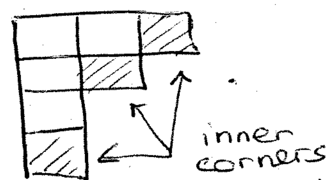
BRANCHING RULES

- We study the restriction of the Specht module S^λ of S_n to S_{n-1}.
[We identify S_{n-1} with the subgroup of S_n consisting of all the partitions that fix n].

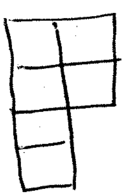
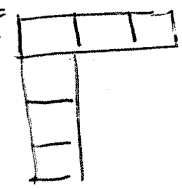
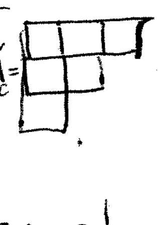
Let λ be a partition of n. Look at the Ferrer diagram T_λ of λ:

$\lambda = (3,2,1,1) \longrightarrow T_\lambda =$



An <u>inner corner</u> of T_λ is a box that sits simultaneously at the end of a row and at the end of a column of T_λ.

- KEY OBSERVATION:
If we remove an inner corner c from T_λ then we obtain a Ferrer diagram for a well defined partition $\tilde{\lambda}_c$ of $n-1$.

inner corner in position $c = (1,3) \longrightarrow \tilde{\lambda}_c =$  ; inner corner in position $c = (2,2) \longrightarrow \tilde{\lambda}_c =$  ; inner corner in position $c = (4,1) \longrightarrow \tilde{\lambda}_c =$ 

Notice that the inner corners are the only boxes of T_λ that can be removed in order to get an $(n-1)$-Tableau.

- ANOTHER KEY OBSERVATION:
If t is any <u>standard</u> Tableau of shape λ, then $\boxed{n}$ can

only sit in an inner corner of λ. Indeed the rows and the columns of t must increase $\Rightarrow$ n must be at the end of a row and at the end of a column.

- at the end of a row and at the end of a column.
Say that n sits in c.

If you remove the inner corner c (with its content $\boxed{n}$) from t, then you obtain a <u>standard</u> tableau $\tilde{t}_c$ of shape $\tilde{\lambda}_c \vdash n-1$.

The map

$$\left\{ \begin{array}{l} \text{standard tableaux } t \text{ of shape } \lambda \vdash n \\ \text{with } n \text{ in the inner corner } \boxed{c} \end{array} \right\} \xrightarrow{\Theta_c} \left\{ \begin{array}{l} \text{standard tableaux} \\ \text{of shape } \tilde{\lambda}_c \vdash n-1 \end{array} \right\}$$

$$t \xrightarrow[\text{remove } c]{} \tilde{t}_c$$

is a bijection. We also notice that

- if $\sigma \in S_{n-1}$ is a permutation of $\{1 \dots n\}$ that fixes n, then

$$\Theta_c(\sigma t) = \sigma \tilde{t}_c .$$

Now we have all the ingredients to describe the restriction of S^λ to S_{n-1}.

Standard polytabloids (of shape λ) are a basis $\mathcal{B}$ of S^λ.

Partition this basis as follows:

- $$\mathcal{B} = \biguplus_{\substack{\text{inner} \\ \text{corners } \boxed{c}}} \{ e_t : t \text{ standard tableau with } n \text{ sitting in the inner corner } \boxed{c} \},$$

We obtain a vector space decomposition:

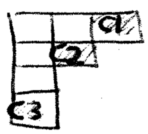
$$S^\lambda = \bigoplus_{\substack{\text{inner} \\ \text{corners } c}} S_c$$

with $S_c = \operatorname{Span}\{e_t : t \text{ standard Tableau with } n \text{ sitting in } \boxed{c}\}$.

Then, for each inner corner c, S_c is stable under the action of S_{n-1} and is "naturally" isomorphic to the Specht module $S^{\tilde\lambda_c}$ of S_{n-1}. An S_{n-1} isomorphism carries $e_t \longrightarrow e_{\tilde t_c}$.

