A scalar product

In this section, we define a scalar product on the space $\Lambda_{\mathbb{Q}}$ of symmetric functions.
[Here $\Lambda_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}\{m_\lambda : \lambda \text{ partitions}\} = \mathrm{Span}_{\mathbb{Q}}\{h_\lambda : \lambda \text{ partitions}\} =$
$= \mathrm{span}_{\mathbb{Q}}\{e_\lambda : \lambda \text{ partitions}\} = \mathrm{Span}_{\mathbb{Q}}\{s_\lambda : \lambda \text{ partitions}\}$].

Let $\langle , \rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Q}$ be the unique bilinear form that satisfies the condition

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \quad (*)$$

for every pair of partitions λ and μ.
Because $\{m_\lambda\}$, $\{h_\mu\}$ are basis of Λ, the bilinear form $\langle , \rangle$ is uniquely defined. The condition $(*)$ means that $\{m_\lambda\}$ and $\{h_\mu\}$ are dual basis of Λ w.r.t. $\langle , \rangle$. It also implies that Λ^n is orthogonal to Λ^m for $n \neq m$. So the bilinear form $\langle , \rangle$ respects the grading $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$.

Proposition – the scalar product $\langle , \rangle$ is symmetric, i.e. $\langle f, g \rangle = \langle g, f \rangle$ $\forall f, g \in \Lambda$.

Proof – Because $\langle , \rangle$ is bilinear, it is enough to prove that $\langle f, g \rangle = \langle g, f \rangle$ $\forall f, g$ in a basis $\mathcal{B}$ of Λ.
Choose $\mathcal{B} = \{h_\lambda : \lambda \text{ partition}\}$. Then we can write:

- $\langle h_\lambda, h_\mu \rangle = \langle h_\lambda, \sum_\rho c_{\mu\rho} m_\rho \rangle = \sum_\rho \underbrace{\langle h_\lambda, m_\rho \rangle}_{=\delta_{\lambda\rho}} c_{\mu\rho} = c_{\mu\lambda}$

and

- $\langle h_\mu, h_\lambda \rangle = \langle h_\mu, \sum_\rho c_{\lambda\rho} m_\rho \rangle = \sum_\rho \underbrace{\langle h_\mu, m_\rho \rangle}_{\delta_{\mu\rho}} c_{\lambda\rho} = c_{\lambda\mu}$,

because $\langle h_\alpha, m_\beta \rangle = \delta_{\alpha,\beta}$ for every pair of partitions α and β

We have already observed that

$$c_{\mu\lambda} = \boxed{\text{numer of } \{0,1\}\text{-matrices with row-sums } \lambda \text{ and column-sums } \mu} = c_{\lambda\mu}.$$

Hence $\langle h_\lambda, h_\mu \rangle = \langle h_\mu, h_\lambda \rangle \quad \forall \lambda, \mu$.

$\Rightarrow \langle , \rangle$ is symmetric. $\square$

We also want to show that $\langle , \rangle$ is positive definite.

Notice that it is enough to prove that there exists a basis $\{f_\lambda\}$ of Λ which is self-dual w.r.t. $\langle , \rangle$:

if $\langle f_\lambda, f_\mu \rangle = \delta_{\lambda,\mu} \quad \forall \lambda, \mu$, then we can write

$$\langle f, f \rangle = \langle \sum_\lambda a_\lambda f_\lambda, \sum_\mu a_\mu f_\mu \rangle = \sum_\lambda a_\lambda^2 \geq 0$$

and $\langle f, f \rangle = 0 \iff a_\lambda = 0 \quad \forall \lambda$.

Once we prove the existence of a self-dual basis, we will have that $\langle , \rangle$ is a well defined scalar product on Λ.

PLAN -

1 Understand the concept of "self-dual basis".

We will show that two basis $\{u_\lambda\}, \{v_\nu\}$ of Λ are dual
$\iff \langle u_\lambda, v_\nu \rangle = \delta_{\lambda\nu} \ \forall \lambda, \nu \iff \prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda u_\lambda(x) v_\lambda(y)$.

So a self-dual basis $\{f_\lambda\}$ is a basis of Λ that satisfies the condition

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda f_\lambda(x) f_\lambda(y).$$

2 Give an alternative definition of Shur functions

3 Prove that the Shur functions satisfy the Cauchy formula:

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

It will follow that the "new" Shur functions $\{s_\lambda\}$ are a self-dual basis of Λ. Hence $\langle , \rangle$ is a scalar product.

4 Prove that the new definition of Shur functions is equivalent to the old one.

STEP $\boxed{1}$ "Understanding self-dual basis of Λ".

Definition Two basis $\{v_\mu\}, \{u_\lambda\}$ of Λ are dual if and only if $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$, $\forall \lambda, \mu$.

Lemma $\{u_\lambda\}$ and $\{v_\mu\}$ are dual if and only if

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda u_\lambda(x) v_\lambda(y).$$

proof- Let S be the matrix with entries

$$S_{\rho\nu} = \langle u_\rho, v_\nu \rangle.$$

Then the basis $\{u_\lambda\}$ and $\{v_\mu\}$ are dual if and only if $S = I$.

