"SHUR'S LEMMA"

LECTURE 2

2.1 Invariant subspaces

Let $\rho: G \to GL(V)$ be a representation of G in V, and let $W \subseteq V$ be a subspace of V. We say that W is "invariant" (or "stable") under the action of G if $\rho(s) W \subseteq W$ for all s in G.

▶ Remark - If $W \subseteq V$ is G-stable, the function

$$\rho^W : G \to GL(W), \; g \mapsto \rho^W(g) \equiv \rho(g)|_W$$

defines a representation of G on W.
We call (ρ^W, W) a "subrepresentation" of (ρ, V).

▶ Examples

① For every representation (ρ, V) of G, the subspaces $\{0\}$ and V of V are G-stable.

② Let ρ be the regular representation on G on $V = \bigoplus_{g \in G} \mathbb{C} e_g$.
Let W be the 1-dimensional subspace of V spanned by

$$w = \sum_{g \in G} e_g .$$

Because
$\rho(s) w = \sum_{g \in G} \rho(s) e_g = \sum_{g \in G} e_{sg} = \sum_{t \in G} e_t = w$, the subspace $W = \mathbb{C} w$ is G-stable. Notice that the subrepresentation

(ρ^W, W) is the trivial representation (of degree 1).

---

③ Let (ρ, V) be a representation of G. We discuss some G-invariant subspaces of $(\rho\otimes\rho, V\otimes V)$.

Consider the linear transformation

$$\theta : V\otimes V \longrightarrow V\otimes V, \quad v\otimes w \longmapsto w\otimes v,$$

Then $\theta^2 = \mathbb{1}_{V\otimes V}$ and θ commutes with the action of G:

$$\theta\left((\rho\otimes\rho)(s)(v\otimes w)\right) = \theta(\rho(s)v \otimes \rho(s)w) =$$

$$= (\rho(s)w \otimes \rho(s)v) = (\rho\otimes\rho)(s)\left(\theta(v\otimes w)\right).$$

It follows that every $\rho\otimes\rho(s)$ preserves the eigenspaces of θ,

Let $Sym^2(V)$ be $(+1)$-eigenspace of θ, and $Alt^2(V)$ be the (-1)-eignspace. The decomposition

$$V\otimes V = Sym^2(V) \oplus Alt^2(V)$$

is a decomposition of $V\otimes V$ as a direct sum of G-invariant subspaces.

The subrepresentations $\rho\otimes\rho^{Sym^2(V)}$ and $\rho\otimes\rho^{Alt^2(V)}$ of $\rho\otimes\rho$ are called "the symmetric square" and the "alternating square" of the representation ρ.

▸ Remark - Let $\{v_j\}_{j=1..n}$ be a basis of V. Then $\mathcal{B}_1 = \{ v_i\otimes v_j + v_j\otimes v_i : i\leq j, i,j=1..n\}$ is a basis of $Sym^2(V)$

and $\mathcal{B}^2 = \{ v_i \otimes v_j - v_j \otimes v_i : i<j, \, i,j = 1..n \}$ is a basis of $Alt^2(V)$. Hence

- $\dim(Sym^2(V)) = \dfrac{n(n+1)}{2}$
- $\dim(Alt^2(V)) = \dfrac{n(n-1)}{2}$

(with $n = \dim V$).

2.2 Irreducible Representations

Let ρ be a representation of G in V. We say that (ρ, V) is "irreducible", or "simple", if the only G-invariant subspaces of V are $W = \{0\}$ and $W = V$.
In other words, V has no proper non-trivial invariant subspaces.
A "reducible" representation of G is a representation which is not irreducible.

▸ Examples / Non-Examples

① The trivial representation of G of degree n is irreducible only for $n = 1$.

② The regular representation of G is always reducible.

2.3 G-homomorphisms

Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G, and let

$T: V_1 \to V_2$ be a linear map of V_1 in V_2. We say that T is a "G-homomorphism" (or that T is "G-linear") if for every $s \in G$ the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(s)} & V_1 \\ T \downarrow & & \downarrow T \\ V_2 & \xrightarrow{\rho_2(s)} & V_2 \end{array}$$

is commutative (i.e. $\rho_2(s) \circ T = T \circ \rho_1(s)$).

We also call T an "intertwining operator" from (ρ_1, V_1) to (ρ_2, V_2).

