DIMENSION OF S^λ

For every partition $\lambda \vdash n$, the permutation module M^μ decomposes as

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda\mu} S^\lambda$$

with diagonal multiplicity $m_{\mu\mu} = 1$.
The coefficients $m_{\lambda\mu}$ have a combinatoric interpretation:
$m_{\lambda\mu}$ = # semistandard Young Tableau of shape λ and content μ. [We will prove this fact later, if time permits].
In this section, we want to determine the dimension of S^λ.

Set $\mu =$  $= (1^n)$. Then M^μ is the regular representation, and it contains every irreducible of S_n with multiplicity equal to its dimension: $M^{\mu=(1^n)} = \bigoplus_{\text{all } \lambda \vdash n} (\dim S^\lambda)\, S^\lambda$.
Compare this with $M^{\mu=(1^n)} = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda\mu} S^\lambda$. Also this direct sum is over all the possible partitions of n (because (1^n) is the "smallest" partition, i.e. it sits in the very bottom of the Hasse diagram). We deduce that $(\dim S^\lambda) = m_{\lambda\mu}$, for $\mu = (1^n)$.

What is $m_{\lambda(1^n)}$?

A generalized Young Tableau of content (1^n) is a Tableau filled up with integers $\{1 .. n\}$ without any repetition, so it's a regular tableau. In particular, a generalized semistan-dard Young Tableau is a regular standard Tableau.

$\Rightarrow m_{\lambda(i)} = \#$ standard Tableaux of shape λ.

$\Rightarrow \forall \lambda \vdash n$, the dimension of S^{λ} is equal to the number of standard Tableaux of shape λ.

[<u>Remark</u>: This is in accordance with the few examples we know ....

- $S^{\square\square\square\square\square\square}$ is the trivial representation, of dimension 1. Indeed there is only one standard Tableau of shape (n), namely $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n \\ \hline\end{array}$.

- $S^{\substack{\square\\ \square\\ \square\\ \square}}$ is the sign representation, also of dimension 1. Indeed there is only one standard Tableau of shape (1^n), namely $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline\end{array}$.

- $S^{(n-1,1)}$ is the standard representation of S_n, of dimension $n-1$. Indeed, there are exactly $n-1$ standard Tableaux of shape $(n-1,1)$:

$$t_2 = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & \cdots & n \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$t_3 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & \cdots & n \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\vdots$$

$$t_n = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n-1 \\ \hline n \\ \cline{1-1}\end{array}.$$

In general t_j has entries $1, 2, \ldots, j-1, \overset{\text{omitted}}{\hat{j}}, j+1, \ldots, n$ in the first row, and entry j in the second row.]

Given this nice combinatorial formula for the dimension of S^λ, we expect to be able to find a basis for S^λ that is parametrized by standard tableaux of shape λ.

A BASIS FOR S^λ

By definition, S^λ is the sub-representation of M^λ generated by the polytabloids of shape λ.
We will prove that the <u>standard</u> polytabloids of shape λ form a basis of S^λ. By the previous remarks, it is sufficient to show that they are linearly independent.

ORDERING OF STANDARD TABLEAUX

We order the standard tableaux of shape λ! in dictionary order:
look at the numberings of

$$t = \begin{array}{|llll} n_{11} & n_{12} & \cdots & n_{1\lambda_1} \\ n_{21} & n_{22} & \cdots & n_{2\lambda_2} \\ \vdots & \vdots & & \\ n_{k1} & \cdots & n_{k\lambda_k} & \end{array} \quad \text{and} \quad t' = \begin{array}{|llll} n'_{11} & n'_{12} & \cdots & n'_{1\lambda_1} \\ n'_{21} & n'_{22} & \cdots & n'_{2\lambda_2} \\ \vdots & & & \\ n'_{k1} & \cdots & n'_{k\lambda_k} & \end{array}.$$