Let's try to compute the matrix S.

For every partition λ and μ, write

$$m_\lambda = \sum_\rho a_{\lambda\rho} u_\rho$$

$$h_\mu = \sum_\nu b_{\mu\nu} v_\nu$$

and consider the matrices A and B with entries $\{a_{\lambda\rho}\}, \{b_{\mu\nu}\}$.
Then

$$\delta_{\lambda\mu} = \langle m_\lambda, h_\mu \rangle \underset{\substack{\uparrow \\ \langle,\rangle \text{ bilinear}}}{=} \sum_{\rho,\nu} a_{\lambda\rho} b_{\mu\nu} \langle u_\rho, v_\nu \rangle =$$

$$= \sum_\rho A_{\lambda\rho} \left[\sum_\nu \langle u_\rho, v_\nu \rangle b_{\mu\nu} \right] = \sum_\rho A_{\lambda\rho} \left[\sum_\nu S_{\rho\nu} (B^T)_{\nu\mu} \right] =$$

$$= \sum_{\rho} A_{\lambda\rho} (SB^T)_{\rho\mu} = (ASB^T)_{\lambda\mu}.$$

$$\Rightarrow \quad ASB^T = I.$$

Notice that both A and B are invertible matrices (because they are changes of basis).

$$\Rightarrow \quad S = A^{-1}(B^T)^{-1}.$$

So we can write:

$$\boxed{\{v_\lambda\}, \{v_\mu\} \text{ are dual}} \Leftrightarrow \boxed{S = I} \Leftrightarrow \boxed{A^{-1}(B^T)^{-1} = I}$$

$$\Updownarrow$$

$$B^T A = I.$$

We will show that the condition

$$B^T A = I \quad \left(\Leftrightarrow \sum_\lambda b_{\lambda\nu}\, a_{\lambda\rho} = \delta_{\nu\rho}\right)$$

is equivalent to the condition

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda u_\lambda(x)\, v_\lambda(y).$$

To show the equivalence, we write down the infinite product $\prod_{ij} \frac{1}{1-x_i y_j}$ in an alternative way...

Recall that, by a lemma proved last time,

$$\prod_{ij} \frac{1}{1-x_i y_j} = \sum_\lambda h_\lambda(x)\, m_\lambda(y) = \sum_\lambda m_\lambda(x)\, h_\lambda(y).$$

Therefore we can write:

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda m_\lambda(x)\, h_\lambda(y) =$$

$$= \sum_\lambda \left(\sum_\rho a_{\lambda\rho}\, u_\rho(x)\right)\left(\sum_\nu b_{\lambda\nu}\, v_\nu(y)\right) =$$

$$= \sum_{\rho,\nu} \left(\sum_\lambda b_{\lambda\nu}\, a_{\lambda\rho}\right) u_\rho(x)\, v_\nu(y) =$$

$$= \sum_{\rho,\nu} \left[\sum_\lambda (B^T)_{\nu\lambda} A_{\lambda\rho}\right] u_\rho(x)\, v_\nu(y) =$$

$$= \sum_{\rho,\nu} (B^T A)_{\nu\rho}\, u_\rho(x)\, v_\nu(y).$$

Because $\{u_\rho\}$ and $\{v_\nu\}$ are basis, $\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\rho u_\rho(x) v_\rho(y)$ if and only if $(B^T A)_{\nu\rho} = \delta_{\nu\rho}$ $\forall \nu, \rho \iff B^T A = I$.

We obtain:

$$\{u_\lambda\}, \{v_\mu\} \text{ are dual} \iff \prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\rho u_\rho(x)\, v_\rho(y),$$

as claimed. $\square$

STEP 2 Give an alternative definition of Shur functions

For the moment, we work in the polynomial ring $\mathbb{Q}[x_1 \dots x_n]$, i.e. we restrict the number of variables to n.

For every sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ of n integers ≥ 0, let x^α be the monomial

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

and let

$$a_\alpha(x_1 \dots x_n) = \det(x_i^{\alpha_j})_{i,j=1 \dots n} = \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \dots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \dots & x_2^{\alpha_n} \\ \vdots & \vdots & & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \dots & x_n^{\alpha_n} \end{pmatrix} =$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x_1^{\alpha_{\sigma(1)}} x_2^{\alpha_{\sigma(2)}} \dots x_n^{\alpha_{\sigma(n)}} =$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\alpha)}.$$

Notice that a_α changes sign if two of the α_i's are interchanged (indeed this corresponds to switching two columns of the matrix $(x_i^{\alpha_j})$). In particular, $a_\alpha = 0$ if two α_i's are equal.

So we restrict the attention to sequences $\alpha \in \mathbb{N}^n$ which have unequal parts.