▸ Notations - The set of all G-homomorphisms from V_1 to V_2 is denoted by $\mathrm{Hom}_G(V_1, V_2)$, and it's a complex vector space.

▸ Remark: We can use G-homomorphisms to study the reducibility of representations, in virtue of the following lemma:

Lemma - Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G, and let $T: V_1 \to V_2$ be a G-homomorphism. Then

(a) $\mathrm{Ker}(T)$ is an invariant subspace of V_1, and

(b) $\mathrm{Im}(T)$ is an invariant subspace of V_2.

proof - Very easy: (a) For all v_1 in $\mathrm{Ker}(T)$ and all s in G,

$T(\rho_1(s) v_1) = (\rho_2(s) T)(v_1) = \rho_2(s)(T(v_1)) = \rho_2(s)\, \underline{0} = \underline{0} \Rightarrow \rho_1(s) v \in \mathrm{Ker}$

(b) For all $v_2 = T(v_1)$ in $\mathrm{Im}\,T$ and all s in G,

$\rho_2(s) v_2 = \rho_2(s)(T(v_1)) = (T \rho_1(s)) v_1 = T(\rho_1(s) v_1) \Rightarrow \rho_2(s) v_2 \in \mathrm{Im}\,T$. ▨

The following lemma is very simple, but crucial.

▸ <u>Schur's lemma</u> - Let $T: V \to W$ be a G-homomorphism between irreducible representations of positive degree of G. Then

(a) Either $T=0$ or T is an isomorphism.

(b) If $V=W$, then T is a multiple of the identity.

▸ <u>proof</u> - (a) By the previous lemma, $\ker T \subseteq V$ and $\operatorname{Im} T \subseteq W$ are G-invariant subspaces. Because both V and W are irreducible, only 2 possibilities hold:

- $\ker T = \{0\}$, $\operatorname{Im} T = W \Rightarrow T$ is an isomorphism
- $\ker T = W$, $\operatorname{Im} T = \{0\} \Rightarrow T = 0$. ✓

[the cases "$\ker T = \{0\}$, $\operatorname{Im} T = \{0\}$" and "$\ker T = W$, $\operatorname{Im} T = W$" are excluded from the fact that both V and W have positive dimension].

(b) Now assume that $V=W$. Because V is a complex vector space, the homomorphism $T: V \to V$ has at least one complex eigenvalue λ. We will show that $T = \lambda \mathbb{1}_V$.

For all s in G, we can write:

$$(T - \lambda I_V)\,\rho(s)(v) = T\rho(s)(v) - \lambda\,\rho(s)(v) = (\rho(s) - \lambda I_V)\,T(v)$$

So $T - \lambda I_V$ is a G-linear map from V to V.

Clearly $(T - \lambda I_V)$ has a non-zero kernel (= the λ-eigenspace of T),

so $(T-\lambda I_V)$ is not an isomorphism.
It follows from part (a) that $(T-\lambda I_V)=0$, so T is a multiple of the identity. ∎

▶ <u>Remark</u> - The first part of the lemma generalizes to representations over non-algebraically closed fields, but the second part does not.

<u>An application:</u>

▶ Let $G = D_8 = \langle a, b : a^4 = b^2 = 1 ; b^{-1}ab = a^{-1} \rangle$.
Consider the representation ρ of G defined by:

$$\rho(a) = \begin{pmatrix} 5 & -6 \\ 4 & -5 \end{pmatrix} ; \quad \rho(b) = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}.$$

[ρ is a representation of G of degree 2, and it's well defined because the matrices $A = \begin{pmatrix} 5 & -6 \\ 4 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}$ satisfy the conditions $A^4 = B^2 = I$, $B^{-1}AB = A^{-1}$.]

<u>Problem</u> : To show that ρ is reducible

By Schur's lemma, it's enough to exhibit a non-constant G-homomorphism from $\mathbb{C}^2$ to $\mathbb{C}^2$ (call it $T: x \mapsto Cx$).
Because G is generated by a and b, this problem reduces to finding a 2×2 matrix C, which is NOT a scalar matrix, and which commutes with both $A=\rho(a)$ and $B=\rho(b)$. A little bit of algebraic manipulation shows that if C satisfies the conditions $\begin{cases} AC = CA \\ BC = CB \end{cases}$, then $C = \begin{pmatrix} \alpha & 3\beta \\ -2\beta & \alpha+5\beta \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{C}$. Clearly $\dim_{\mathbb{C}}(\mathrm{Hom}_G(V,V)=2)$, so ρ can't be irreducible ∎