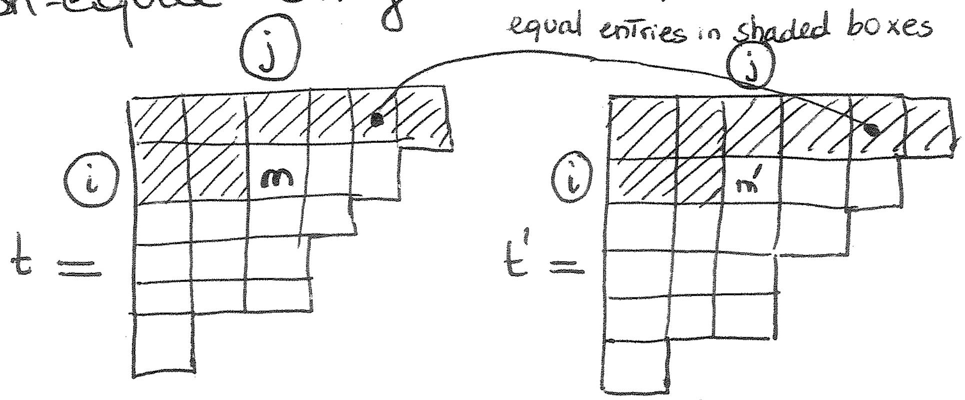
We say that $t < t'$ if the first non-zero entry of $t - t'$ is negative, i.e. the first of $n_{11} - n'_{11}$, $n_{12} - n'_{12}$, $\ldots$ $n_{1\lambda_1} - n'_{1\lambda_1}$, $\ldots$, $n_{k\lambda_k} - n'_{k\lambda_k}$ which is non-zero, is negative. This is a <u>total</u> order on standard tableaux.

<u>Example</u> : $n = 5$, $\lambda = (3,2)$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

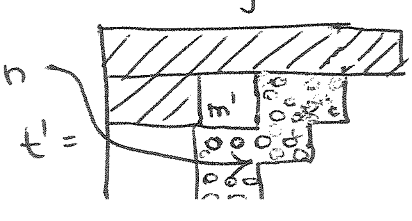
<u>Lemma</u> Let t, t' be two standard tableaux of shape λ. If $t < t'$, then $t \frown t'$ (i.e. there are two integers that are in the same row of t and in the same column of t').

<u>proof</u>- By assumption $t < t'$. Suppose that the first non-equal entry is in position $\{i, j\}$



Set $m = t_{i,j}$, $m' = t'_{i,j}$. Because $t < t'$, $m < m'$.

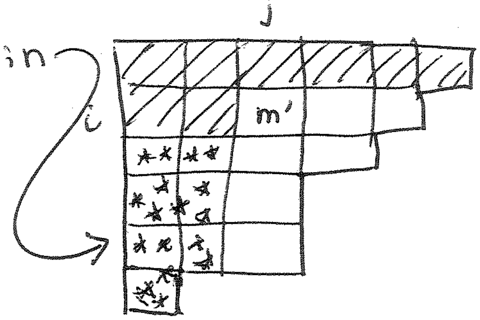
The integer m sits in one of the boxes of t'. It cannot be in the shaded area (because that is reserved for integers that also lie in the shaded area of t, and m appears in a box of t outside the shaded area ...). It cannot be in



The region either

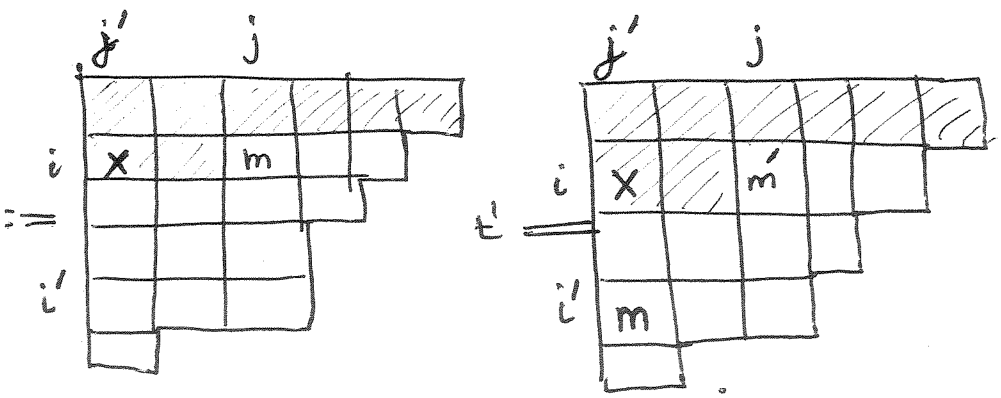
because t' is standard, so both the rows and the columns of t' must increase. [Recall that $m < m'$].

