REPRESENTATIONS OF S_n

The number of conjugacy classes of S_n is equal to the number of partitions of n

(the bijection sends $\underset{\lambda_1}{(---)}\underset{\lambda_2}{(--)}\underset{\lambda_3}{(--)} \longleftrightarrow (\lambda_1 \geq \lambda_2 \geq \lambda_3)$)

hence the number of irreducible inequivalent representations of S_n is also equal to the number of partitions of n.

$\Rightarrow$ for each partition $\lambda \vdash n$, we must produce an irreducible representation of S_n. This is not an obvious task, and will take us a couple of lectures ...

PERMUTATION REPRESENTATION OF $\lambda \vdash n$

Given any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k) \vdash n$, consider the Young subgroup

$$S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k} \subseteq S_n$$

and set

$$M^\lambda = \mathrm{Ind}_{S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}}^{S_n} (\text{trivial}).$$

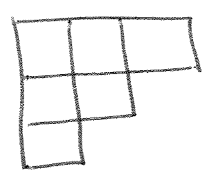
"PERMUTATION REPR. OF $\lambda \vdash n$"

M^λ is a well defined representation of S_n (of dimension $= [S_n : S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}] = \dfrac{n!}{\lambda_1! \lambda_2! \dots \lambda_k!}$), and is

naturally associated to the partition λ.
Although M^λ is almost always reducible, a good understanding of M^λ will give insigth on how to construct the irreducible representation associated to λ.

The best way to understand M^λ is to give an alternative construction...
We start with some notations/definitions.

Def- If $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k > 0)$ is a partition of n, we define the FERRER DIAGRAM OF λ to be an array with λ_j left-justified boxes in the j^{th} row:

$\lambda = (3,2,1) \longleftrightarrow$ 

If we fill up the n boxes of the Ferrer diagram with the integers $\{1 \dots n\}$ (allowing no repetitions) we obtain a Young Tableau of shape λ.

For instance

$$t_1 = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \quad \text{and} \quad t_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

are two tableaux of shape $\lambda = (2,1,1)$.
If $\lambda \vdash n$, there are exactly $n!$ tableaux of shape λ.

Definition – Let t_1 and t_2 be two tableaux of shape λ. We say that t_1 and t_2 are row-equivalent if corresponding rows have the same elements.

For instance

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array} \quad \text{and} \quad t_2 = \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array}$$

are row-equivalent, but $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ are not.

Row-equivalence is an equivalence relation among tableaux of the same shape. The equivalence class of a tableau t is called a TABLOID, and is denoted by $\{t\}$:

$$\{t\} = \{ t' : t' \sim t \}.$$

If $t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array}$, we write $\{t\} = \begin{array}{c}\hline 1\,2\,3 \\ \hline 4\,5 \\ \hline\end{array}$ for the corresponding tabloid. The vertical lines are omitted to suggest that you can permute the entries in each row, without modifying the tabloid.

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0) \vdash n$, there are $\dfrac{n!}{\lambda_1!\lambda_2!\dots\lambda_k!}$ different tabloids of shape λ.

For instance if $\lambda = (n)$, then there is only 1 tableau $\begin{array}{c}\hline 1\,2\,3 \cdots n \\ \hline\end{array}$, if $\lambda = 1^n$, then there are $n!$ tableaux of shape λ.

We notice that there is a natural action of S_n on the set of tableaux (of each given shape $\lambda \vdash n$):

$$\sigma \cdot \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline \sigma(1) & \sigma(2) & \sigma(3) \\ \hline \sigma(5) & \sigma(6) \\ \cline{1-2} \sigma(4) \\ \cline{1-1}\end{array}.$$

This action preserves row-equivalence, so S_n also acts on tabloids.

This action is transitive: give any $\{t_1\}, \{t_2\}$ of shape λ, there exists $\sigma \in S_n$: $\sigma \cdot \{t_1\} = \{t_2\}$.

Equivalently, the orbit of each tabloid has cardinality

$$\frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_k!} = \# \text{ tabloids of shape } \lambda.$$

The stabilizer of a tabloid consists of those permutations that only permutes element within the same rows (never mixing the entries of different rows).