Important remarks

Schur's lemma completely characterizes the space of G-homomorphisms between 2 irreducible representations (ϱ, V), (ρ, W):

$$\operatorname{Hom}_G(V, W) = \begin{cases} \{0\} & \text{if } V \not\cong W \\ \mathbb{C} & \text{if } V \cong W. \end{cases}$$

Indeed, let $T: V \to W$ be a G-homomorphism.
Assume that V is inequivalent to W. Then T can't be an isomorphism (or $V \cong W$), and T must be 0.
Viceversa, assume that $V \cong W$, and fix any nonzero G-homomorphism $T_0 : V \to W$. To show that $\operatorname{Hom}_G(V, W)$ is one-dimensional, we prove that T is a multiple of T_0.
Look at the composition $T_0^{-1} \circ T : V \to V$.
Then $T_0^{-1} \circ T$ is a G-homomorphism from V to V

(because $T_0^{-1} T \varrho(g) v = T_0^{-1} \rho(g) T(v) = \varrho(g) T_0^{-1} T(v)$ $\forall v \in V, g \in G$).

It follows from Schur's lemma that $T_0^{-1} T = c \mathbb{1}_V$ for some $c \in \mathbb{C}$. Hence $T = c T_0$, as claimed. ∎

IMPORTANT: For irreducible representations
$$\dim_G \operatorname{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \not\cong W \\ 1 & \text{if } V \cong W \end{cases}$$

Next, we discuss some important corollaries of Shur's lemma.

▶ Corollary 1 - Let ρ be an IRREDUCIBLE representation of G in V, and let z be a central element in G. Then there exist a complex number λ s.t. $\rho(z)v = \lambda v$ for all v in V.

proof - If z belongs to the center of G, then $\rho(z)$ is an intertwining operator between ρ and itself. Indeed for all $g \in G$ we have a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\rho(z)"'] & V \arrow[d, "\rho(z)"] \\ V \arrow[r, "\rho(g)"] & V \end{tikzcd} \iff \underbrace{\rho(z)\rho(g)}_{\rho(zg)} = \underbrace{\rho(g)\rho(z)}_{\rho(gz)}.$$

It follows from Schur's lemma, and from the irreducibility of ρ, that $\rho(z)$ must be a multiple of the identity. So the claim follows. ■

▶ Corollary 2 - Every irreducible representation of an abelian group is 1-dimensional.

proof - Because $Z(G) = G$, every element of G acts on V by a scalar. It follows that every 1-dimensional subspace of V is G-stable, and that V is irreducible if and only if it has dimension one. ■

. Example - Find all the irreducible representations of C_3 (= the cyclic group of order 3).

▸ Let (ρ, V) be an irreducible representation. Then V is 1-dimensional and ρ is given by a group homomorphism $\rho: G \to \mathbb{C}^*$. Call "a" a generator of G. Then ρ is completely determined by the value of $\rho(a)$.
We notice that, because a has order 3, $\rho(a)$ must be a cubic root of unit. This gives us only 3 possibilities:

$$\rho_1 : G \to \mathbb{C}^*, \quad a \mapsto 1 \quad \text{(trivial representation)}$$

$$\rho_2 : G \to \mathbb{C}^*, \quad a \mapsto e^{2\pi i/3}$$

$$\rho_3 : G \to \mathbb{C}^*, \quad a \mapsto e^{4\pi i/3}.$$

these three representations are inequivalent, and exhaust all the irreducible representations of C_3. ∎

For a generalization of this example, see the classification of irreducible representation of any finite abelian group (at the end of the lecture).

▸ Corollary 3 - If G has a faithful irreducible representation, then the center of G is cyclic.

▷ proof - Let ρ be an irreducible faithful representation of G in V. For each element $z \in Z(G)$ there exists a complex number λ_z s.t. $\rho(z) = \lambda_z \mathbb{1}_V$ (This constant λ_z is non-zero because $\rho(z)$ must be invertible).
We obtain a map $\lambda : Z(G) \to \mathbb{C}^*$, $z \mapsto \lambda_z$, that is also a group homomorphism. Because ρ is faithful, λ is injective; hence $Z(G) \cong \lambda(G) \leq \mathbb{C}^*$. The claim

in the corollary follows from the fact that every finite subgroup of $\mathbb{C}^*$ (or, if you prefer, S^1) is cyclic, and generated by a root of unity. ∎

- **Application**: $C_2 \times D_8$ does not have any faithful irreducible representation.

