Orderings of Partitions

- **Definition** – Let A be a set. A partial order on A is a relation $\leq$ such that
  1) $a \leq a$
  2) $a \leq b$ and $b \leq a$ implies $a = b$
  3) $a \leq b$ and $b \leq c$ implies $a \leq c$,

  for all a, b, c in A.
  Notice that if a and b are any two elements in A, it migth happen that none of the relations "$a \leq b$" or "$b \leq a$" is true. Indeed $\leq$ is just a "partial" order.
  If, for any pair $\{a, b\}$ in A, we have either $a \leq b$ or $b \leq a$, then we say that $\leq$ is a "total" order on A.

- Let $A = \mathcal{P}_n = \{$all partitions of $n\}$.
  We define a total order on A and also a partial order.

TOTAL ORDER ($\leq$)

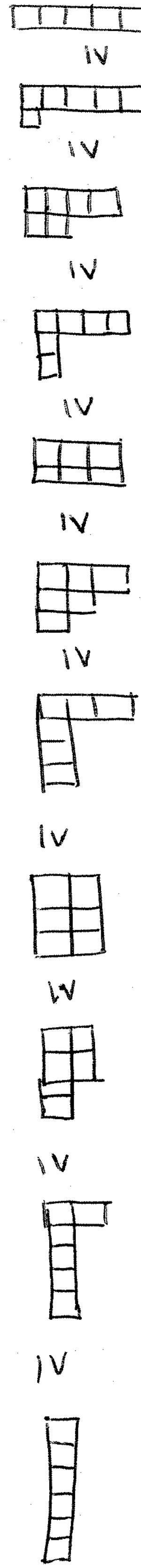
Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_e)$ and $\mu = (\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_m)$ be partitions of n. You can assome that $l = m$ by allowing some parts to be zero.
We say that $\lambda < \mu$ is the first non-zero entry of $\lambda - \mu$ is negative, i.e. there exists an index K st

- $$\lambda_i = \mu_i \quad \forall\, i = 1 \dots K$$

  and $\lambda_K < \mu_K$.

This is the lexicografic order of partitions, and is a 2 Total order.

- Example (n=6)



$$6 \geq (5,1) \geq (4,2) \geq (4,1,1) \geq (3,3) \geq (3,2,1) \geq (3,1,1,1) \geq (2,2,2) \geq (2,2,1,1) \geq (2,1,1,1,1) \geq (1,1,1,1,1,1)$$

- Because its a Total order, The Hasse diagram is always a chain.

PARTIAL ORDER ($\trianglelefteq$)

- Suppose that $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell)$ and $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_m)$ are partitions of n. [Assume $l = m$.]
We say that $\lambda \trianglelefteq \mu$ if

$$\lambda_1 + \lambda_2 + \dots + \lambda_i \leq \mu_1 + \mu_2 + \dots + \mu_i \quad \forall i = 1 \dots l.$$

This is just a partial order. For instance the partitions $\lambda = (4,1,1)$ and $\mu = (3,3,0)$ of $n = 6$ are incomparable. Indeed