Then m must sit somewhere in



in other words, it must sit in position (i', j') with $\boxed{i' > i}$ and $\boxed{j' < j}$.

We obtain the following picture:



By construction, the elements of t and t' in position (i, j') coincide. Call x this element.

The pair of integers $\{x, m\}$ lies in the i^{th} ~~row~~ of t and in the j'^{th} column of t'.

$\Longrightarrow$ $t \circ\!\!-\, t'$, as claimed. $\square$

<u>Corollary</u>. If t and t' are standard tableaux of shape λ and $t < t'$, then $\kappa_{t'} \cdot e_t = 0$.

Theorem - The standard polytabloids $\{e_t : t \text{ standard}\}$ are linearly independent in M^λ.

proof - Let $t_1 < t_2 < \dots < t_m$ be the standard Tableaux of shape λ, in increasing order. We want to prove that the corresponding polytabloids $e_{t_1}, e_{t_2} \dots e_{t_m}$ are l.i.

Consider a l.c. of these polytabloids: $x = \sum_{i=1}^{m} a_i e_{t_i}$.

If $x = 0$, then $\langle x, \{t_1\} \rangle = 0$ (here $\langle , \rangle$ is the usual inner product in M^λ, defined by imposing that the tabloids are orthonormal ...). So we can write:

$$0 = \langle x, \{t_1\} \rangle = \langle \sum_{i=1}^{m} a_i e_{t_i}, \{t_1\} \rangle = \sum_{i=1}^{m} a_i \langle e_{t_i}, \{t_1\} \rangle =$$

$$= \sum_{i=1}^{m} a_i \langle K_{t_i} \{t_i\}, \{t_1\} \rangle =$$

$$= \sum_{i=1}^{m} a_i \langle \{t_i\}, K_{t_i} \cdot \{t_1\} \rangle =$$

$$= a_1 \langle \{t_1\}, K_{t_1} \cdot \{t_1\} \rangle$$

because $K_{t_i} \circ \{t_1\} = 0 \quad \forall i \geq 2$, for $t_i > t_1$ for $i \geq 2$.

We also notice that

$$\langle \{t_1\}, K_{t_1} \{t_1\} \rangle = \langle \{t_1\}, \{t_1\} + \sum_{\substack{\sigma \in C_{t_1} \\ \sigma \neq id}} (\operatorname{sgn} \sigma) \{\sigma t_1\} \rangle =$$

$K_{t_1} = \sum_{\sigma \in C_{t_1}} (\operatorname{sgn} \sigma) \sigma$

$$= \underbrace{\langle \{t_1\}, \{t_1\} \rangle} + \sum_{\sigma \in C_{t_1}} \operatorname{sgn} \sigma \underbrace{\langle \{t_1\}, \{\sigma t_1\} \rangle}_{=0} = 1.$$

[We use the fact that tabloids are orthonormal, and $\{\sigma t_1\}$ is a tabloid different from t_1 for every non trivial permutation in the <u>column</u> stabilizer of t_1.]

This gives:

$$\sum_{i=1}^{m} a_i \underline{e}_{t_i} = 0 \implies a_1 = 0.$$

Start again: $\sum_{i=2}^{m} a_i \underline{e}_{t_i} = 0 \implies \langle \sum_{i=2}^{m} a_i \underline{e}_{t_i}, \{t_2\}\rangle = 0$

$\implies$ $a_2 = 0$ and so on ....

$t_2 < t_i \ \forall i \geq 2$

We obtain the linear independence of $\underline{e}_{t_1} \dots \underline{e}_{t_m}$ $\left(\sum_{i=1}^{m} a_i e_t = 0 \implies a_1 = a_2 = \dots = a_m = 0\right)$. ▨

<u>Corollary</u> - $\boxed{\text{The standard polytabloids are a basis of } S^\lambda}$

<u>YOUNG'S NATURAL REPRESENTATION</u>

The matrices for S^λ with respect to the basis of standard polytabloids form the "Young's natural representation". Let's try to describe these matrices.

