DEF. OF S^λ

Let $\boxed{\lambda}$ be a partition of n, let $\boxed{\Gamma}$ be the corresponding Ferrers diagram and let $\boxed{t}$ be a tableau of shape Γ.

<u>Example:</u> $n = 4$; $\lambda = (2,1,1)$; $\Gamma = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$; $t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$.

Denote by $R_1^t, R_2^t \dots R_e^t$ the rows of t, and by $C_1^t \dots C_k^t$ the columns of t. [They are subsets of $\{1,2,\dots n\}$].

<u>Example</u>: $t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \Rightarrow$ $R_1^t = \{1,4\}$, $R_2^t = \{3\}$, $R_3^t = \{2\}$
$C_1^t = \{1,3,2\}$, $C_2^t = \{4\}$

The row-stabilizer of t is the group

$$R^t = S_{R_1^t} \times S_{R_2^t} \times \dots \times S_{R_e^t} = \{\sigma \in S_n : \sigma \text{ preserves each row of } t\} \leq S_n$$

(For each set A we have denoted by S_A the symmetric group on A).

Similarly, we define the column-stabilizer of t to be the group

$$C^t = S_{C_1^t} \times S_{C_2^t} \times \dots \times S_{C_k^t} = \{\sigma \in S_n : \sigma \text{ preserves each column of } t\} \leq S_n$$

<u>Example</u>: $t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$\Rightarrow R^t = \{\mathbb{1}, (14)\} \leq S_4$.

$C^t = \{\mathbb{1}, (132), (123), (12), (13), (23)\} \subsetneq S_4$.

<u>Remark</u> For every Tableau t and every permutation $\pi \in S_n$, we have:

- $R_{\pi t} = \pi R_t \pi^{-1}$

- $C_{\pi t} = \pi C_t \pi^{-1}$.

<u>Proof</u> - Just notice that if $R_j^t = \{i_1, \dots, i_{m_j}\}$ is the j^{th} row of t, then

$$\pi R_j^t = \{\pi(i_1), \dots, \pi(i_{m_j})\} = R_j^{\pi t}$$

is the j^{th} row of πt, and that for every permutation ρ of $\{i_1 \dots i_{m_j}\}$,

$\pi \rho \pi^{-1}$ is a permutation of $\{\pi(i_1) \dots \pi(i_{m_j})\}$.

So we have:

$$\underbrace{S_{\underbrace{\pi R_j^t}_{\overset{\shortparallel}{R_j^{\pi t}}}}}_{\text{permutations of the } j^{th} \text{ row of } \pi t} = \pi \underbrace{S_{R_j^t}}_{\text{permutations of the } j^{th} \text{ row of } t} \pi^{-1}$$

<u>idea</u> =

$$\{\pi(i_1) \dots \pi(i_{m_j})\} \xrightarrow{\pi^{-1}} \{i_1 \dots i_{m_j}\} \xrightarrow{\sigma} \{i_{\sigma(1)} \dots i_{\sigma(m)}\} \xrightarrow{\pi} \{\pi(i_{\sigma(1)}), \dots \pi(i_{\sigma(m)})\}$$

is a permutation of $\{\pi(i_1) \dots \pi(i_m)\}$

for every j.

$$\Rightarrow R^{\pi t} = S_{R_1^{\pi t}} \times S_{R_2^{\pi t}} \times \dots \times S_{R_e^{\pi t}} =$$

$$= \left(\pi S_{R_1^t} \pi^{-1}\right) \times \left(\pi S_{R_2^t} \pi^{-1}\right) \times \dots \times \left(\pi S_{R_e^t} \pi^{-1}\right) =$$

$$= \pi \left(S_{R_1^t} \times S_{R_2^t} \times \dots \times S_{R_e^t} \right) \pi^{-1} =$$

$$= \pi R^t \pi^{-1} \checkmark$$

Same for the column stabilizer ... $\square$

POLYTABLOIDS

Given a Tableau t, we also define:

$$k_t = \sum_{\sigma \in C^t} \operatorname{sgn}(\sigma)\, \sigma .$$

column stabilizer

(k_t is an element of The group algebra $\mathbb{C}[S_n]$, so it acts on Tableaux <u>and</u> on Tabloids).