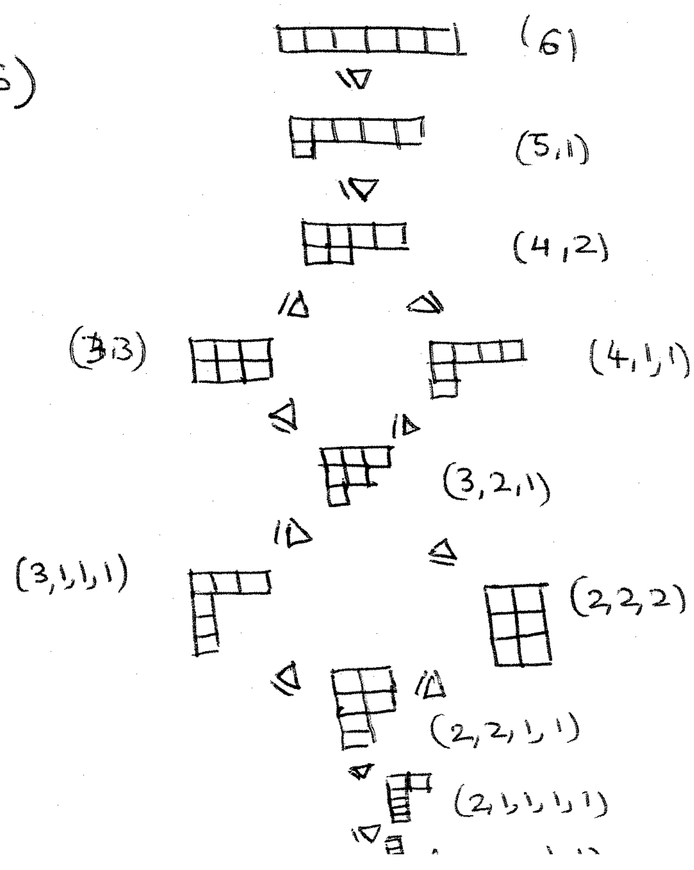
$$\lambda_1 = 4 > \mu_1 = 3$$

but

$$\lambda_1 + \lambda_2 = 5 < \mu_1 + \mu_2 = 6 .$$

- Notice that, because there could be some incomparable partitions, the Hasse diagram does not need to be a chain.

Example (n=6)

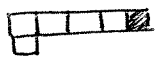


Intuitively, $\lambda \trianglerighteq \mu$ if the Ferrer diagram of λ is short and fat, and the Ferrer diagram of μ is long and skinny.

Another way to think about this order is in terms of moving boxes: $\lambda \trianglerighteq \mu$ if the Ferrer diagram of μ is obtained from the Ferrer diagram of λ by moving down some boxes through a series of "elementary operations".

There are two elementary operations:

OPERATION 1. Moving a box down to the next row:

 $(5,1)$

$\trianglerighteq$

 $(4,2)$

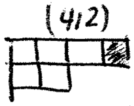
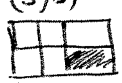
OPERATION 2 Moving a box down to the next column:

 $(2,1^5)$

$\trianglerighteq$

 (1^6)

In many cases these two operations produce the same result:

(4,2)  $\trianglerighteq$ (3,3) .

(#1 & #2) are the same

Some other times only one of the two operation is doable...

REMARK #1 - The total order $\leq$ is a refinement of the partial order $\trianglelefteq$. Indeed, if $\lambda \trianglelefteq \mu$ then $\lambda \leq \mu$.

Proof - Assume that $\lambda \trianglelefteq \mu$. If $\lambda = \mu$ there's nothing to prove, because certainly $\lambda \leq \mu$.

Assume that $\lambda \triangleleft \mu$. There exists an index k st

$$\lambda_1 + \lambda_2 + \dots + \lambda_i = \mu_1 + \mu_2 + \dots + \mu_i \quad \forall i = 1 \dots k$$

but $\lambda_1 + \lambda_2 + \dots + \lambda_k < \mu_1 + \mu_2 + \dots + \mu_k$.

Then we have:

$$\lambda_i = \mu_i \quad \forall i = 1 \dots k-1$$

$$\lambda_k < \mu_k.$$

this implies that $\lambda \leq \mu$ (in the lexicographical order)

$\vdash\!\dashv$

Notations If λ and μ are partitions of n, and let t_λ and t_μ be tableaux of shape λ and μ respectively, write $\boxed{t_\lambda \circ\!\!-\, t_\mu}$ if there are two (distinct) integers $\{a, b\}$ in $\{1, \dots, n\}$ that are included in the same row of λ and the same column of μ.

[this condition means that the transposition (a,b) belongs to the row stabilizer of t_λ and the column stabilizer of t_μ]. Otherwise, write $t_\lambda \circ\!\!\!\not\!-\, t_\mu$.

Remark #2 - If $\underbrace{\lambda > \mu}_{\text{STRICTLY GREATER IN THE TOTAL ORDER}}$, then $t_\lambda \circ\!\!-\, t_\mu$.

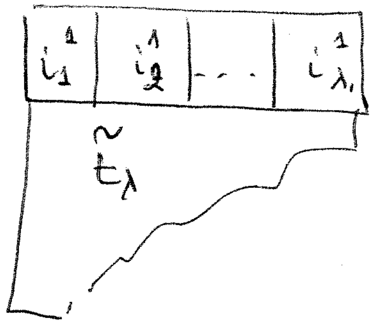
Proof - By contradiction. Assume that $t_\lambda \circ\!\!\!\not\!-\, t_\mu$, i.e.