[$Z(D_8) \cong C_2$, so $Z(C_2 \times D8) = C_2 \times C_2$ is not cyclic]

write $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

then $Z(D_8) = \{1, a^2\}$.

Suppose $|H| = n$ and $H \leq \mathbb{C}^*$.
Denote by U_n the subgroup of $\mathbb{C}^*$ consisting of the n^{th}-roots of 1. Then $U_n \leq \mathbb{C}^*$, U_n is cyclic and $|U_n| = n$.
For all $h \in H$, $h^n = 1 \Rightarrow h \in U_n$.
So $H \subseteq U_n$, but they have both cardinality n
$\Rightarrow H = U_n \Rightarrow H$ is cyclic.

Representations of finite abelian groups.

Let G be any finite abelian group.

$\Rightarrow G \cong C_{n_1} \times C_{n_2} \times \dots \times C_{n_r}$

(every finite abelian group is isomorphic to a direct product of cyclic groups).

For all $j = 1..r$, let a_j be a generator of C_{n_j}, and set

$$g_j = (1, 1, \dots, a_j, 1, \dots, 1) \in G.$$

$\uparrow$ position j

Then $\{g_1, \dots, g_r\}$ generate G. They satisfy the conditions $g_j^{n_j} = 1$ and $g_j g_i = g_i g_j \ \forall i,j = 1..r$.

Let $\varphi : G \to \mathbb{C}^*$ be any irreducible representation of G ($\Leftarrow$ 1-dimensional, by Schur's lemma). Then φ is completely determined by the value of $\varphi(g_j)$, for all $j = 1..r$.

Because g_j has order n_j in G, $\varphi(g_j)$ must be an n_j^{th} root of unity.

For each choice of $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ satisfying $\lambda_j^{n_j} = 1$, we obtain a well defined irreducible representation of G, carrying every g_j into λ_j, so carrying

$$g = g_1^{h_1} g_2^{h_2} \cdots g_r^{h_r} \longmapsto \lambda_1^{h_1} \lambda_2^{h_2} \cdots \lambda_r^{h_r}.$$

Because there are exactly n_j choices for every λ_j, we obtain exactly $n_1 n_2 \cdots n_r = |G|$ irreducible inequivalent representations.

Examples: Representations of $C_2 \times C_2$

There are 4 irreducible inequivalent representations:

$$\rho : C_2 \times C_2 \to \mathbb{C}^*, \quad a^h b^h \longmapsto \lambda_1^h \lambda_2^h$$

with $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$.

	ρ_1	ρ_2	ρ_3	ρ_4
$a \mapsto$	1	-1	1	-1
$b \mapsto$	1	-1	-1	1
$ab \mapsto$	1	1	-1	-1
$1 \mapsto$	1	1	1	1

You can check that there are no faithful representations, in accordance to the fact that $Z(C_2 \times C_2) = C_2 \times C_2$ is not cyclic.

Remark: If G is a finite abelian group, then every irreducible representation of G is 1-dimensional. We will prove shortly that the reverse is also true: if all the irreducible representations of a finite group G are 1-dim. then G is abelian.

Important Remark:

If you consider representations of a finite group in a vector space which is <u>not</u> algebraically closed, then only the first part of Schur's lemma holds:

(1) If $V \not\cong W \Rightarrow \mathrm{Hom}_G(V, W) = \{0\}$

but

(2) If $V = W \Rightarrow T \in \mathrm{Hom}_G(V, V)$ is not necessarily a multiple of $\mathbb{1}_V$.

Of course, also the corollaries that we discuss are no longer valid. For instance, <u>real</u> representations of an abelian group are no longer (necessarily) 1-dimensional.

COUNTEREXAMPLE

- $G = C_3$ (cyclic group). If a is a generator, we write $G = \{1, a, a^2\}$
- $V = \mathbb{R}\underline{e}_1 \oplus \mathbb{R}\underline{e}_a \oplus \mathbb{R}\underline{e}_{a^2}$

(regular representation <u>over $\mathbb{R}$</u>, not over $\mathbb{C}$)

An element $g \in G$ acts on V by

$$\rho(g)(x_1 \underline{e}_1 + x_2 \underline{e}_a + x_3 \underline{e}_{a^2}) = x_1 \underline{e}_g + x_2 \underline{e}_{ga} + x_3 \underline{e}_{ga^2}$$

Just like in the complex case, the 1-dimensional vector subspace $U = \mathbb{R}\left[\sum_{g \in C_3} e_g\right]$ is G-stable and it defines a 1-dimensional irreducible subrepresentation; but we will show that not every irreducible subrepresentation is one-dimensional.