Given such a sequence α, we can reorder the entries so that $\alpha_{j_1} > \alpha_{j_2} > \dots > \alpha_{j_n}$. This corresponds to reordering the columns of the matrix $(x_i^{\alpha_j})$, so the determinant only changes by a factor (-1).

$\Rightarrow$ Up to sign, we can assume that $\alpha \in \mathbb{N}^n$ is a sequence that satisfies $\alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0$ (ie. a partition of $N = \sum_{i=1}^n \alpha_i$ with strictly decreasing entries). Notice that every such sequence can be written in the form

$$\alpha = \lambda + \delta$$

where $\delta = (n-1, n-2, \dots, 1, 0)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) =$
$= (\alpha_1 - (n-1), \alpha_2 - (n-2), \dots, \alpha_{n-1} - 1, \alpha_n)$. Here λ is again partition (ie. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$) but the parts of λ are not necessarily distinct.

<u>Example</u>: $n = 4$

$\alpha = (5, 3, 2, 0) \leftarrow$ distinct parts

$\delta = (3, 2, 1, 0)$

$\lambda = (2, 1, 1, 0) \leftarrow$ not necessarily distinct parts

We are interested in the functions

$$a_\alpha = a_{\lambda+\delta}(x_1, \dots, x_n)$$

where λ is <u>any partition in at most n parts</u>

$(\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$.

Notice that a_α is a homogeneous skew-symmetric function in $x_1 \dots x_n$ of degree $\sum_{i=1}^{n} \alpha_i$.

When $\lambda = 0$, we call a_δ <u>the Vandermonde determinant</u>

$$a_\delta = \det(x_i^{\delta_j})_{i,j=1\dots n} = \det(x_i^{n-j}) = \det \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2 & 1 \\ \vdots & & & & \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n & 1 \end{vmatrix}$$

<u>Claim</u> $a_\delta(x_1 \dots x_n) = \prod_{i<j} (x_i - x_j)$.

<u>Proof</u>- Notice that a_δ vanishes if you set $x_i = x_j$ (because two rows of the matrix (x_i^{n-j}) become equal).

$\Rightarrow a_\delta(x_1 \dots x_n)$ is divisible by $(x_i - x_j)$ $\forall i \neq j$

$\Rightarrow a_\delta(x_1 \dots x_n)$ is divisible by $\prod_{i<j}(x_i - x_j)$.

Also notice that

- $a_\delta(x_1 \dots x_n)$ is a homogeneous polynomial of degree $(n-1) + (n-2) + (n-3) + \dots + 2 + 1 + 0 = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$
- $\prod_{i<j}(x_i - x_j)$ is also a homogeneous polyn. of degree

$\frac{n^2-n}{2}$

$\Rightarrow \exists\, c$ s.t. $a_\delta(x_1 \ldots x_n) = c \prod_{i<j} (x_i - x_j)$.

Comparing the coefficients of

$$x_1^{n-1}\, x_2^{n-2} \cdots x_{n-1}^{1}\, x_n^{0}$$

in the two polynomials, you see that $c = 1$.

Hence $a_\delta(x_1 \ldots x_n) = \prod_{i<j} (x_i - x_j)$, as claimed. $\square$

<u>Remark</u>- For all $\alpha = \lambda + \delta$ (λ: any partition in at most n parts), $a_\alpha = a_{\lambda+\delta}$ is divisible by a_δ.

Indeed $a_\alpha(x_1 \ldots x_n) = a_{\lambda+\delta}(x_1 \ldots x_n) = \det\left(x_i^{\lambda_j + n - j}\right)$

vanishes if $x_i = x_j$.

$\Rightarrow a_\alpha(x_1 \ldots x_n)$ is divisible by $\prod_{i<j}(x_i - x_j) = a_\delta$ (in the polynomial ring $\mathbb{Q}[x_1 \ldots x_n]$).

Define
$$s_\lambda(x_1 \ldots x_n) = \frac{a_{\lambda+\delta}(x_1 \ldots x_n)}{a_\delta(x_1 \ldots x_n)}.$$

Because both $a_{\lambda+\delta}$ and a_δ are skew-symmetric functions, the quotient s_λ is a symmetric function.

is an isomorphism, so the "algebraically defined" s_λ's (where λ is a partition in at most n parts) are a basis of the space of symm. fns in $x_1 \dots x_n$.

<u>NOTE</u> - [Defining $\tilde{s}_\lambda$] If $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a monomial in s_λ with coefficient c_α, then every monomial $x_{j_1}^{\alpha_1} \cdots x_{j_n}^{\alpha_n}$ (with $(j_1, \dots, j_n) \in \mathbb{N}^n$) should appear in $\tilde{s}_\lambda$ with the same coefficient. If you construct $\tilde{s}_\lambda$ according to this rule, you obtain a symmetric function with the property that $\tilde{s}_\lambda(x_1 \dots x_n\, 0\, 0 \dots 0 \dots) = s_\lambda(x_1 \dots x_n)$. $\tilde{s}_\lambda$ is uniquely determined by s_λ.