For instance, if $\{t\} = \begin{array}{ccc}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 8 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$, then

$$St_{\{t\}} = S_{\{1,2,3\}} \times S_{\{4,5,8\}} \times S_{\{6\}} \times S_{\{7\}} \simeq \underbrace{S_3 \times S_3 \times S_1 \times S_1}_{\text{row-lengths !!!}}.$$

In general, the stabilizer of a tabloid of shape $\lambda=(\lambda_1,\dots,\lambda_k)$ is isomorphic to $S_{\lambda_1}\times S_{\lambda_2}\times\dots\times S_{\lambda_k}$ and has cardinality $\lambda_1!\,\lambda_2!\dots\lambda_k!$.

[This is in accordance to the fact that the cardinality of the orbit is $\frac{n!}{\lambda_1!\,\lambda_2!\dots\lambda_n!}$].

Define $\tilde{M}^\lambda$ to be the permutation representation associated to the action of S_n on tabloids of shape λ.

[It's a well defined representation of S_n, of dimension $\frac{n!}{\lambda_1!\,\lambda_2!\dots\lambda_n!}$, and an element $\sigma\in S_n$ acts on $x=\sum_{\{t\}} a_{\{t\}}\{t\}$

by $\sigma\cdot x = \sum_{\{t\}} a_{\{t\}}\,\sigma.\{t\} = \sum_{\{t\}} a_{\{t\}}\{\sigma.t\}$.]

Let's show that $\tilde{M}^\lambda = M^\lambda = \mathrm{Ind}_{S_{\lambda_1}\times S_{\lambda_2}\times\dots\times S_{\lambda_k}}^{S_n}$ (trivial).

Because $\tilde{M}^\lambda$ and M^λ have the same dimension, it is enough to show that the restriction of $\tilde{M}^\lambda$ to the Young subgroup $S_{\lambda_1}\times S_{\lambda_2}\times\dots\times S_{\lambda_k}$ contains the trivial representation. [The equality will follow from Frobenius reciprocity]. This is easy to do:

WRONG ARGUMENT! (but easy to fix see next set of notes)

realize $S_{\lambda_1}\times S_{\lambda_2}\times\dots\times S_{\lambda_k}$ as a subgroup of S_n, by letting S_{λ_1} act on $\{1 \dots \lambda_1\}$, S_{λ_2} act on $\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}$ and so on...

Then $S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}$ acts trivially on the tabloid

$$\{t_0\} = \begin{array}{l} \hline 1 \quad 2 \dots \dots \dots \lambda_1 \\ \hline \lambda_1+1 \quad \lambda_1+2 \dots \lambda_1+\lambda_2 \\ \hline \vdots \\ \hline \lambda_{n-1}+1 \dots \lambda_{n-1}+\lambda_n \\ \hline \end{array} .$$

$V = \mathbb{C}\{t_0\}$ is a subspace of $\tilde{M}^\lambda$ on which $S_{\lambda_1} \times \dots \times S_{\lambda_k}$ acts as the trivial representation.

$\Rightarrow \tilde{M}^\lambda = M^\lambda$.

SOME EXAMPLES OF PERMUTATION MODULES

<u>Set $n=3$</u>. Possible partitions: (3), $(2,1)$, (1^3).

$\boxed{\lambda=(3)}$ If λ is the trivial partition, then there is only 1 tabloid of shape λ: $\{t\} = \dfrac{\quad}{1\ 2\ 3}$.

Clearly S_3 acts trivially on $\{t\}$. So $M^{(3)}$ is the trivial representation of S_3.

$\boxed{\lambda=(2,1)}$ If $\lambda=(2,1)$, there are $6=3!$ tableaux of shape λ:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

row equivalent   row equivalent   row-equivalent

and 3 Tabloids of shape λ :

$$\begin{array}{c}\hline 1\,2 \\ \hline 3 \\ \hline\end{array} \; ; \; \begin{array}{c}\hline 1\,3 \\ \hline 2 \\ \hline\end{array} \; ; \; \begin{array}{c}\hline 2\,3 \\ \hline 1 \\ \hline\end{array} .$$

An element $\sigma \in S_3$ acts on $\begin{array}{c}\hline i\;j \\ \hline k \\ \hline\end{array}$ by : $\sigma . \begin{array}{c}\hline i\;j \\ \hline k \\ \hline\end{array} = \begin{array}{c}\hline \sigma(i)\;\sigma(j) \\ \hline \sigma(k) \\ \hline\end{array}$.