So we get:

$$\boxed{\operatorname{Res}^{S_n}_{S_{n-1}}(S^\lambda) = \bigoplus_{\substack{\text{inner} \\ \text{corners } c}} S^{\tilde\lambda_c}}.$$

<u>Example</u>: If $\lambda =$ 

$= (3,2,1,1)$, Then There are exactly three inner corners. ← partition of $n=7$. Their removal gives rise to the partitions

$$\tilde\lambda_{c_1} = (2,2,1,1)\ ; \quad \tilde\lambda_{c_2} = (3,1,1,1)\ ; \quad \tilde\lambda_{c_3} = (3,2,1)$$

of $n-1=6$.

Hence

$$\operatorname{Res}^{S_7}_{S_6}\left[S^{(3,2,1,1)}\right] = S^{(2,2,1,1)} \oplus S^{(3,1,1,1)} \oplus S^{(3,2,1)}.$$

Using Frobenius reciprocity, we can transfer these "results on restrictions" to "results on inductions".

Let $\lambda \vdash n$. Write $\mathrm{Ind}_{S_n}^{S_{n+1}}(S^\lambda)$ as a direct sum of irreducible representations of S_{n+1}:

$$\mathrm{Ind}_{S_n}^{S_{n+1}}(S^\lambda) = \bigoplus_{\mu \vdash n+1} c_\mu S^\mu .$$

By Frobenius reciprocity

$$c_\mu = \text{multiplicity of } S^\mu \text{ in } \mathrm{Ind}_{S_n}^{S_{n+1}}(S^\lambda) =$$

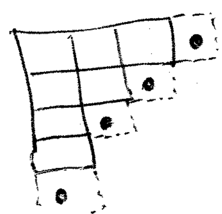
$$= \text{multiplicity of } S^\lambda \text{ in } \mathrm{Res}_{S_n}^{S_{n+1}}(S^\mu) =$$

$$= \begin{cases} 1 & \text{if } \Pi_\lambda \text{ can be obtained from } \Pi_\mu \text{ by removing an inner corner} \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient to introduce the notion of "outer corner"

An outer corner of λ is a node $(i,j) \notin \lambda$ whose addition to Π_λ produces the Ferrer diagram of a partition of $n+1$.

<u>Example</u>: $\lambda = (3,2,1,1)$

$\Pi_\lambda =$  ← there are 4 outer corners.

By adding a box in an outer corner, you obtain the partitions $(4,2,1,1)$; $(3,3,1,1)$; $(3,2,2,1)$ and $(3,2,1,1,1)$ of $n+1 = 8$.

It's clear that for all partitions $\lambda \vdash n$ and $\mu \vdash n+1$ we can write

μ is obtained from λ by adding a box in an outer corner $\iff$ λ is obtained from μ by removing a box from an inner corner.

Therefore the multiplicity c_μ of S^μ in $\mathrm{Ind}_{S_n}^{S_{n+1}} S^\lambda$ is given by:

$$c_\mu = \begin{cases} 1 & \text{if } T_\mu \text{ can be obtained from } T_\lambda \text{ by adding a box in an outer corner} \\ 0 & \text{otherwise.} \end{cases}$$

For every outer corner o.c. of $\lambda \vdash n$, denote by $\tilde{\lambda}_{oc}$ the corresponding partition of $n+1$ obtained from λ by adding the outer corner

Then

$$\boxed{\mathrm{Ind}_{S_n}^{S_{n+1}}(S^\lambda) = \bigoplus_{\text{outer corners}} S^{\tilde{\lambda}_{oc}}.}$$

<u>Example</u>: $\mathrm{Ind}_{S_7}^{S_8}\left(S^{(3,2,1,1)}\right) = S^{(4,2,1,1)} \oplus S^{(3,3,1,1)} \oplus S^{(3,2,2,1)} \oplus S^{(3,2,1,1,1)}$.

The ring of symmetric functions

Let $x = \{x_1, x_2, x_3, \dots\}$ be an infinite set of variables.

For each sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers s.t. $\sum_{i \geq 1} \alpha_i = n$, we say that

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots$$

is a monomial of degree n.

