Shur functions

[A combinatorial description]

Let λ be a partition of n, and let T be a generalized semistandard Young Tableau (SSYT) of shape λ.
To obtain T, we fill up the boxes of the Ferrer diagram of λ with positive integers $1, 2, \ldots, n, \ldots$ in a way that entries increase weakly along rows and strictly along columns. Notice that we allow the entries to be arbitrarily big, so for instance

$$T = \begin{array}{|c|c|}\hline 19 & 19 \\ \hline 23 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

is a well defined generalized semistandard Young tableau (SSYT) of shape $(2,1)$.

To each SSYT $\boxed{T}$ we associate a composition $\boxed{\mu_T}$ $(\mu_T = (\mu_1, \mu_2, \ldots))$ by setting

$$\mu_j = \#\text{ of entries of } T \text{ that are equal to } j$$
$$= \#\text{ of occurrences of } j \text{ in } T.$$

In our example

$$T = \begin{array}{|c|c|}\hline 19 & 19 \\ \hline 23 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \Rightarrow \mu_T = (\underbrace{0, \ldots, 0}_{18}, \underset{\substack{\uparrow \\ 19}}{2}, 0, 0, 0, \underset{\substack{\uparrow \\ 23}}{1}, 0, 0, \ldots).$$

Given μ_T, we set

$$x^T = x_1^{\mu_1} x_2^{\mu_2} \cdots$$

and we call $\boxed{x^T}$ the monomial associated to the SSYT $\boxed{T}$. So $x_{19}^2 x_{23}$ is the monomial associated to $\begin{array}{|c|c|}\hline 19 & 19 \\ \hline 23 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$.

Notice that if $\lambda \vdash n$, then every monomial x^T has degree n.

<u>Definition</u> The Shur function s_λ associated to a partition λ of n is

$$s_\lambda = \sum_T x^T.$$

The sum is over all SSYT of shape λ.

<u>EXAMPLE 1</u> : $\lambda = (1^n)$. A SSYT of shape λ is of the form

$$T = \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array} \quad \text{with} \quad i_1 < i_2 < \dots < i_n.$$

Hence $s_\lambda = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = e_n(x) \Rightarrow \boxed{s_{(1^n)} = e_n}$

<u>EXAMPLE 2</u> : $\lambda = (n)$. A SSYT of shape λ is of the form

$$T = \begin{array}{|c|c|c|c|} \hline j_1 & j_2 & \cdots & j_n \\ \hline \end{array} \quad \text{with } j_1 \leq j_2 \leq \dots \leq j_n.$$

Hence $s_\lambda = \sum_{j_1 \leq j_2 \leq \dots \leq j_n} x_{j_1} x_{j_2} \cdots x_{j_n} = \sum$ all monomials of degree n

$= h_n(x)$.

$\Rightarrow \boxed{s_{(n)} = h_n}$.

Let's try to understand the general form of s_λ.

- For every $\lambda \vdash n$, the coefficient of $x_1 \dots x_n = x^{(1^n)}$ in s_λ is equal to the number of standard Young tableau of shape λ (so it's the dimension of the Specht module S^{λ}). Indeed, every SSYT of content (1^n) is actually a standard Young Tableau.

- If $\alpha = (\alpha_1, \alpha_2, \dots)$ is any composition of n, i.e. any sequence of non negative integers whose sum is n, then the monomial $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \dots$ appears in s_λ with coefficient

$$K_{\lambda\alpha} = \#\text{ of SSYT of shape } \lambda \text{ and content } \alpha.$$

[A SSYT T is said to have content α if for all $j \geq 1$, j occurs in T with multiplicity α_j].

Then we can write:

$$s_\lambda = \sum_{\substack{\text{all compositions} \\ \alpha \text{ of } n}} K_{\lambda\alpha} x^{\alpha}.$$

<u>Remark</u>: This is a combinatorial description of the Shur functions. It does not show the fact that every s_λ is symmetric, but it's a very easy and natural definition. We will give later another analytical definition of Shur functions, which is more complicated, but has the advantage of showing the symmetric nature of s_λ.