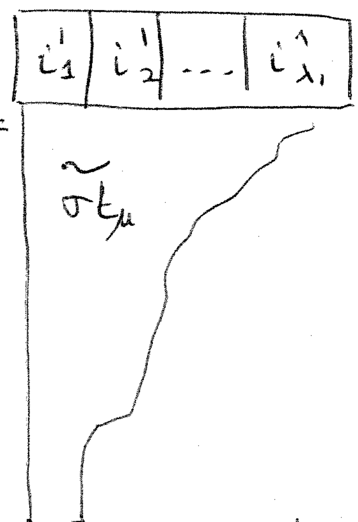
that there is no pair of indices $a, b \in \{1, \dots, n\}$ st a, b are in the same row of t_λ and the same column of t_μ.

Let $i_1^1, i_2^1, \dots, i_{\lambda_1}^1$ be the integers in the first row of t_λ. They must lie in different columns of t_μ. So the first row of μ has length $\mu_1 \geq \lambda_1$.

If $\mu_1 > \lambda_1$ we reach a contradiction.

Assume $\mu_1 = \lambda_1$. By applying a column permutation σ to t_μ we can bring $i_1^1, i_2^1, \dots, i_{\lambda_1}^1$ to the first row of t_μ.

$t_\lambda =$ 

$\sigma t_\mu =$ 

The first row of t_λ and of σt_μ coincide.

Compare the rest the two tableaux...

Again, there are no integers $a, b \in \{1 \dots n\}$ that are in the same row of $\tilde{t}_\lambda$ and the same column of $\widetilde{\sigma t_\mu}$.

So by applying another column permutation σ' you can make sure that all the elements in the second row of t_λ appear in the second row of $\sigma' \cdot \sigma \cdot t_\mu$.

This implies that $\lambda_2 \geq \mu_2$. To avoid a contradiction,

we must assume $\lambda_2 = \mu_2$.
Procede like this, until you prove that $\lambda_i = \mu_i \ \forall i$. So $\lambda = \mu$, which is itself a contradiction.
This proves that if $\lambda > \mu$ then $t_\lambda \multimap t_\mu$. ◻

Remark #3 We have proven that if $t_\lambda \multimap t_\mu$ then $k_\mu \cdot \{t_\lambda\} = 0$. So, by the previous remark, we can also say that

we proved it in the case in which the two tableaux had the same shape, but the results holds in general, with the same proof.

$$\lambda \rhd \mu \Rightarrow \lambda > \mu \Rightarrow t_\lambda \multimap t_\mu \Rightarrow k_\mu \cdot \{t_\lambda\} = 0$$

strictly greater in the partial order

strictly greater in the total order

$$\boxed{\lambda \rhd \mu \Rightarrow k_\mu \cdot \{t_\lambda\} = 0}$$

If λ and μ are any two partition st $k_\mu \cdot \{t_\lambda\} \neq 0$, then $\lambda \leq \mu$.

If λ and μ are any two comparable partitions, then we can also say that $\lambda \trianglelefteq \mu$.
these facts implies that the decomposition of M^λ in irreducible only involves Spech modules S^μ with $\mu \geq \lambda$.

We start by proving that:

Proposition - If $\mathrm{Hom}_{S^n}(S^\mu, M^\lambda) \neq \{0\}$, then $\mu \trianglerighteq \lambda$.
Moreover, $\mathrm{Hom}_{S^n}(S^\lambda, M^\lambda) = \mathbb{C}$.

[This proposition says that if S^μ is a summand of M^λ then $\mu \trianglerighteq \lambda$. It also says that S^λ appears in M^λ with multiplicity exactly one.
What's missing is only the fact that every irreducible in M^λ is a Specht module ...].

so S^μ is an irreducible summand of M^λ.

proof – Assume that $\mathrm{Hom}_{S^n}(S^\mu, M^\lambda) \neq \{0\}$, and let θ be a non-zero S^n-homomorphism $S^\mu \to M^\lambda$.
We can extend θ to an S^n-homomorphism $M^\mu \to M^\lambda$ as follows: let $\langle , \rangle$ be the inner product on M^μ defined by $\langle \{t\}, \{s\} \rangle = \delta_{\{t\},\{s\}}$ (tabloids are orthonormal) – then $\langle , \rangle$ is a non-degenerate inner product on M^λ and it's easy to check that $\langle , \rangle$ is S^n-invariant:

$$\langle \sigma\{t\}, \sigma\{s\} \rangle = \langle \{\sigma t\}, \{\sigma s\} \rangle = \delta_{\{\sigma t\},\{\sigma s\}} = \delta_{\{t\},\{s\}}$$