[You can clearly ignore the zero entries of the sequence α, and write x^α as a product of only a finite number of variables:

$$x^\alpha = x_{i_1}^{\alpha_{i_1}} x_{i_2}^{\alpha_{i_2}} \dots x_{i_k}^{\alpha_{i_k}} .]$$

A formal power series is an infinite linear combination of monomials:

$$f(x) = \sum_\alpha c_\alpha x^\alpha .$$

We take coefficients in $\mathbb{C}$, and we denote the ring of formal power series by $\mathbb{C}[[x]]$.

Def - A formal power series is called homogeneous of degree n if all the monomials have degree n, and is called symmetric if

$$f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots)$$

for every permutation of the set $P = \mathbb{Z}_{\geq 0}$ of positive integers.

<u>Example</u>: $f(\underline{x}) = \sum_{i,j} x_i^3 x_j$ is a homogeneous symmetric function of degree 4.

A more concrete definition: a homogeneous function of degree n

$$f = \sum_{\substack{\alpha:\text{ composition} \\ \text{of } n}} c_\alpha x^\alpha$$

is <u>symmetric</u> if $c_\alpha = c_{\alpha'}$ for every permutation $\alpha' = (\alpha'_1, \alpha'_2, \dots)$ of $\alpha = (\alpha_1, \alpha_2, \dots)$.

Equivalently, for every partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k > 0)$ of n, and every permutation λ' of $\lambda = (\lambda_1, \lambda_2, \dots)$, $c_\lambda = c_{\lambda'}$.

For all $n > 0$, set

$$\Lambda^n = \left\{ \begin{array}{l} \text{homogeneous symmetric formal} \\ \text{power series of degree } n \end{array} \right\} \cup \{0\}.$$

Clearly Λ^n is a vector space. We will prove that the dimension of Λ^n equals the number of partitions of n.

Also set $\Lambda^0 = \mathbb{C}$ and $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$.

Λ is an infinite-dimensional vector space, and also a graded ring: $\Lambda^n \Lambda^m \subseteq \Lambda^{n+m}$.

We call Λ the ring of symmetric functions,

- The purpose of this lecture is to describe many bases of Λ.

[Remark: We will notice shortly that Λ is not the set of all formal power series that satisfy the condition

$$f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots) \quad \forall \pi.$$

So the name "ring of symmetric functions" for Λ is somehow an abuse of notations...].

Monomial Symmetric functions

Given $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$, define a symmetric function $m_\lambda(x) \in \Lambda^n$ by

$$m_\lambda = \sum_\alpha x^\alpha$$

where the sum ranges over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \dots)$ of the entries of the vector $\lambda = (\lambda_1, \lambda_2, \dots)$.

[Remark: if $\lambda \vdash n$, m_λ is a homogeneous symmetric function of degree n in an unlimited number of variables].

Examples
$$m_\emptyset = 1 \quad \leftarrow n = 0$$

$$m_1 = \sum_i x_i \quad \leftarrow n = 1$$

$$\left.\begin{aligned} m_2 &= \sum_i x_i^2 \\ m_{11} &= \sum_{i<j} x_i x_j \end{aligned}\right] \leftarrow n = 2$$

$$\left.\begin{aligned} m_3 &= \sum_i x_i^3 \\ m_{21} &= \sum_{i \neq j} x_i^2 x_j \\ m_{1^3} &= \sum_{i<j<k} x_i x_j x_k \end{aligned}\right] \leftarrow n = 3$$

We call m_λ a "monomial symmetric function", because it is obtained by symmetrizing the monomial $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ (here $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$).

Theorem The set $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ^n.

proof - Let $f = \sum_{\substack{\alpha \\ \text{composition} \\ \text{of } n}} c_\alpha x^\alpha$ be an element of Λ^n.

If $\alpha = (\alpha_1, \alpha_2, \dots)$ is a permutation of $\lambda = (\lambda_1, \lambda_2, \dots)$, then $c_\alpha = c_\lambda$.

Then we can write

$$f(x) = \sum_{\lambda \vdash n} c_\lambda \left(\sum_{\substack{\text{distinct} \\ \text{permutations } \alpha \\ \text{of } \lambda}} x^\alpha \right) = \sum_{\lambda \vdash n} c_\lambda m_\lambda .$$

$\square$

Corollary 1 - $\dim(\Lambda^n) = \#$ partitions of n.