<u>Example</u> $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array}$

- $C^t = \{\mathbb{1}, (43) ; (15) ; (43)(15)\}$

  $\mathbb{1}$, $(43)(15)$: even; (43), (15): odd

- $k_t = \mathbb{1} - (43) - (15) + (43)(15)$

We are particularly interested in The action of k_t on The Tabloid $\{t\}$ (associated to the tableau t).

<u>example</u> $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} \Rightarrow \{t\} = \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array}$

$\Rightarrow k_t \{t\} = \{t\} - (43)\{t\} - (15)\{t\} + (43)(15)\{t\}$

$$= \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 3 & 1 & 2 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 4 & 5 & 2 \\ \hline 3 & 1 & \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 3 & 5 & 2 \\ \hline 4 & 1 & \\ \cline{1-2}\end{array} .$$

$k_t\{t\}$ is a linear combination of tabloids, with coefficients ± 1. So is an element of M^λ

<u>Definition</u>: if t is a tableau, we call $e_t = k_t\{t\}$ the polytabloid associated to t.

Remark: If t is any tableau

- $\kappa_{\pi t} = \pi \kappa_t \pi^{-1}$

- $\underline{e}_{\pi t} = \pi \underline{e}_t$

proof - $\kappa_{\pi t} = \sum_{\rho \in C^{\pi t}} sgn(\rho)\,\rho \underset{C^{\pi t} = \pi C^t \pi^{-1}}{=} \sum_{\sigma \in C^t} sgn(\pi\sigma\pi^{-1})(\pi\sigma\pi^{-1}) =$

$= \sum_{\sigma \in C^t} sgn(\sigma)(\pi\sigma\pi^{-1}) = \pi\left(\sum_{\sigma\in C^t} sgn(\sigma)\,\sigma\right)\pi^{-1} = \pi \kappa_t \pi^{-1}.$ ✓

Also:

$\kappa_{\pi t}\cdot\{\pi t\} = (\pi \kappa_t \pi^{-1})\cdot\{\pi t\} = \{\pi\kappa_t\pi^{-1}\cdot(\pi t)\} =$

$= \{\pi \kappa_t\cdot(t)\} = \pi\kappa_t\cdot\{t\} = \pi\cdot(\kappa_t\cdot\{t\}) = \pi\cdot\underline{e}_t$ ✓

SPECHT MODULE S^λ

Let $\lambda \vdash n$ be a partition of n. For each tableau t of shape λ we can construct the polytabloid $\underline{e}_t$ (which is an element of M^λ). Define S^λ to be the subspace of M^λ generated by all the polytabloids $\underline{e}_t$ (of shape λ).

S^λ is stable under the action of S_n: indeed if $\underline{e}_t$ is a polytabloid of shape λ, and π is a permutation then $\pi\cdot\underline{e}_t = \underline{e}_{\pi t}$ is again a polytabloid of shape λ

$\Rightarrow S_n$ permutes the generators of S^λ

$\Rightarrow S_n$ stabilizes S^λ

$\Rightarrow S^\lambda$ is a subrepresentation of M^λ. We call S^λ the Specht module corresponding to λ

<u>Example1</u>

- $\lambda = (n)$
- $t = \boxed{1 \mid 2 \mid 3 \mid \cdots \mid n}$
- $\{t\} = \overline{\underline{1\ 2\ 3 \cdots n}}$
- $C^t = \{\mathbb{1}\}$ (only the trivial permutation stabilizes every column of t)

- $k_t = \sum_{\sigma \in C^t} \mathrm{sgn}(\sigma)\, \sigma = \mathbb{1}$

$\Rightarrow \underline{e}_t = k_t \{t\} = \{t\}$

(This is the only polytabloid of shape λ).
$\Rightarrow S^{(n)} \underset{\text{generated by all polytabloid}}{\overset{\text{subspace of } M^{(n)}}{=}}$ ⇝ 1 dimensional.
Notice that S_n acts trivially on $\underline{e}_t$ (because $S_n \equiv R^t =$ row-stabilizer of t):

$$\sigma \cdot \underline{e}_t = \sigma\{t\} = \{t\} \quad \forall \sigma \in S_n .$$

$\Rightarrow S^\lambda$ is the trivial representation of S_n.