$$= \langle \{t\}, \{s\} \rangle.$$

It follows that $S^\mu \subseteq M^\mu$ has an S^n-invariant complement: $M^\mu = S^\mu \oplus (S^\mu)^\perp$, with $S^\mu, (S^\mu)^\perp$ both S^n-stable.
Define $\tilde\theta : M^\mu \to M^\lambda$ by $\begin{cases} \tilde\theta|_{S^\mu} = \theta \\ \tilde\theta|_{(S^\mu)^\perp} = 0. \end{cases}$

then $\tilde\theta$ is again an S^n-homomorphism.
The advantage of passing from θ to $\tilde\theta$ is the fact that $\tilde\theta$ is now defined on every tabloid of shape μ (not only on polytabloids, which are special l.c. of tabloids...)

By hypothesis, $\theta : S^\mu \to M^\lambda$ is non zero.
Then there exists a polytabloid $\underline{e}_{t_\mu}$ of shape μ
s.t. $\theta(\underline{e}_{t_\mu}) \neq 0$. [$\Leftarrow$ Polytabloids are a basis of S^μ !!].

We can write:

$$0 \neq \theta(\underline{e}_{t_\mu}) = \tilde{\theta}(\underline{e}_{t_\mu}) = \tilde{\theta}(K_{t_\mu} \cdot \{t_\mu\}) =$$

$$= \tilde{\theta}\Big(\sum_{\sigma \in C_{t_\mu}} (\operatorname{sgn}\sigma) \underbrace{\sigma}_{\text{in } S^n} \cdot \underbrace{\{t_\mu\}}_{\text{in } M^\mu}\Big) \underset{\substack{\uparrow \\ \tilde{\theta} \text{ linear} \\ \text{and } S^n\text{-homom.}}}{=} \sum_{\sigma \in C_{t_\mu}} (\operatorname{sgn}\sigma)\, \sigma \cdot \underbrace{\tilde{\theta}\{t_\mu\}}_{\text{well defined!}} =$$

$$= \Big[\sum_{\sigma \in C_{t_\mu}} \operatorname{sgn}\sigma\ \sigma\Big] \tilde{\theta}\{t_\mu\} = K_{t_\mu} \cdot \tilde{\theta}\{t_\mu\}.$$

Write $\tilde{\theta}\{t_\mu\} = \sum_{\substack{\text{Tabloids} \\ \text{of shape } \lambda}} c_{\{s\}} \{s\}$. [Such decomposition exists because $\tilde{\theta}\{t_\mu\} \in M^\lambda$, and a basis of M^λ consists of tabloids of shape λ].

$$\Rightarrow\ 0 \neq K_{t_\mu} \cdot \Big[\sum c_{\{s\}} \{s\}\Big] = \sum c_{\{s\}} K_{t_\mu} \cdot \{s\}.$$

$\Rightarrow$ there exists a tableau $\{s_\lambda\}$ of shape λ such that $K_{t_\mu} \cdot \{s\} \neq 0$.

By our previous remarks, we have $\mu \trianglerighteq \lambda$ (as claimed).

Next, we prove that $\operatorname{Hom}_{S^n}(S^\lambda, M^\lambda) = \mathbb{C}$, i.e. every S^n-homomorphism from S^λ to M^λ is given by scalar multiplication.

Let $\theta \in \mathrm{Hom}_{S^n}(S^\lambda, M^\lambda)$, and let $\tilde{\theta}$ be its extension $M^\lambda \to M^\lambda$. [10]

If $\theta \neq 0$, let t_λ be a tableau s.t. $\theta(\underline{e}_{t_\lambda}) \neq 0$.

then, with the same notations used before, we can write:

$$0 \neq \theta(\underline{e}_{t_\lambda}) = k_{t_\lambda} \cdot \tilde{\theta}\{t_\lambda\} = k_{t_\lambda} \cdot \left(\sum c_s \{s\} \right) =$$

$$= \underbrace{\sum c_{\{s\}} k_{t_\lambda} \cdot \{s\}}_{= \varepsilon_{\{s\}} \underline{e}_{t_\lambda}} = \underbrace{\left[\sum c_s \, \varepsilon_{\{s\}} \right]}_{\text{some constant}} \underline{e}_{t_\lambda} = c \, \underline{e}_{t_\lambda}.$$

± 1 or 0

because s and t_λ have the same shape $(=\lambda)$

By construction, $c \neq 0$ and θ acts on $\underline{e}_{t_\lambda}$ as multiplication by c. Let's prove that θ acts this way everywhere on S^λ...