Corollary 2 - The set $\{m_\lambda\} = \bigcup_{n \geq 0} \{m_\lambda : \lambda \vdash n\}$ is a basis of $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$.

It follows that every element of Λ is a finite linear combination of m_λ's. Notice that this implies that

$$\Lambda \subsetneq \{ f \in \mathbb{C}[[x]] : f \text{ "symmetric"} \}.$$

For instance, $g(x) = \prod_{i \geq 1} (1 + x_i)$ is a formal power series which satisfies the condition

$$g(x_1, x_2, \dots) = g(x_{\pi(1)}, x_{\pi(2)}, \dots) \quad \forall \pi \text{ permutation of } \mathbb{Z}_{\geq 0}$$

but g is not a finite linear combination of m_λ's.

Elementary Symmetric functions

For every $K \geq 1$ define

$$e_K = m_{1^K} = \sum_{i_1 < i_2 < \dots < i_K} x_{i_1} x_{i_2} \cdots x_{i_K}$$

and set $e_0 = 1$.

$\boxed{e_K}$ is a homogeneous symmetric function of degree K, and is called the "elementary K^{th} symmetric function".

For every partition $\lambda \vdash n$, say $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_e > 0$, set

$$\boxed{e_\lambda} = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_e}.$$

<u>Example</u>:

$$e_{2,1} = \Big(\sum_{i \neq j} x_i x_j\Big)\Big(\sum_k x_k\Big)$$

$$e_3 = \sum_{i<j<k} x_i x_j x_k$$

$$e_{1^3} = \Big(\sum_i x_i\Big)\Big(\sum_j x_j\Big)\Big(\sum_k x_k\Big)$$

We will prove that the set $\{e_\lambda : \lambda \vdash n\}$ is a basis of Λ^n, so the set $\{e_\lambda\}$ is a basis of Λ (as a vector space).
[Equivalently, we can say that $\{e_j\}_{j \in \mathbb{N}}$ is a multiplicative basis for Λ: every element of Λ can be expressed as a polynomial in the e_j's.]

Because the set $\{m_\mu : \mu \vdash n\}$ forms a basis of Λ^n, for every partition $\lambda \vdash n$, we must be able to express e_λ as a linear combination of m_μ's.

<u>theorem</u> - Let λ' be the dual partition of λ.

then

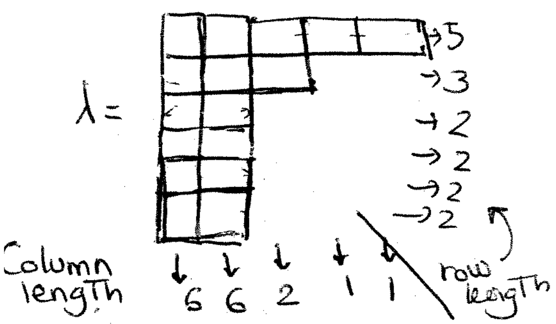
$$e_\lambda = m_{\lambda'} + \sum_{\mu \triangleleft \lambda'} a_{\lambda\mu} m_\mu .$$

[$\trianglelefteq$ is the dominance order of partitions. It's just a partial order ...]

DEFINITION OF DUAL PARTITION:

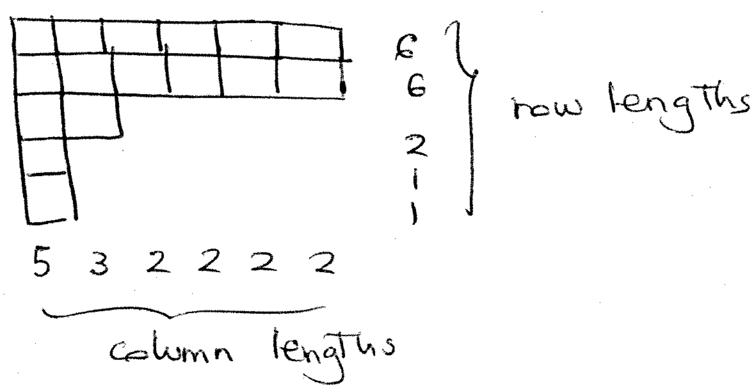
If $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$, the parts of the dual partition $\lambda' = (\lambda'_1 \geq \dots \geq \lambda'_\ell)$ are the lengths of the columns of the Ferrer diagram of λ.

