm{Ind}_H^G(\mathrm{triv.})}(C) = \frac{|G|}{|H|} \sum_{s=1}^{r} \frac{\# D_s}{\# C} \underbrace{\chi_{\mathrm{triv}}(D_s)}_{1} =$$

$$= \frac{n!}{\lambda_1!\, \lambda_2! \dots \lambda_n!} \sum_{s=1}^{r} \frac{\# D_s}{\# C} =$$

$$= \frac{1^{l_1} 2^{l_2} \dots n^{l_n}\, l_1!\, l_2! \dots l_n!}{\lambda_1!\, \lambda_2! \dots \lambda_n!} \sum_{s=1}^{r} \# D_s.$$

$\# C = \#$(conjugacy class parametrized by π) $= \frac{n!}{z_\pi} =$

$= \frac{n!}{1^{l_1} 2^{l_2} \dots n^{l_n}\, l_1!\, l_2! \dots l_n!}$

We just need to understand the decomposition

$$C \cap H = D_1 \uplus \dots \uplus D_r$$

of $C \cap H$ in H-conjugacy classes. A permutation $\sigma \in C \cap H$ consists of l_j cycles of length j:

$$\underbrace{(-)(-)\dots(-)}_{l_1}\ \underbrace{(--)(--)\dots(--)}_{l_2}\ \dots\ \underbrace{(-----)(-----)\dots(-----)}_{l_n},$$

and can be written as a product $\sigma_1 \sigma_2 \dots \sigma_n$ with $\sigma_j \in S_{\lambda_j}$.

To get σ_j we pick $l_k(j)$ K-cycles from σ, $\forall K = 1 \dots n$.

This choice gives a sequence

$\{ l_K(j) \}_{j=1\ldots n}$ s.t. $\sum_{K=1}^{n} K l_K(j) = \lambda_j$, $\forall j = 1 \ldots n$.

- We also need: $\sum_{j=1}^{n} l_k(j) = l_k$, $\forall K = 1 \ldots n$.

Every conjugancy class D_s in $C_n H$ is given by sequences $\{ l_K(j) \}_{j=1\ldots n}$ $(k = 1..n)$ s.t.

$$\sum_{K=1}^{n} K l_K(j) = \lambda_j \; ; \; \sum_{j=1}^{n} l_K(j) = l_K ,$$

and is obtained by taking the conjugacy class

$1^{l_1(j)} \, 2^{l_2(j)} \ldots n^{l_n(j)}$ in each S_{λ_j}.

- It follows that

$$\# D_s = \prod_{j=1}^{n} \frac{\lambda_j !}{\prod_{K=1}^{n} \left[K^{l_K(j)} \; l_K(j)! \right]} = \frac{\prod_{j=1}^{n} \lambda_j !}{\left[\prod_{K=1}^{n} K^{\sum_{j=1}^{n} l_K(j)} \right] \left[\prod_{K,j=1\ldots n} l_K(j)! \right]} = \frac{\prod_{j=1}^{n} \lambda_j !}{\left[\prod_{K=1}^{n} K^{l_K} \right] \left[\prod_{j,K=1}^{n} l_K(j)! \right]}$$

Therefore we obtain:

$$\chi_{M1}(\pi) = \frac{\cancel{\left(1^{l_1} 2^{l_2} \ldots n^{l_n} \right)} (l_1! \, l_2! \ldots l_n!)}{\cancel{(\lambda_1! \, \lambda_2! \ldots \lambda_n!)}} \sum_{\substack{\text{sequences } \{l_K(j)\} \\ \text{satisfying} \\ \sum_K K l_K(j) = \lambda_j \\ \sum_j l_K(j) = l_K}} \frac{\cancel{(\lambda_1! \, \lambda_2! \ldots \lambda_n!)}}{\cancel{\left[1^{l_1} 2^{l_2} \ldots n^{l_n} \right]} \left[\prod_{K,j} l_K(j)! \right]}$$

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$\longrightarrow$

$$= \sum_{\substack{\text{sequences } \{l_k(j)\} \text{ satisfying} \\ \sum_k k\, l_k(j) = \lambda_j;\ \sum_j l_k(j) = l_k}} \frac{\prod_{k=1}^{n} l_k!}{\prod_{j,k=1}^{n} l_k(j)!}.$$

this is exactly the coefficient of x^λ in $P_\pi(x)$.
So the proof is complete! $\blacksquare$

<u>Corollary</u>: $P_\pi(x) = \sum_{\lambda \vdash n} \chi_{M^\lambda}(\pi)\, m_\lambda(x)$