It's easy to check that the map

$$M^{\square\square} \longrightarrow \mathbb{C}^3$$

$$\begin{array}{c}\hline i\;j \\ \hline k \\ \hline\end{array} \longmapsto \underline{e}_k$$

is an intertwining operator from $M^{\square\square}$ to $\mathbb{C}^3$ ($\leftarrow \mathbb{C}^3$ being the permutation representation of S_3).

$\Rightarrow M^{\square\square} = \mathbb{C}^3$.

We know that $\mathbb{C}^3$ is reducible, and equal to the direct sum of a copy of the trivial and a copy of the standard representation.

$$\Rightarrow M^{\square\square} = \text{permut. repr. associated to the action of } S_3 \text{ on } \{1,2,3\} = \underbrace{M^{\square\square\square}}_{\text{trivial}} \oplus V .$$

$\boxed{\lambda = 1^3}$ then there are $6 = 3!$ Tabloids (because for every Tableau $\boxed{t}$, ~~xxxxx~~ the only tableau $t' \sim t$

is t itself !).

Notice that $\sigma \cdot \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array} = \begin{array}{|c|}\hline \sigma(i) \\ \hline \sigma(j) \\ \hline \sigma(k) \\ \hline\end{array}$ hence $\sigma \cdot \begin{array}{c}\underline{i} \\ \underline{j} \\ \underline{k}\end{array} = \begin{array}{c}\underline{\sigma(i)} \\ \underline{\sigma(j)} \\ \underline{\sigma(k)}\end{array}$,

$\Rightarrow$ if $\sigma \neq \mathbb{1}$, then σ does not fix any Tabloid.

$$\Rightarrow \chi_{M^{(1^3)}}(\sigma) = \begin{cases} 6 & \text{if } \sigma = \mathbb{1} \\ 0 & \text{ow.} \end{cases}$$

We can recognize the character of the regular representation : $\boxed{M^{(1^3)} \cong \text{REGULAR REPRESENTATION}}$.

$$\Rightarrow M^{(1^3)} = M^{(3)} \oplus 2 \underbrace{V}_{\text{an irr. in } M^{(2,1)}} \oplus \underbrace{V'}_{\text{a new irred.}}.$$

Generalizing these examples (to all n):

<u>FACT #1</u> : $M^{(n)}$ = trivial of S_n

same proof

$M^{(n-1,1)}$ = permutation repr. associated to the action of S_n on $\{1 \dots n\} = \mathbb{C}^n$

$\vdots$

$M^{(1^n)}$ = regular representation.

<u>FACT #2</u> $M^{(\lambda)}$ is irreducible $\Longleftrightarrow \lambda = (n)$

needs a proof

lso needs a proof

FACT#3 There exists a partial ordering of the partitions of n st

$$M^{\lambda} = \begin{bmatrix} \text{some irreducibles} \\ \text{Coming from } M^{\mu} \text{ with} \\ \mu \trianglelefteq \lambda \end{bmatrix} \oplus \text{one new irreducible}$$

this new irreducible (That appears in M^{λ} with multiplicity one) is The irreducible representation associated To λ.

FOR $n=3$, the ordering is :

$$\ydiagram{3} \trianglelefteq \ydiagram{2,1} \trianglelefteq \ydiagram{1,1,1} .$$

And indeed we see that :

- $M^{\ydiagram{3}}$ is irreducible ($\Rightarrow$ The irred repr. associated To $\ydiagram{3}$ is The Trivial repr)

- $M^{\ydiagram{2,1}} = \underset{\substack{\uparrow \\ \text{an irred.} \\ \text{in } M^{\ydiagram{3}}}}{U} \oplus \underset{\substack{\uparrow \\ \text{a new irred.} \\ \text{(with multiplicity one)}}}{V}$ ($\Rightarrow$ The irred repr. associated To $\ydiagram{2,1}$ is V = standard)

- $M^{\ydiagram{1,1,1}} = \underbrace{U \oplus 2V}_{\text{irred. that already appear in } M^{\ydiagram{3}} \text{ and } M^{\ydiagram{2,1}}} \oplus U'$ $\longrightarrow$ a new irreducible

($\Rightarrow$ we associate To $\ydiagram{1,1,1}$ The sign representation)