<u>Remark</u>: S_n is generated by the transpositions

$$(12), (23), \dots, (k\ k+1), \dots, (n-1, n)$$

so it is enough to understand how these elements act.

For every $k = 1 \ldots n-1$, and every standard tableau t, we must express $(k\ k+1)\,\underline{e}_t$ as a linear combination of standard polytabloids.

Two cases are "easy":

<u>case 1</u>: k and $k+1$ lie in the same column of t, i.e. $(k\ k+1) \in C_t$. Then $(k\ k+1)\,\underline{e}_t = -\underline{e}_t$.

[This follows from the definition of polytabloid:

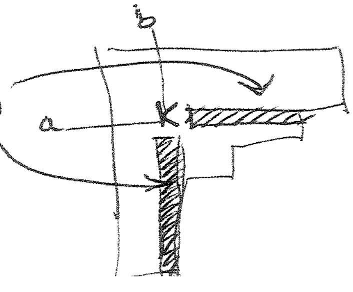
$$\underline{e}_t = \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\, \pi \{t\} \Longrightarrow (k\ k+1)\,\underline{e}_t = \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\, \underbrace{(k\ k+1)\pi}_{\text{in } C_t} \{t\}$$

$$= - \sum_{\pi \in C_t} \operatorname{sgn}[\underbrace{(k\ k+1)\pi}_{\sigma}]\, \underbrace{(k\ k+1)\pi}_{\sigma} \{t\}$$

$$= -\Big(\sum_{\sigma \in C_t} \operatorname{sgn}\sigma\ \sigma\Big)\{t\} = -\underline{e}_t.\ ]$$

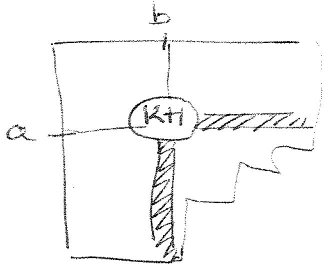
<u>Case 2</u>: k and $k+1$ are not in the same row or column of t. then $(k\ k+1)t$ is again a <u>standard</u> tableau and $(k\ k+1)\,\underline{e}_t = \underline{e}_{(k\ k+1)t} = \underline{e}_{t'}$.

[Say that k is in position (a,b) and $(k+1)$ is in position (a',b'). Then because no repetitions are allowed, and t is standard every element in



is strictly greater than $k+1$.

. . . .



So if we replace k by $k+1$, the a^{th} row and the b^{th} column remain increasing.

The a^{th} row and b^{th} column remain increasing if you replace $k+1$ by k.

The last case ("k and $k+1$ are in the same row of t") is harder.

<u>Claim</u> - If k and $(k+1)$ are in the same row of t, then

$$(k\ k+1)\,\underline{e}_t = \underline{e}_t \pm \text{other basis elements } \underline{e}_{t'} \text{ with } t' > t.$$

[More explicitely: $(k\ k+1)\,\underline{e}_t = \underline{e}_t + \sum_{t'>t} a_{t'}\,\underline{e}_{t'}$

and each $a_{t'}$ is $0, 1,$ or -1].

We just give an idea of the proof, and show how to find $(k\ k+1)\,\underline{e}_t$ in practice ...

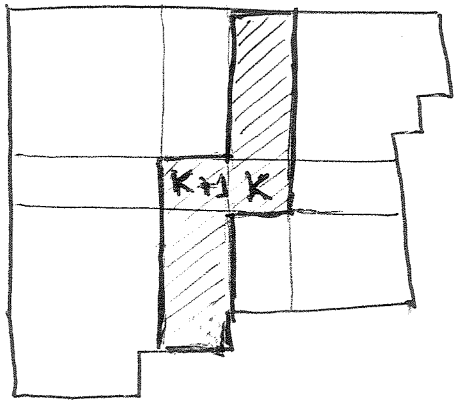
If $\underline{e}_t$ is a standard tableau and $k, k+1$ are in the same row of $\underline{e}_t$, then they must be in adjacent position [i.e. $\underline{e}_t$ contains $\boxed{k \mid k+1}$ in some row]

$\Rightarrow$ $(k\ k+1)\,\underline{e}_t$ is no longer a standard tableau and one of its rows contain the sequence $\boxed{k+1 \mid k}$

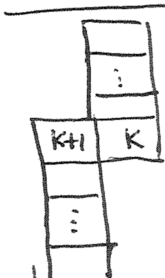
PROBLEM: not increasing !!!