Look at the subspace

$$W = \mathrm{Span}_{\mathbb{R}}(1 - \underline{e}_a ; 1 - \underline{e}_{a^2}).$$

Claim - W is G-stable

- $\rho(a)(1-\underline{e}_a) = \underline{e}_a - \underline{e}_{a^2} = (\underline{e}_a - 1) + (1-\underline{e}_{a^2}) =$

$= -(1-\underline{e}_a) + (1-\underline{e}_{a^2}) \in W$

- $\rho(a)(1-\underline{e}_{a^2}) = \underline{e}_a - 1 = -(1-\underline{e}_a) \in W$ ✓

Claim - W is irreducible.

Suppose by contradiction that $W = W_1 \oplus W_2$. Then W_1 and W_2 are subrepresentations of W.

If $W_i = \mathrm{Span}(w_i)$, then w_i must be an eigenvector of $\rho(a)$ with real eigenvalue (because you're dealing with real representations). We reach a contradiction, because

$$\underset{\substack{\text{w.r.t. basis} \\ \{1-e_a,\ 1-e_{a^2}\}}}{\rho(a)} \rightsquigarrow \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

only has complex eigenvalues: $e^{2\pi i/3}$, $e^{4\pi i/3}$.

$\Rightarrow$ W is irreducible.

Notice: C_3 is abelian, but it has one real repr. which irreducible and of degree 2

We also notice that $T = \rho(a): W \to W$ is a G-homomorphism from W to itself, and it's NOT trivial (i.e. it's not a multiple of the identity on W).

Summary of Lecture 2

Let G be a finite group, and let (ρ, V) be a representation of G.

1. A subspace W of V is G-invariant if $\rho(s)W \subseteq W$, for all s in G
2. (ρ, V) is irreducible if the only G-invariant subspaces are $\{0\}, V$
3. A G-homomorphism $T: V_1 \to V_2$ between two representations is a linear map s.t. $\rho_2(s) \circ T = T \circ \rho_1(s)$, $\forall s \in G$.
   If T is a G-homomorphism, $\operatorname{Ker} T \subseteq V_1$ and $\operatorname{Im} T \subseteq V_2$ are G-invariant subspaces.
4. If V_1 and V_2 are inequivalent irreducible representations, then every G-homomorphism from V_1 to V_2 is zero.
5. If V is irreducible, every G-homomorphism from V to V is a multiple of the identity.
6. If V_1 and V_2 are equivalent irreducible representations, then every non-zero G-homomorphism from V_1 to V_2 is an isomorphism.
7. $Z(G)$ acts by scalars on any irreducible representation of G.
8. If G is abelian, every irreducible representation of G has degree
9. If G has a faithful irreducible representation, then $Z(G)$ is cyclic

Suggested problems for Lecture 2

1. Find all irreducible representations of $C_2 \times C_2$ and all irreducible faithful representations of C_8.
[C_n = cyclic group of order n].

2. Let $G = C_4 \times C_4$.
(a) Find a non trivial irreducible representation of G s.t. $\rho(g^2) = 1$ for all $g \in G$.
(b) Prove that there is no irreducible representation σ of G s.t. $\sigma(g) = -1$ for all elements g of order 2 in G.

3. Verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\mathrm{Hom}(V,W)^G$ of elements of $\mathrm{Hom}(V,W)$ fixed under the action of G.

4. Construct, if possible, an irreducible faithful representation of the following groups:
(a) C_4
(b) $C_2 \times C_2$
(c) $C_2 \times C_3$.

5. Let $G = S_3$ (the symmetric group on 3 letters). Define a representation of G on $\mathbb{C}^3$ by:

$$\rho(\sigma)(x_1 e_1 + x_2 e_2 + x_3 e_3) = x_1 e_{\sigma(1)} + x_2 e_{\sigma(2)} + x_3 e_{\sigma(3)}$$

[Equivalently, $\rho(\sigma)\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ x_{\sigma^{-1}(3)} \end{vmatrix}$]. Check that ρ is well defined and find all the G-invariant subspaces of $\mathbb{C}^3$.