<u>Example 2</u>

- $\lambda = (1^n)$
- $t = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline n \\ \hline \end{array} \Rightarrow \{t\} = \begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$ (is the same as the tableau t)

- $C^t =$ column-stabilizer of $t = S^n$
- $k_t = \sum_{\sigma \in C^t} (\mathrm{sgn}\,\sigma)\, \sigma = \sum_{\sigma \in S^n} (\mathrm{sgn}\,\sigma)\, \sigma .$
- $e_t = k_t \{t\}$ (is a linear combination of tabloids)

$$\underline{e}_t = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \begin{array}{c} \hline \sigma(1) \\ \hline \sigma(2) \\ \hline \vdots \\ \hline \sigma(n) \\ \hline \end{array}.$$

Now choose another Tableau $t' = \begin{array}{|c|} \hline \pi(1) \\ \hline \pi(2) \\ \hline \vdots \\ \hline \pi(n) \\ \hline \end{array}$. Clearly, $t' = \pi t$ for some $t \in S_n$.

$$\Rightarrow \underline{e}_{t'} = \pi \underline{e}_t = \pi \left(\sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) \begin{array}{c} \hline \sigma(1) \\ \hline \sigma(2) \\ \hline \vdots \\ \hline \sigma(n) \\ \hline \end{array} \right) =$$

$$= \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) \begin{array}{c} \hline \pi\sigma(1) \\ \hline \pi\sigma(2) \\ \hline \vdots \\ \hline \pi(\sigma(n)) \\ \hline \end{array} = (\operatorname{sgn}\pi) \underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\pi\sigma) \begin{array}{c} \hline \pi\sigma(1) \\ \hline \pi(\sigma)(2) \\ \hline \vdots \\ \hline \pi\sigma(n) \\ \hline \end{array}}_{\text{it's again } \underline{e}_t} =$$

$$= (\operatorname{sgn}\pi)\, \underline{e}_t.$$

$\Rightarrow$ Up to sign, there's a unique polytabloid

$\Rightarrow$ The subspace $S^{(1^n)}$ of $M^{(1^n)}$ is 1-dimensional.

Let's describe the action of S_n on the Specht modules

Because $S^{(1^n)} = \mathbb{C}\underline{e}_t$, it's enough to specify the action of S_n on $\underline{e}_t$.

If $\rho \in S_n$, then $\rho \underline{e}_t \underset{\text{as above}}{=} (\operatorname{sgn}\rho)\underline{e}_t$. So S_n acts on $S^{(1^n)}$ via the sign representation.

<u>Example 3</u>

- $\lambda = (n-1, 1)$
- $t = \begin{array}{|c|c|c|c|c|} \hline i_1 & \cdots & \cdot & \cdots & i_{n-1} \\ \hline i_n \\ \cline{1-1} \end{array}$ a tableau

- $\{t\} = \begin{array}{l}\overline{i_1\, i_2 \cdots i_{n-1}} \\ \underline{i_n}\end{array}$

(notice that we can identify t with its last entry i_n).

- C^t = column stabilizer of t $= \{(i_1\, i_n)\,;\, \mathbb{1}\}$
- $k_t = \sum_{\sigma \in C^t} (\operatorname{sgn}\sigma)\,\sigma = \mathbb{1} - (i_1\, i_n)$
- $e_t = k_t \{t\} = \begin{array}{l}\overline{i_1\, i_2 \cdots i_{n-1}} \\ \underline{i_n}\end{array} - \begin{array}{l}\overline{i_n\, i_2 \cdots i_{n-1}} \\ \underline{i_1}\end{array}$

We want to describe the subspace of M^λ generated by all polytabloids.