If we want to express $(k\ k{+}1)e_t$ as a l.c. of standard polytabloids we need to fix this problem.

- The way to fix it is to apply an appropriate "Garnir element".

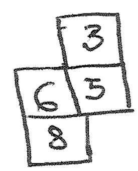
$(k\ k{+}1)t = t' =$ 

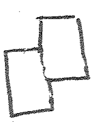
For simplicity of notations, set $(k\ k{+}1)t = t'$.
Inside t', look at the elements in the boxes that sit <u>above k</u> and <u>below k+1</u>.

This gives you a "funny shape" .

- Let S be the set of all indices in the funny shape and define a "Garnir element" g to be:

$$g = \sum_{\substack{\text{all permutation } \sigma \\ \text{of } S \text{ that make both} \\ \text{columns of the funny} \\ \text{shape increasing}}} (\operatorname{sgn} \sigma)\, \sigma .$$

<u>Example</u>: $t' = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 5 \\ \hline 7 & 8 & \\ \hline \end{array} \Rightarrow$ the funny shape is  and

- $S = \{3, 5, 6, 8\}$. We need to look for all possible ways to fill up  using $\{3, 5, 6, 8\}$ in such a way that the two columns of the funny shape are increasing.

$\ldots \begin{array}{cc} & 3 \\ 6 & 5 \\ 8 & \end{array} ; \begin{array}{cc} & 3 \\ 5 & 6 \\ 8 & \end{array} ; \begin{array}{cc} & 3 \\ 5 & 8 \\ 6 & \end{array} ; \begin{array}{cc} & 5 \\ 3 & 6 \\ 8 & \end{array} ; \begin{array}{cc} & 5 \\ 3 & 8 \\ 6 & \end{array} ; \begin{array}{cc} & 6 \\ 3 & 8 \\ 5 & \end{array}$

This is the one that comes from t', and should be the reference point ...

These arrays are obtained from $\begin{array}{cc} & 3 \\ 6 & 5 \\ 8 & \end{array}$ by applying the permutations

$$\text{id} ; (56) ; (658) ; (356) ; (3586) ; (36)(58) \quad (*)$$

of sign:

$$+ ; - ; + ; + ; - ; + .$$

the Garnir element of t' is the sum of all the permutations $(*)$ with their sign:

$$g = \text{id} - (56) + (658) + (356) - (3586) + (36)(58)$$

It has the property that $\boxed{g \cdot e_{t'} = 0}$.

More explicitely:

$$e_{t'} - e_{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{array}} + e_{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 7 & 6 & \end{array}} + e_{\begin{array}{ccc} 1 & 2 & 5 \\ 4 & 3 & 6 \\ 7 & 8 & \end{array}} - e_{\begin{array}{ccc} 1 & 2 & 5 \\ 4 & 3 & 8 \\ 7 & 6 & \end{array}} + e_{\begin{array}{ccc} 1 & 2 & 6 \\ 4 & 3 & 8 \\ 7 & 5 & \end{array}} = 0$$

$(56)t$ $\qquad t$

$$\Rightarrow (56)\, \underline{e}_t = \underline{e}_t - e_{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 7 & 6 & \end{array}} - e_{\begin{array}{ccc} 1 & 2 & 5 \\ 4 & 3 & 6 \\ 7 & 8 & \end{array}} + e_{\begin{array}{ccc} 1 & 2 & 5 \\ 4 & 3 & 8 \\ 7 & 6 & \end{array}} - e_{\begin{array}{ccc} 1 & 2 & 6 \\ 4 & 3 & 8 \\ 7 & 5 & \end{array}}$$

Notice that each one of these 4 tableaux is bigger than t !!!