<u>Example</u>: $\lambda = (5, 3, 2, 2, 2, 2)$

$\lambda =$ 

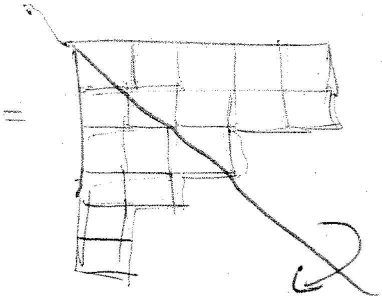
$\Rightarrow$ Columns of λ have length 6, 6, 2, 1, 1

$\Rightarrow \lambda' = (6, 6, 2, 1, 1)$



You must swicht column lengths with row lengths!!

Equivalently, the Ferrer diagram of λ' is obtained from the Ferrer diagram of λ by reflecting across

The main diagonal: $\lambda =$ 

[To prove this Theorem, we first notice that if x^μ (μ is a partition of n and) is a monomial in e_λ, then $\mu \trianglelefteq \lambda'$.

This will give us: $e_\lambda = \sum_{\mu \trianglelefteq \lambda'} a_{\lambda'\mu}\, m_\mu$, because e_λ is symmetric. Then we show that $a_{\lambda'\lambda'} = 1$].

By definition, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k} =$

$$= \Big(\sum_{i_1<\dots<i_{\lambda_1}} x_{i_1}\cdots x_{i_{\lambda_1}}\Big)\Big(\sum_{j_1<\dots<j_{\lambda_2}} x_{j_1} x_{j_2}\cdots x_{j_{\lambda_2}}\Big)\cdots\Big(\sum_{l_1<\dots<l_k} x_{l_1}\cdots x_{l_{\lambda_k}}\Big).$$

Suppose that $i_1<\dots<i_{\lambda_1}, \dots, l_1<\dots<l_{\lambda_k}$ are indices s.t. $(x_{i_1}\cdots x_{i_{\lambda_1}})(x_{j_1}\cdots x_{j_{\lambda_2}})\cdots(x_{l_1}\cdots x_{l_k}) = x^\mu = x_1^{\mu_1}\cdots x_s^{\mu_s}$.

Enter the indices $i_1 \dots i_{\lambda_1}$ in the first column of the Ferrer diagram of λ', the indices $j_1 \dots j_{\lambda_2}$ in the second column and so on ...

i_1	j_1	...	l_1
i_2	j_2	...	$\vdots$
$\vdots$	$\vdots$	...	l_{λ_k}
	j_{λ_2}		
i_{λ_1}			

(λ' is the dual partition so the columns have length $\lambda_1, \lambda_2, \dots \lambda_k$). We obtain a generalized Tableau of shape λ' and content μ.

Notice that if $i_a = t$, then $a \le t$.

[For instance you cannot have $i_3 = 2$, because $i_1 < i_2 < i_3$].

Similarly, if $j_2 \le t$ then $a \le t$ and so on...

$\Rightarrow$ <u>All the numbers $\le r$ are entered in the first r rows.</u>

Because we enter a total of μ_1 one's, μ_2 two's (etc) we need:

$$\mu_1 \le \text{length of } 1^{st} \text{ row of } \lambda' = \lambda'_1$$

$$\mu_1 + \mu_2 \le \text{sum of the lengths of first two rows of } \lambda' = \lambda'_1 + \lambda'_2$$

$$\vdots$$

$$\mu_1 + \dots + \mu_k \le \lambda'_1 + \lambda'_2 + \dots + \lambda'_k$$

$$\vdots$$

This is the dominance order!!!

this proves that $\mu \trianglelefteq \lambda'$. <u>ok</u>

$\Rightarrow$ $e_\lambda = \sum_{\mu \trianglelefteq \lambda'} a_{\lambda'\mu} m_\mu$ for some coefficients $a_{\lambda'\mu}$.