Let's fix some notations:

$\boxed{1} = \begin{array}{l}\overline{2\,3\,4 \cdots n} \\ \underline{1}\end{array}$; $\boxed{2} = \begin{array}{l}\overline{1\,3\,4 \cdots n} \\ \underline{2}\end{array}$; $\boxed{3} = \begin{array}{l}\overline{1\,2\,4 \cdots n} \\ \underline{3}\end{array}$

$\cdots$ $\boxed{n-1} = \begin{array}{l}\overline{1\,2 \cdots n-2\;\, n} \\ \underline{n-1}\end{array}$; $\boxed{n} = \begin{array}{l}\overline{1\,2 \cdots n-2\;\, n-1} \\ \underline{n}\end{array}$

so that

$M^\lambda = \mathbb{C}\langle \boxed{1}, \boxed{2}, \ldots, \boxed{n} \rangle$ ← n-dimensional

and every polytabloid is of the form $\boxed{i} - \boxed{j}$ for some $i, j = 1 \ldots n$, $i \neq j$.

$\Rightarrow$ S^λ = subspace of M^λ generated by all polytabloids

$= \{ c_1 \boxed{1} + c_2 \boxed{2} + \ldots + c_n \boxed{n} \; : \; c_1 + c_2 + \ldots + c_n = 0 \}$

↳ $(n-1)$-dimensional

As a basis for $S^{\lambda=(n-1,n)}$ we can choose, for instance, the polytabloids:

$$\boxed{2}-\boxed{1}\ ;\ \boxed{3}-\boxed{1}\ ;\ \cdots\ ;\ \boxed{n-1}-\boxed{1}\ ;\ \boxed{n}-\boxed{1}.$$

Let's describe the action of S_n on this basis.

$$\pi\circ\boxed{i} = \pi\circ\begin{array}{l}\overline{j_1\,j_2\cdots j_{n-1}}\\ \underline{i}\end{array} = \begin{array}{l}\overline{\pi(j_1)\ \pi(j_2)\cdots\pi(j_{n-1})}\\ \underline{\pi(i)}\end{array} = \boxed{\pi(i)}$$

so

$$\pi\circ\left(\boxed{i}-\boxed{j}\right) = \boxed{\pi(i)}-\boxed{\pi(j)},\ \text{for all } i,j=1\ldots n\ (i\neq j)$$

and in particular

$$\pi\circ\left(\boxed{i}-\boxed{1}\right) = \boxed{\pi(i)}-\boxed{1}.$$

DESCRIPTION OF THE SPECHT MODULE $S^{(n-1,n)}$

Let $V=\{\underline{x}\in\mathbb{C}^n : x_1+x_2+\cdots+x_n=0\}$.

Let S_n act on V by

$$\sigma\cdot\begin{vmatrix}x_1\\x_2\\\vdots\\x_n\end{vmatrix} = \begin{vmatrix}x_{\sigma^{-1}(1)}\\x_{\sigma^{-1}(2)}\\\vdots\\x_{\sigma^{-1}(n)}\end{vmatrix}.$$

PERMUTATION REPRESENTATION

$$\Rightarrow V\cong S^{(n-1,n)}.$$

IRREDUCIBILITY OF S^λ

We need a few lemmas.

Lemma 1 - $\boxed{S^\lambda \text{ is cyclic}}$

This is very easy: choose a tableau t of shape λ.
If s is another tableau of shape λ, then we can write
$s = \pi_s \cdot t$ for some $\pi_s \in S_n$.

$$\Rightarrow \quad e_s = \pi_s \cdot e_t \in S_n \cdot e_t .$$

Because the polytabloids (of shape λ) are a basis for S^λ, it's clear that $S^\lambda = S_n \cdot e_t$ (for each fixed polytabloid t of shape λ).

$\longmapsto$

remark

Next we introduce an inner product in $M^\lambda \supseteq S^\lambda$.

Recall that M^λ is the vector space $/_{\mathbb{C}}$ with basis the various tabloids. So the rule:

$$\langle \{t\}, \{s\} \rangle = \begin{cases} 0 & \text{if } \{t\} \neq \{s\} \\ 1 & \text{if } \{t\} = \{s\} \end{cases}$$

extends uniquely to an inner product on M^λ.

Let's show that (wrt to this inner product), every K_t acts as a symmetric operator:

$$\langle K_t \underline{x}, \underline{y} \rangle = \langle \underline{x}, K_t \underline{y} \rangle \quad \text{for all } \underline{x}, \underline{y} \text{ in } M^\lambda.$$

Again, it's enough to prove this for $\underline{x}$ and $\underline{y}$ tabloids ---
So choose s_1, s_2 tableaux of shape λ, and set
$\underline{x} = \{s_1\}$, $\underline{y} = \{s_2\}$.