<u>Theorem</u>- For every partition $\lambda \vdash n$, s_λ is a symmetric function.

<u>Proof</u>- It is sufficient to prove that

$$(i\ i{+}1)\, s_\lambda = s_\lambda$$

for all $i \geq 1$.

If we write $s_\lambda = \sum_{\substack{\text{all compos.}\\ \alpha \text{ of } n}} K_{\lambda\alpha} x^\alpha$, this accounts to prove that

$$K_{\lambda\alpha} = K_{\lambda\,(i\ i+1)\alpha} \qquad (*)$$

for every $i \geq 1$ and every composition α of n.

Notice that if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \alpha_{i+2}, \dots)$, then $(i\ i{+}1)\,\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \underbrace{\alpha_{i+1}, \alpha_i}, \alpha_{i+2}, \dots)$. Basically, we just interchange the i^{th} and $(i+1)^{th}$ entries of the composition.

Recall that the Kostka number $K_{\lambda\alpha}$ counts the number of SSYT of shape λ and content α, so to prove $(*)$ we just need to exhibit a bijection

$$\Theta : \left\{ \begin{matrix} \text{SSYT of shape } \lambda \\ \text{and content } \alpha \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{SSYT of shape } \lambda \\ \text{and content } (i\ i{+}1)\alpha \end{matrix} \right\}.$$

If T is a SSYT of content $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_i, \alpha_{i+1}, \dots)$, then ΘT should be a SSYT of shape $(\alpha_1, \alpha_2, \dots \dots \alpha_{i+1}, \alpha_i, \dots)$. So Θ should interchange the number of i's and the number of $(i{+}1)$'s inside T, and leave the number of occurrences of every other integer fixed. It's not hard to construct a bijection Θ

with these properties.

If T is a SSYT of shape λ, and content α then every column of T contains at most one i and at most one $(i+1)$.

If a column contains a pair $(i\ i{+}1)$, we call this pair "FIXED".

If a column contains only i or only $i+1$, we call this entry "FREE".

If k is the number of fixed pairs, then

$$\alpha_i = k + \#\text{free } i\text{'s}$$

$$\alpha_{i+1} = k + \#\text{free } (i+1)\text{'s}.$$

So, in order to interchange α_i and α_{i+1}, we just need to interchange the number of free i's and the number of free $(i+1)$'s.

With this in mind, we let Θ act as follows:

if a <u>row</u> of T contains a sequence

$$\underbrace{i\ i \dots i}_{\ell}\ \ \underbrace{i{+}1\ \ i{+}1 \dots i{+}1}_{k}$$

of "<u>free</u>" i's and "<u>free</u>" $(i+1)$'s, we replace this sequence with

$$\underbrace{i\ i \dots i}_{k}\ \ \underbrace{i{+}1\ \ i{+}1 \dots i{+}1}_{\ell}.$$

Every other entry of T (included the "fixed" i's and the "fixed" $(i+1)$'s) is fixed by Θ.

Call T' the new Tableau. It's clear that the rows of T' are weakly increasing.

The fact that the columns of T' are strongly increasing is less obvious, and it depends on the fact that we only move the free i's.

So if a column $\begin{array}{|c|}\hline a \\ \hline b \\ \hline \vdots \\ \hline c \\ \hline i \\ \hline d \\ \hline \vdots \\ \hline e \\ \hline \end{array}$ is replaced by $\begin{array}{|c|}\hline a \\ \hline b \\ \hline \vdots \\ \hline c \\ \hline i+1 \\ \hline d \\ \hline \vdots \\ \hline e \\ \hline \end{array}$

Then $a < b < \dots < c < i < i+1 \underset{\substack{\uparrow \\ \text{because } i \text{ is free, so} \\ \text{there's no } (i+1) \text{ in the column}}}{<} d < \dots < e$, and the column remains increasing.

Example : $\lambda = (6,5,1)$; $\alpha = (4,3,4,1)$, $i = 1$ and

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array}.$$

then are fixed pairs. Only the 3^{nd}, 4^{th} and 5^{th} column of T contain free i's and free 2's.