We obtain:

$$(56)\ \underline{e}_t = \underline{e}_t \pm \text{polytabloids } e_{\tilde{t}} \text{ with } \tilde{t} > t.$$

In general, when you apply this method, Some of the $\tilde{t}$'s that you get are standard, some are not...
[In this example none of them is...]
Each tableau that is not standard needs to be fixed by applying an appropriate Garnir element...
Eventually, this will give you:

$$(56)\ \underline{e}_t = \underline{e}_t \pm \underline{\text{standard}} \text{ polytabloids } \underline{e}_{\tilde{\tilde{t}}} \text{ with } \tilde{\tilde{t}} > t.$$

REFERENCE: $\boxed{\text{SAGAN}}$'s book
(The symmetric group)

MATRICES FOR THE ACTION OF S_5 ON $S^{(3,2)}$

- standard tableaux of shape $(3,2)$

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \quad t_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \quad t_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \quad t_4 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \quad t_5 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

- lexicographical ordering:

$$t_1 < t_2 < t_3 < t_4 < t_5$$

ACTION OF (12)

for simplicity start from the highest ...

- $(12)\underline{e}_{t_5} = -\underline{e}_{t_5}$ bc. 1,2 same column of t_5
- $(12)\underline{e}_{t_4} = -\underline{e}_{t_4}$ bc 1,2 same column of t_4

- $(12)\underline{e}_{t_3} = ?$

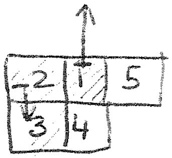
we expect: $(12)\cdot\underline{e}_{t_3} = \underline{e}_{t_3} + a\,\underline{e}_{t_4} + b\,\underline{e}_{t_5}$

with $a, b \in \{0, 1, -1\}$

because 1,2 are in the same row of t_3 and because t_4 and t_5 are higher than t_3 ...

- $(12)\underline{e}_{t_3} = \begin{array}{|c|c|c|}\hline 2 & 1 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$

Let's look for a Garnir element that "fixes" the descent $\boxed{2\ 1}$ of $(12)\underline{e}_{t_3}$

$\boxed{t'} = $  $\rightsquigarrow$ $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \hline\end{array}$ $\rightsquigarrow$ now look at all possible permutations of $\{1,2,3\}$ that make all the columns increasing [Keep track of the signs...]

Take things on top of 1 and below 2

- id : $\begin{matrix} 2 & 1 \\ 3 & \end{matrix} \to \begin{matrix} 2 & 1 \\ 3 & \end{matrix}$ sign $+$
- (12) : $\begin{matrix} 2 & 1 \\ 3 & \end{matrix} \to \begin{matrix} 1 & 2 \\ 3 & \end{matrix}$ sign -1
- (132) : $\begin{matrix} 2 & 1 \\ 3 & \end{matrix} \to \begin{matrix} 1 & 3 \\ 2 & \end{matrix}$ sign $+$

$$g = \mathrm{id} - (12) + (132)$$

$$\Rightarrow \quad g \cdot e_{t'} = 0$$

$$\Rightarrow \quad e_{t'} - \underbrace{(12)\, e_{t'}} + (132)\, e_{t'} = 0$$

$$\Rightarrow \quad e_{t'} - e_{t_3} + e_{t_5} = 0 \Longrightarrow e_{t'} = e_{t_3} - e_{t_5} \quad \underline{\text{ok}}$$

$2)$ $\begin{array}{|c|c|c|}\hline 2 & 1 & 5 \\ \hline 3 & 4 & \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline\end{array} = t_3$

$32)$ $\begin{array}{|c|c|c|}\hline 2 & 1 & 5 \\ \hline 3 & 4 & \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline\end{array} = t_5$

- $(12)\, t_2 = \begin{array}{|c|c|c|}\hline 2 & 1 & 4 \\ \hline 3 & 5 & \\ \hline\end{array}$ call it t'

Let's compute the Garnir element of t'.

Which permutations of $\{1,2,3\}$ make the columns of $\begin{matrix} 2 & 1 \\ 3 & \end{matrix}$ increasing?

Just like before: $g = id - (12) + (132)$.

We have:

- $id \cdot t' = t'$
- $(12)\, t' = t_2$
- $(132)\, t' = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} = t_4$

$\Rightarrow \quad e_{t'} - e_{t_2} + e_{t_4} = 0$

$\Rightarrow \boxed{e_{t'} = e_{t_2} - e_{t_4}} \Rightarrow \quad (12)\, e_{t_2} = e_{t_2} - e_{t_4}$.