Let t' be any tableau of shape λ.

$\Rightarrow \exists \sigma \in S^n : t' = \sigma \cdot t_\lambda$. We can write:

$$\theta(\underline{e}_{t'}) = \theta(\underline{e}_{\sigma \cdot t_\lambda}) = \theta(\sigma \cdot \underline{e}_{t_\lambda}) = \sigma \, \theta(\underline{e}_{t_\lambda}) =$$

$$= \sigma (c \, \underline{e}_{t_\lambda}) = c(\sigma \underline{e}_{t_\lambda}) = c \, \underline{e}_{t'}$$

$\Rightarrow \theta$ is multiplication by c.

This proves that $\dim_{\mathbb{C}} \mathrm{Hom}_{S^n}(S^\lambda, M^\lambda) = 1$, as claimed. $\square$

<u>Remark</u> - So far we get:

$$M^\lambda = S^\lambda \oplus \left[\bigoplus_{\mu \rhd \lambda} \underbrace{m_{\mu\lambda}}_{\text{some multiplicity} \geq 0} S^\mu \right] \oplus \{???\}$$

a priori there could be something else (that is NOT a Specht module)

We still need to prove that there are no other irreducible" summands in M^λ.

This will follow from a more general fact:

<u>theorem</u> - The set $\{S^\lambda : \lambda \vdash n\}$ forms a complete list of irreducible inequivalent representations of S^n.

<u>Proof</u> - It's enough to show that $S^\lambda \not\cong S^{\lambda'}$ if $\lambda \neq \lambda'$. Then we will have a total number of irreducible inequivalent representations of S^n equal to the # of partitions of n (= # conjugacy classes of S^n) and our list will be complete.

Assume $S^\lambda \cong S^\mu$, say via $T: S^\lambda \to S^\mu$, with $T \neq 0$. Because $S^\mu \subseteq M^\mu$, you can regard T as an S^n-homomorphism $T: S^\lambda \longrightarrow M^\mu$. By construction, $T \neq 0$ so $\lambda \trianglerighteq \mu$. Similarly, T^{-1} gives $\mu \trianglerighteq \lambda$. Hence $\lambda = \mu$. $\boxtimes$

<u>Corollary</u> - $M^\lambda = \bigoplus_{\mu \trianglerighteq \lambda} m_{\mu\lambda} S^\mu$, and $m_{\lambda\lambda} = 1$.

KOSTKA NUMBERS

Let $\mu = (\mu_1, \dots, \mu_m)$ be a composition of n, i.e. a sequence of positive integers s.t. $\sum_{i=1}^{m} \mu_i = n$.

Let λ be a partition of n.

Def – A <u>generalized Young Tableau</u> of shape λ is an array T obtained by filling up the boxes of the Ferrer diagram of T with positive integers. [No conditions are imposed on the ordering of these integers are imposed. Also notice that repetitions <u>are</u> allowed].

For instance, $T = \begin{array}{|c|c|c|} \hline 4 & 1 & 4 \\ \hline 1 & 3 & \\ \cline{1-2} \end{array}$ is a generalized Young tableau.

Def – Given a generalized Young Tableau T, we say that T has <u>content</u> μ if μ_i is equal to the number of i's in T.

In the example, T has content $\mu = (2, 0, 1, 2)$.

If T has shape λ and content μ, we also write $T \in \Upsilon_{\lambda\mu}$, in other words we denote by $\Upsilon_{\lambda\mu}$ the set of all generalized Young tableau of shape λ and content μ.

Def – A generalized Young Tableau T is called <u>semistandard</u> if the rows of T are weakly increasing and the columns of T are strictly increasing.

[Rows should weakly increase from left to right, Columns should strictly increase from top to bottom].

Example $\quad T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$ is semistandard

but $T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 1 & 4 \\ \cline{1-2}\end{array}$ is not semi-standard.

We denote the set of semistandard generalized Young Tableau of shape λ and content μ by $\mathcal{T}^{o}_{\lambda\mu}$.

Def- The Kostka numbers are

$$K_{\lambda\mu} = \# \mathcal{T}^{o}_{\lambda\mu}.$$

Example: Suppose that $\mu = (221)$ and $\lambda = (3,2)$.

then there are 2 generalized Young Tableaux of shape λ and content μ:

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \qquad T_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

Remark - Suppose that both λ and μ are partitions. Then

$$K_{\lambda\mu} = \begin{cases} 0 & \text{if } \lambda < \mu \\ 1 & \text{if } \lambda = \mu \end{cases}$$

Proof- Suppose that $K_{\lambda\mu} \neq 0$. So there exists at least one generalized Young Tableau of shape λ and content μ.