We replace the sequence $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array}$ of free 1's and 2's in the 1^{st} row by the sequence $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \end{array}$ (switching the number of free 1's and free 2's), and

... we define

$$T' = \Theta T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array}$$

Clearly T' is still semistandard.

Let's go back to our proof. The map Θ

$$\left\{ \begin{array}{l} \text{SSYT of shape } \lambda \\ \text{and content } \alpha \end{array} \right\} \xrightarrow{\Theta} \left\{ \begin{array}{l} \text{SSYT of shape } \lambda \\ \text{and content } (i\ i{+}1)\alpha \end{array} \right\}$$

$$T \longmapsto T'$$

is well defined. Θ is bijective (it's actually an involution), so the two sets must have the same cardinality.

$\Rightarrow K_{\lambda\alpha} = K_{\lambda (i\, i+1)\alpha}$, and s_λ is symmetric. $\square$

Using the symmetry of s_λ, we can write:

$$s_\lambda = \sum_{\substack{\text{all composition} \\ \alpha \text{ of } n}} K_{\lambda\alpha}\, x^\alpha =$$

$$= \sum_{\substack{\text{all partitions} \\ \mu \text{ of } n}} K_{\lambda\mu} \Bigg(\sum_{\substack{\alpha:\ \text{distinct} \\ \text{permutations of } \mu}} x^{\alpha} \Bigg)$$

$$= \sum_{\mu \vdash n} K_{\lambda\mu}\, m_\mu .$$

$$\boxed{s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu}\, m_\mu}$$

To simplify this expression, we notice that if $K_{\lambda\mu} \neq 0$ then $\mu \trianglelefteq \lambda$ (in the dominance order).

Indeed, if T is any SSYT of shape λ and content μ (with λ and μ partitions of n), then T contains μ_1 one's, μ_2 two's and so on. Because the columns of T must increase strictly, all the entries $\leq i$ must sit in the first i rows.

$$\Longrightarrow \quad \underbrace{\mu_1 + \mu_2 + \dots + \mu_i}_{\text{number of entries } \leq i} \leq \underbrace{\lambda_1 + \lambda_2 + \dots + \lambda_i}_{\substack{\text{number of boxes in the} \\ \text{first } i \text{ rows}}}$$

This is true for all $i \geq 1$.

Hence, by definition of dominance order, $\mu \trianglelefteq \lambda$.

$\Longrightarrow$ we can also write $\quad s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} m_\mu$.

Finally, we observe that if $\lambda = \mu$, then the Kostka number $K_{\lambda\lambda}$ is equal to one. Indeed there's only one SSYT of shape λ and content λ:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 \cdots & 1 & \cdots & 1 \\
\hline
2 & 2 & 2 & 2 \cdots & 2 \\
\cline{1-5}
\vdots & \vdots & \vdots & \vdots \\
\cline{1-4}
\ell & \ell & \cdots & \ell \\
\cline{1-4}
\end{array} \qquad \lambda = (\lambda_1, \dots, \lambda_\ell).$$

<u>Corollary</u>: $\boxed{s_\lambda = m_\lambda + \sum_{\mu \triangleleft \lambda} K_{\lambda\mu} m_\mu}$

· Choose an ordering of the partitions of n which is compatible with the partial order $\trianglelefteq$ (for instance the lexicographical order).

Then the system

integer ≥ 0

$$\left\{ s_\lambda = 1\, m_\lambda + \sum_{\mu \triangleleft \lambda} K_{\lambda\mu}\, m_\mu \right\}_{\lambda \vdash n}$$

is triangular, with one's on the diagonal.

$\Rightarrow$ we can solve this system, and express each m_λ as a linear combination of Shur fenctions

(with integer coefficients).

$\Rightarrow \{s_\lambda\}_{\lambda \vdash n}$ is another basis of Λ^n.

$\Rightarrow$ The set $\{s_\lambda\}_{\text{all partitions}}$ is another basis of Λ.