Next, we want to compute the coefficient $a_{\lambda'\lambda'}$ (and show it's i

Recall that for all $\mu \le \lambda'$

$$a_{\lambda'\mu} = \# \text{ of ways of writing } x^\mu = x_1^{\mu_1} \cdots x_k^{\mu_k}$$

as $(x_{i_1} \cdots x_{i_{\lambda_1}}) \cdots (x_{l_1} \cdots x_{l_{\lambda_k}})$

with $i_1 < \dots < i_{\lambda_1}; \dots ; l_1 < \dots < l_{\lambda_k}$.

If $\mu = \lambda'$ and $(x_{i_1} \cdots x_{i_{\lambda_1}}) \cdots (x_{l_1} \cdots x_{l_k}) = x_1^{\lambda'_1} \cdots x_k^{\lambda'_k}$

then the tableau associated to the monomial has shape λ' and content $\mu = \lambda'$.

There's only one tableau with this property, namely

1	1	1	1	...	1
2	2	2	...	2	

$\Rightarrow$ There's only 1 monomial with

This property, namely

$$(x_1 x_2 \ldots x_{\lambda_1})(x_1 x_2 \ldots x_{\lambda_2}) \cdots (x_1 \ldots x_{\lambda_k}).$$

this proves that $a_{\lambda'\lambda'} = 1$ and

$$e_\lambda = m_{\lambda'} + \sum_{\mu \lhd \lambda'} a_{\lambda'\mu}\, m_\mu . \quad \square$$

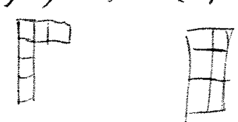
non-negative integers

<u>Remark</u> – the partial ordering $\trianglerighteq$ has the property that if $\lambda_1 \trianglerighteq \lambda_2$ then $\lambda_1' \trianglelefteq \lambda_2'$, but it is <u>not</u> a total ordering (if $n \geq 6$). On the other hand, the lexicographical order is a total order but does not have the nice property that $\lambda_1 \geq \lambda_2 \Rightarrow \lambda_1' \leq \lambda_2'$.

<u>example</u>: $\lambda_1 = (4, 1, 1) > (3, 3) = \lambda_2$

taking duals we get:

$\lambda_1' = (3, 1, 1, 1) > (2, 2, 2) = \lambda_2'$



instead of $\lambda_1' < \lambda_2'$ as desired.

Nonetheless, we can always define a total ordering of partitions

$$\lambda_1 < \lambda_2 < \ldots < \lambda_r$$

which is compatible with the dominance order

($\text{ie. } \lambda \trianglelefteq \mu \Rightarrow \lambda \leq \mu$)

s.t. $\lambda_1' \geq \lambda_2' \geq \dots \geq \lambda_r'$ is also compatible with the dominance order.

For instance, for $n=6$, you can choose

$$6 > (5,1) > (4,2) > (3,3) > (4211) > (321) > (31^3) > 2^3 >$$
$$> (2^2 1^2) > (21^4) > (1^6).$$

If you choose this ordering of the partitions of n, the matrix that expresses the e_λ's in terms of the $(m_{\mu'})$'s is triangular, with 1 on the diagonal.

Therefore, you can solve the triangular system

$$e_\lambda = m_{\lambda'} + \sum_{\mu \triangleleft \lambda'} \underbrace{\tilde{a}_{\lambda'\mu}}_{\text{in } \mathbb{Z}_{\geq 0}} m_\mu \qquad \forall \lambda \vdash n$$

to find $m_{\lambda'}$ as a linear combination of e_μ's:

$$m_{\lambda'} = e_\lambda + \sum_{\mu > \lambda} \underbrace{b_{\lambda\mu}}_{\text{in } \mathbb{Z}} e_\mu.$$

It follows that the elements $\{e_\lambda : \lambda \vdash n\}$ are also a basis of Λ^n.

$\Rightarrow$ The set $\{e_\lambda\}$ is a basis of Λ.