- $(12)\, t_1 = \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} = t'$

We have to look for all the permutations of $\{1,2,4\}$ that ~~make~~ make the columns of $\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 \\ \cline{1-1} \end{array}$ increasing....

$g = id - (12) + (142)$

We get:

$0 = e_{t'} - e_{t_1} + e_{\begin{array}{|c|c|c|} \hline 1 & 4 & 3 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}} \Rightarrow \quad e_{t'} = e_{t_1} - e_{\begin{array}{|c|c|c|} \hline 1 & 4 & 3 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}}$

Problem: $\begin{array}{|c|c|c|} \hline 1 & 4 & 3 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} = s$ is <u>not</u> standard.

We have to apply the algorithm again.

- $g = (id) - (34) + (543) \Longrightarrow e_s - e_{\substack{134 \\ 25}} + e_{\substack{135 \\ 24}} = 0$

(corresponding to $\substack{4\ 3 \\ 5}$; $\substack{3\ 4 \\ 5}$; $\substack{3\ 5 \\ 4}$)

$e_s - e_{t_4} + e_{t_5} = 0$

$\Downarrow$

$$\Rightarrow \underline{e}_{t'} = \underline{e}_{t_1} - (\underline{e}_{t_4} - \underline{e}_{t_5}) = \underline{e}_{t_1} - \underline{e}_{t_4} + \underline{e}_{t_5}.$$

We obtain:

$$\begin{array}{cc} & \begin{array}{ccccc} \underline{e}_{t_1} & \underline{e}_{t_2} & \underline{e}_{t_3} & \underline{e}_{t_4} & \underline{e}_{t_5} \end{array} \\ (12) \rightsquigarrow & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

Next, we compute the matrix of (23):

$$(23)\,\underline{e}_{t_1} = e_{\begin{smallmatrix}1&3&2\\4&5\end{smallmatrix}}$$

we need a Garnir element...

$$\begin{smallmatrix}3&2\\5\end{smallmatrix};\ \begin{smallmatrix}2&3\\5\end{smallmatrix};\ \begin{smallmatrix}2&5\\3\end{smallmatrix} \Rightarrow g = id - (23) + (325)$$

$$\Rightarrow e_{\begin{smallmatrix}1&3&2\\4&5\end{smallmatrix}} = e_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} - e_{\begin{smallmatrix}1&2&5\\4&3\end{smallmatrix}} = \underline{e}_{t_1} - e_{\begin{smallmatrix}1&2&5\\4&3\end{smallmatrix}}$$

We need a Garnir element...

$$\begin{smallmatrix}&2\\4&3\end{smallmatrix};\ \begin{smallmatrix}&3\\2&4\end{smallmatrix};\ \begin{smallmatrix}&2\\3&4\end{smallmatrix} \Rightarrow g = id - (34) + (234)$$

$$\Rightarrow e_{\begin{smallmatrix}1&2&5\\4&3\end{smallmatrix}} = e_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} - e_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} = \underline{e}_{t_3} - \underline{e}_{t_5}$$

$$\Rightarrow (23)\,\underline{e}_{t_1} = \underline{e}_{t_1} - \underline{e}_{t_3} + \underline{e}_{t_5}$$

$$(23)\,\underline{e}_{t_2} = e_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} = \underline{e}_{t_4}$$

$$(23)\,\underline{e}_{t_3} = e_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} = \underline{e}_{t_5}$$

$$(23)\, e_{t_4} = e_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} = e_{t_2}$$

$$(23)\, e_{t_5} = e_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} = e_{t_3}$$

$$\Rightarrow \quad (23) \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

ACTION OF (34)

- $(34)\cdot t_1 = (34)\begin{array}{|lll|}1&2&3\\4&5\end{array} = \begin{array}{|lll|}1&2&4\\3&5\end{array} = t_2 \Rightarrow (34)\cdot e_{t_1} = e_{t_2}$
- $(34)\cdot t_2 = (34)\begin{array}{|lll|}1&2&4\\3&5\end{array} = \begin{array}{|lll|}1&2&3\\4&5\end{array} = t_1 \Rightarrow (34)\cdot e_{t_2} = e_{t_1}$ (obviously!)
- $(34)\cdot t_3 = (34)\cdot\begin{array}{|lll|}1&2&5\\3&4\end{array} = \begin{array}{|lll|}1&2&5\\4&3\end{array}$