Say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$.

We notice that an integer $\boxed{i}$ can only occur in the first i rows of T (if i sits in position (l, j) with $l > i$, then the entries of the j^{th} column cannot be strictly increasing).

So the first row of T contains at least all the one's:

$$\underbrace{\lambda_1}_{\text{length of } 1^{st} \text{ row of } T} \geq \underbrace{\mu_1}_{\text{Total } \# \text{ of } 1\text{'s}}$$

the first two rows of T contain all the one's and all the two's:

$$\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$$

and, in general, $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \forall i$.

Equivalently, $\lambda \trianglerighteq \mu \implies \lambda \geq \mu$.

We have shown that $K_{\lambda\mu} \neq 0 \Rightarrow \lambda \geq \mu$. Hence $K_{\lambda\mu} = 0$ if $\lambda < \mu$.

Next, we look at the case $\lambda = \mu$.

the only possibility to fill up a tableau of shape $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$ with λ_1 many ones, λ_2 many twos ... in a way that integers increase weakly along rows and strictly along columns is to put all of the i's in the i^{th} row.

1	1	1	1
2	2	2	
3	3		
4			
5			
6			

$\leftarrow$ example: $\lambda = \mu = (4, 3, 2, 1, 1, 1)$.

Therefore, $K_{\lambda\lambda} = 1$.

<u>theorem</u> (Young's rule)
The multiplicity of S^λ in M^μ is equal to the number of semistandard Tableau of shape λ and content μ, i.e.

$$M^\mu = \bigoplus_\lambda K_{\lambda\mu} S^\lambda.$$

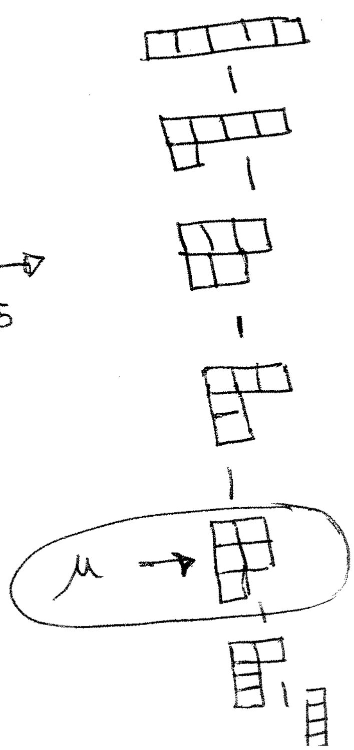
By the propeities of Kostka numbers, we can also write:

$$M^\mu = S^\mu \oplus \left[\bigoplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^\lambda \right].$$

We will not prove this now.

<u>Example</u>: $\mu = (2,2,1)$. What's the decomposition of M^μ?
μ is a partition of 5. We start by looking at all the partitions of 5 that are greater than μ:

Hasse diagram for $n=5$ →



these are the λ's that dominate μ.

For each $\lambda \trianglerighteq \mu$, we must compute the Kostka number $K_{\lambda\mu}$ (= # semi-standard ~~generalized~~ Y. Tableaux of shape λ and content μ.

Recall that $\mu = (2,2,1)$. So we must fill up a Tableau of shape λ with 2 one's, 2 Two's and 1 Three. Rows should weakly increase, columns should strictly increase.

- $\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline\end{array} \longrightarrow K_{\lambda,\mu} = 1$ if $\lambda = (5)$

- $\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2\\\cline{1-1}\end{array}$ or $\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3\\\cline{1-1}\end{array} \longrightarrow K_{\lambda\mu} = 2$ if $\lambda = (4,1)$

- $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$ or $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array} \longrightarrow K_{\lambda\mu} = 2$ if $\lambda = (3,2)$

- $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \longrightarrow K_{\lambda\mu} = 1$ if $\lambda = (3,1,1)$

- $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \longrightarrow K_{\lambda\mu} = 1$ if $\lambda = (2,2,1)$

So we obtain:

$$M^{(2,2,1)} = S^{(2,2,1)} \oplus S^{(3,1,1)} \oplus \underline{\underline{2}}\, S^{(3,2)} \oplus \underline{\underline{2}}\, S^{(4,1)} \oplus S^{(5)}.$$