Then:

$$\langle K_t \{s_1\}, \{s_2\} \rangle = \langle \sum_{\pi \in C(t)} (\operatorname{sgn} \pi)\, \pi \{s_1\}, \{s_2\} \rangle =$$

$$= \sum_{\pi \in C(t)} \operatorname{sgn} \pi \langle \pi \{s_1\}, \{s_2\} \rangle =$$

1 if $\pi\{s_1\} = \{s_2\}$, 0 otherwise

$$= \sum_{\pi \in C(t)} (\operatorname{sgn} \pi)\, \delta_{\pi\{s_1\}, \{s_2\}} =$$

$$= \sum_{\pi \in C(t)} (\operatorname{sgn} \pi)\, \delta_{\{s_1\}, \pi^{-1}\{s_2\}} =$$

$$= \sum_{\pi \in C(t)} (\operatorname{sgn} \pi^{-1})\, \delta_{\{s_1\}, \pi^{-1}\{s_2\}} =$$

$$= \sum_{\tilde{\pi} \in C(t)} \operatorname{sgn} \tilde{\pi}\, \delta_{\{s_1\}, \tilde{\pi}\{s_2\}}$$

$$= \langle \{s_1\}, \sum_{\tilde{\pi} \in C(t)} (\operatorname{sgn} \tilde{\pi})\, \tilde{\pi}\{s_2\} \rangle =$$

$$= \langle \{s_1\}, K_t \{s_2\} \rangle \checkmark$$

$\dashv$

The following lemma is crucial:

Lemma: Let t and s be Tableaux of the same shape.

① if there is a pair of indices that belong both to row of t and a column of s, then $K_s \cdot \{t\} = 0$.

② otherwise, $K_s \cdot \{t\} = \pm e_s$.

Proof — Suppose that t and s are both Tableau of shape λ.

<u>CASE 1</u> There is a pair of indices $\{b, c\}$ that appear in one row of t and in one column of s.

Then we can consider the transposition (bc), which is an element of $C(s)$ ← the column stabilizer of s.

Notice that $K_s \cdot (bc) = \left[\sum_{\sigma \in C(s)} (\operatorname{sgn} \sigma)\, \sigma\right] (bc) =$

$$= \sum_{\sigma \in C(s)} \underbrace{(\operatorname{sgn} \sigma)}_{-\operatorname{sgn} \tau} \underbrace{\sigma (bc)}_{\tau \in C(s)} = - \sum_{\tau \in C(s)} [\operatorname{sgn} \tau]\, \tau = - K_s .$$

It follows that

$$K_s \cdot \{t\} = K_s \cdot \left[(bc) \cdot \{t\}\right] = \left[K_s \cdot (bc)\right] \cdot \{t\} = - K_s \cdot \{t\}$$

(bc) belongs to $R(t)$ ← row stabilizer of t so it does not affect the Tabloid…

$\Rightarrow K_s \cdot \{t\} = 0.$

<u>CASE 2</u> There is no pair of indices that appear both in a row of t and a column of s.

Look at the first row of t : $\{i_1 \dots i_m\}$. These elements belong to distinct columns of s, so by applying some $\pi_1 \in C(s)$ we can arrange that they all appear in the first row of s (possibly in a different ordering w.r.t. the 1st row of t).

$\pi_1 s$ is a new tableau. but $C(\pi_1 s) = C(s)$.

~~By applying~~ Notice that $\pi_1 s$ again satisfies the property that there is no pair of indices in the

we can iterate the construction, and say that for some $\pi_2 \in C(\pi_1 s) = C(s)$, $\pi_2 \pi_1 s$ and t have the same elements in the first two rows.
Keep doing. Arrange that t and $\pi_k \pi_{k-1} \cdots \pi_2 \pi_1 s$ have the same elements in each of the k-rows. [Here we have chosen k to be the # of rows].
By construction $\pi = \pi_k \pi_{k-1} \cdots \pi_2 \pi_1 \in C(s)$. So we have shown that t and πs (for $\pi \in C(s)$) only differ by the ordering of the elements in each row $\Rightarrow$ $\{t\} = \{\pi s\}$. Because $\{\pi s\} = \pi\{s\}$, it follows that $\{t\} = \pi\{s\}$, for π in $C(s)$.
Then we can write:

$$K_s \cdot \{t\} = K_s \cdot \pi \cdot \{s\} = \left[\sum_{\sigma \in C(s)} (\operatorname{sgn} \sigma)\sigma\right]\pi \cdot \{s\} =$$

$$= (\operatorname{sgn}\pi)\left[\sum_{\sigma \in C(s)} (\operatorname{sgn}\sigma)(\operatorname{sgn}\pi)\,\sigma\pi\right]\cdot\{s\} =$$

$$" \sum_{\sigma' \in C(s)} (\operatorname{sgn}\sigma')\sigma' = K_s$$

$$= (\operatorname{sgn}\pi)\cdot K_s\cdot\{s\} =$$

$$= (\operatorname{sgn}\pi)\,\underline{e}_s = \pm\underline{e}_s .$$

∎

Now we are ready to prove the irreducibility of S^λ. Theorem: S^λ irreducible. Proof:
Pick $U \subseteq S^\lambda$ S_n-invariant. To show: $U = S^\lambda$ or $U = \{0\}$.

- Fix any Tableau s of shape λ. Then $S^\lambda = S_n \cdot e_s$. Choose an element $\underline{u}$ in U. Because $U \subseteq S^\lambda \subseteq M^\lambda$, we can write $\underline{u}$ as a l.c. of tabloids:

$$\underline{u} = \sum_{t} \underbrace{c_t}_{\text{in } \mathbb{C}} \{t\}.$$

t Tableau of shape λ

then $K_s \cdot \underline{u} = K_s \cdot \left(\sum_t c_t \{t\} \right) = \sum_t \underbrace{c_t}_{\text{in } \mathbb{C}} \underbrace{K_s \cdot \{t\}}_{\text{either } 0 \text{ or } \pm e_s} =$

$= \underbrace{c_1}_{\text{some } c_1 \text{ in } \mathbb{C}} e_s$.

We distinguish Two cases:

- $c_1 \neq 0 \Rightarrow$ we can write $e_s = \frac{1}{c_1} \underbrace{(K_s \cdot \underline{u})}_{} \in U$

$\underline{u} \in U$ and U is S_n-stable $\Rightarrow K_s \cdot \underline{u} \in U$

$\Rightarrow \underbrace{S_n \cdot e_s}_{= S^\lambda} \subseteq U$

$\Rightarrow S^\lambda \subseteq U \underset{U \subseteq S^\lambda}{\Rightarrow} \boxed{U \equiv S^\lambda}$

- $c_1 = 0 \Rightarrow K_s \cdot \underline{u} = \underline{0}$. Then we can write:

$\langle \underline{u}, e_s \rangle = \langle \underline{u}, K_s \cdot \{s\} \rangle = \langle \underbrace{K_s \underline{u}}_{\underline{0}}, \{s\} \rangle = \langle \underline{0}, \{s\} \rangle =$

$\Rightarrow \underline{u}$ is perpendicular To e_s

but the choice of s (t Tableau of shape λ) was arbitrary. So we can say that $\underline{u} \perp \underline{e}_s$ for every polytabloid $\underline{e}_s$ of shape λ. these objects are a basis for $M^\lambda \Rightarrow \underline{u} \perp M^\lambda$

By assumption $\underline{u} \in S^\lambda \subseteq M^\lambda$, so $\underline{u} \perp \underline{u} \Rightarrow \underline{u} = \underline{0}$.

(write $\underline{u} = \sum_t c_t \{t\}$ in $\mathbb{C}$, then $\langle \underline{u}, \underline{u} \rangle = \sum c_t^2 \underbrace{\langle \{t\}, \{t\} \rangle}_{1}$
$= \sum c_t^2 = 0 \iff c_t = 0$ for all t).

Because $\underline{u}$ was an arbitrary elem.t of U, it follows that $U = \{\underline{0}\}$.

$\blacksquare$

NEXT: $M^\lambda = \underbrace{S^\lambda}_{\text{mult } 1} \oplus \sum_{\lambda_i \trianglerighteq \lambda} (S^{\lambda_i})^{\oplus a_i}$