Garnir element for $\begin{array}{|ll|} &2\\4&3\end{array}$: $\mathrm{Id} - (34) + (234)$.

corresponding to:

$$\left(\begin{smallmatrix} &2\\4&3\end{smallmatrix} \;;\; \begin{smallmatrix} &2\\3&4\end{smallmatrix} \;;\; \begin{smallmatrix} &3\\2&4\end{smallmatrix} \right)$$

$$\Rightarrow e_{\begin{smallmatrix}1&2&5\\4&3\end{smallmatrix}} - e_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} + e_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} = 0$$

$$\Rightarrow e_{\begin{smallmatrix}1&2&5\\4&3\end{smallmatrix}} = e_{t_3} - e_{t_5}$$

$$\Rightarrow (34)\cdot e_{t_3} = e_{t_3} - e_{t_5}$$

- $(34)\cdot t_4 = (34)\cdot\begin{array}{|lll|}1&3&4\\2&5\end{array} = \begin{array}{|lll|}1&4&3\\2&5\end{array}$

$$\left(\begin{smallmatrix}4&3\\5\end{smallmatrix} \quad \begin{smallmatrix}3&4\\5\end{smallmatrix} \quad \begin{smallmatrix}3&5\\4\end{smallmatrix} \right)$$

We need a Garnir element for $\begin{smallmatrix}4&3\\5\end{smallmatrix} \rightarrow g = e - (34) + (354)$

$$\Rightarrow \underline{e}_{\begin{smallmatrix}1&4&3\\2&5\end{smallmatrix}} = \underline{e}_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} - \underline{e}_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} \Rightarrow (34)\cdot \underline{t}_4 = \underline{t}_4 - \underline{t}_5 .$$

- $(34)\cdot \underline{e}_{t_5} = -\underline{e}_{t_5}$ (3 and 4 are on the same column)

$$\Rightarrow (34) \leadsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

ACTION OF (45)

$(45)\, \underline{e}_{t_1} = ?$

$(45)\, t_1 = \begin{smallmatrix}1&2&3\\5&4\end{smallmatrix}$

- Garnir element for $\begin{smallmatrix}&2\\5&4\end{smallmatrix}$??

$$\rightarrow \quad g = e - (45) + (245)$$

$$\begin{smallmatrix}&2\\5&4\end{smallmatrix} \qquad \begin{smallmatrix}&2\\4&5\end{smallmatrix} \qquad \begin{smallmatrix}&4\\2&5\end{smallmatrix}$$

We get:

$$\underline{e}_{\begin{smallmatrix}1&2&3\\5&4\end{smallmatrix}} = \underline{e}_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} - \underline{e}_{\begin{smallmatrix}1&4&3\\2&5\end{smallmatrix}} = \underline{e}_{t_1} - \underline{e}_{\begin{smallmatrix}1&4&3\\2&5\end{smallmatrix}}$$

- Garnir element for $\begin{smallmatrix}4&3\\5\end{smallmatrix}$:

$$g = e - (34) + (435) \quad \begin{smallmatrix}3&5\\4\end{smallmatrix}$$

$$\Rightarrow \underline{e}_{\begin{array}{|l}\hline 143 \\ 25 \\ \hline \end{array}} = \underline{e}_{\begin{array}{|l}\hline 134 \\ 25 \\ \hline \end{array}} - \underline{e}_{\begin{array}{|l}\hline 135 \\ 24 \\ \hline \end{array}} = \underline{e}_{t_4} - \underline{e}_{t_5}$$

$$\Rightarrow (45)\underline{e}_{t_1} = \underline{e}_{t_1} - \left(\underline{e}_{t_4} - \underline{e}_{t_5}\right) = \underline{e}_{t_1} - \underline{e}_{t_4} + \underline{e}_{t_5}$$

- $(45)\,\underline{e}_{t_2} = \underline{e}_{t_3}$
- $(45)\,\underline{e}_{t_3} = \underline{e}_{t_2}$
- $(45)\,\underline{e}_{t_4} = \underline{e}_{t_5}$
- $(45)\,\underline{e}_{t_5} = \underline{e}_{t_4}$.

So we get:

$$(45) \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$